

Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy

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**Volume 31
Theoretical and Experimental Aspects
of Controlled Nuclear Fusion**



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PROCEEDINGS OF THE SECOND
UNITED NATIONS INTERNATIONAL CONFERENCE
ON THE PEACEFUL USES OF ATOMIC ENERGY

Volume 31
THEORETICAL AND EXPERIMENTAL ASPECTS
OF CONTROLLED NUCLEAR FUSION

PREFACE

More than 2,100 papers were submitted by the nations, the specialized agencies, and the International Atomic Energy Agency, which participated in the Second United Nations International Conference on the Peaceful Uses of Atomic Energy. The number of papers was thus about twice that involved in the First Conference. Provision was therefore made to hold five concurrent technical sessions in comparison with the three that were held in 1955. Even so, the percentage of orally presented papers was less in 1958 than in 1955.

In arranging the programme, the Conference Secretariat aimed at achieving a balance, allowing adequate time for presentation of as many papers as possible and, nevertheless, leaving time for discussion of the data presented. Three afternoons were left free of programme activities so that informal meetings and discussions among smaller groups could be arranged. No records of these informal meetings were made.

A scientific editorial team assembled by the United Nations checked and edited all of the material included in these volumes. This team consisted of: Mr. John H. Martens, Miss L. Ourom, Dr. Walter M. Barss, Dr. Lewis G. Bassett, Mr. K. R. E. Smith, Martha Gerrard, Mr. F. Hudswell, Betty Guttman, Dr. John H. Pomeroy, Mr. W. B. Woollen,

Dr. K. S. Singwi, Mr. T. E. F. Carr, Dr. A. C. Kolb, Dr. A. H. S. Matterson, Mr. S. Peter Welgos, Dr. I. D. Rojanski and Dr. David Finkelstein.

The speedy publication of such a vast bulk of literature obviously presents considerable problems. The efforts of the editors have therefore been primarily directed towards scientific accuracy. Editing for style has of necessity been kept to a minimum, and this should be noted particularly in connection with the English translations of certain papers from French, Russian and Spanish.

The Governments of the Union of Soviet Socialist Republics and of Czechoslovakia provided English translations of the papers submitted by them. Similarly, the Government of Canada provided French-language versions of the Canadian papers selected for the French edition. Such assistance from Governments has helped greatly to speed publication.

The task of printing this very large collection of scientific information has been shared by printers in Canada, France, Switzerland, the United Kingdom and the United States of America.

The complete Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy are published in a 33-volume English-language edition as follows:

<i>Volume No.</i>		<i>Sessions Included</i>
1	Progress in Atomic Energy	1, 2, 23a, 23b, 23c
2	Survey of Raw Material Resources	E-5, E-7b, E-9
3	Processing of Raw Materials	E-10, E-6 and E-7a
4	Production of Nuclear Materials and Isotopes	E-11, E-12, C-14, C-15
5	Properties of Reactor Materials	E-14, E-15
6	Basic Metallurgy and Fabrication of Fuels	E-13, E-17, E-18
7	Reactor Technology	E-19, E-21, E-22
8	Nuclear Power Plants, Part 1	3, 6, 7
9	Nuclear Power Plants, Part 2	B-9, B-10, B-11
10	Research Reactors	B-5, B-12
11	Reactor Safety and Control	B-13, B-14a, A-14
12	Reactor Physics	B-17, B-18, B-21
13	Reactor Physics and Economics	B-19, B-15, B-14b
14	Nuclear Physics and Instrumentation	A-18, A-19
15	Physics in Nuclear Energy	A-21, A-22

<i>Volume No.</i>		<i>Sessions Included</i>
16	Nuclear Data and Reactor Theory	A-11, A-12, A-13
17	Processing Irradiated Fuels and Radioactive Materials	C-17, C-18, C-19
18	Waste Treatment and Environmental Aspects of Atomic Energy	C-21, C-22, D-19
19	The Use of Isotopes: Industrial Use	5b, D-7
20	Isotopes in Research	D-6
21	Health and Safety: Dosimetry and Standards	5a, D-15
22	Biological Effects of Radiation	D-9, D-10
23	Experience in Radiological Protection	D-11, D-12
24	Isotopes in Biochemistry and Physiology, Part 1	D-13
25	Isotopes in Biochemistry and Physiology, Part 2	D-14
26	Isotopes in Medicine	D-17, D-18
27	Isotopes in Agriculture	D-21, D-22
28	Basic Chemistry in Nuclear Energy	C-9, C-10, C-11
29	Chemical Effects of Radiation	C-12, C-13
30	Fundamental Physics	15, A-17
31	Theoretical and Experimental Aspects of Controlled Nuclear Fusion	4, A-5, A-6
32	Controlled Fusion Devices	A-7, A-9, A-10
33	Index of the Proceedings	

TABLE OF CONTENTS

Volume 31

Session 4: Possibility of Controlled Fusion			<i>page</i>
P/145	Alfvén	Magnetohydrodynamics and Thermonuclear Problems	3
P/2298	Artsimovich	Controlled Fusion Research in the USSR.....	6
P/1056	Biermann	Recent Work on Controlled Fusion in Germany (Federal Republic)	21
P/2410	Teller	Peaceful Uses of Fusion	27
P/78	Thonemann	Controlled Thermonuclear Research in the United Kingdom	34
Record of session			39
 Session A-5 Theoretical Aspects of Plasma Physics			
P/2500	Braginsky, Shafranov	High-Temperature Pinches	43
P/2292	Dreicer	Runaway Electrons	57
P/383	Garren	Individual Particle Motion in Axial Magnetic Fields	65
P/1799	Hoyaux <i>et al.</i>	Effect of Distant Collisions on Autostriction	72
P/1861	Longmire	Static Pinch.....	84
P/347	Rosenbluth	Stability and Heating in the Pinch Effect	85
P/2213	Trubnikov, Kudryavtsev ...	Plasma Radiation in a Magnetic Field	93
P/2300	Akhiezer <i>et al.</i>	High Frequency Plasma Oscillations	99
P/357	Berger <i>et al.</i>	Plasma Heating by Oscillating Electromagnetic Fields	112
P/2215	Sagdeyev, Shafranov	Absorption of High-Frequency Electromagnetic Energy	118
P/361	Stix	Generation and Thermalization of Plasma Waves	125
P/1307	Ciulli, Micu	Plasma Oscillations in External Magnetic Fields	134
P/365	Kruskal, Oberman	Stability of a Plasma in Static Equilibrium	137
P/349	Rosenbluth, Rostoker	Theoretical Structure of Plasma Equations	144
P/2214	Sagdeyev <i>et al.</i>	Dynamics of a Rarefied Plasma in a Magnetic Field ..	151
P/354	Suydam	Stability of a Linear Pinch	157
P/33	Taylor	Stability of a Constricted Discharge	160
P/1538	Berkowitz <i>et al.</i>	Cusped Geometries	171
P/376	Berkowitz <i>et al.</i>	Magnetohydrodynamic Stability	177
P/386	Grad, Rubin	Hydromagnetic Equilibria and Force-Free Fields	190

		<i>page</i>
P/1875	Johnson Some Stable Hydromagnetic Equilibria	198
P/1876	Kruskal, Kulsrud Equilibrium of a Plasma in a Toroid	213
P/1	Adlam, Allen Large Amplitude Disturbances	221
P/2509	Akhiezer <i>et al.</i> Simple Waves and Shock Waves in Magnetohydrodynamics	225
P/374	Gardner <i>et al.</i> Hydromagnetic Shock Waves in High-Temperature Plasmas	230
P/1368	Barsanti <i>et al.</i> Possibility of Fusion with 4π Focusing	238
P/2532	Boulegue <i>et al.</i> Slowing Down of Charged Particles	242
P/1183	Braffort, Chaigne Mechanization of Some Fusion Reactor Calculations	247
P/2358	Geller <i>et al.</i> Engineering Aspects of a Thermonuclear Power Plant	257
P/1765	Grad Thermonuclear Reaction Rate in a Discharge	263
P/2430	Greyber Transient Temperature Variations in Self-Heated Plasma	267
P/1941	Gryzinski Fusion Chain Reaction	270
P/1184	Magnac-Valette <i>et al.</i> Energy Development in a Thermonuclear Plasma ...	275
	Record of session	278
 Session A-6: Experimental Aspects of Plasma Physics		
P/360	Stix, Palladino Ion Cyclotron Resonance	282
P/2211	Dubovoi <i>et al.</i> Ion Cyclotron Oscillations and New Injection Methods	288
P/2536	Sinelnikov <i>et al.</i> Electron Resonance, Plasma Wave-Guides and Plas- moids	292
P/344	Barnett <i>et al.</i> The Oak Ridge Thermonuclear Experiment	298
P/1790	Luce Intense Gaseous Discharges	305
P/2396	Neidigh, Weaver Effect of a Pressure Gradient on a Magnetically Colli- mated Arc	315
P/2383	Boyer <i>et al.</i> Ixion	319
P/1708	Dattner <i>et al.</i> Liquid Conductor Model of Instabilities	325
P/345	Kolb Magnetic Compression of Shock Pre-heated Deuterium	328
P/355	Marshall Acceleration of a Plasma into Vacuum	341
P/2301	Andrianov <i>et al.</i> High Current Pulsed Discharges	348
P/1460	Curren <i>et al.</i> High Current Gas Discharges	365
P/2302	Komelkov <i>et al.</i> Development of a Power Fuel Discharge in Deuterium	374
P/2504	Komelkov <i>et al.</i> Plasma Motion in Powerful Discharges	382
	Record of session	389

Session 4

POSSIBILITY OF CONTROLLED FUSION

LIST OF PAPERS

		<i>Page</i>
P/145	Magnetohydrodynamics and the thermonuclear problem H. Alfvén	3
P/2298	Research on controlled thermonuclear reactions in the USSR L. A. Artsimovich	6
P/1056	Recent work on controlled thermonuclear fusion in Germany (Federal Republic) L. Biermann	21
P/2410	Peaceful uses of fusion E. Teller	27
P/78	Controlled thermonuclear research in the United Kingdom P. C. Thonemann	34

Magnetohydrodynamics and the Thermonuclear Problem

By H. Alfvén *

At the first Geneva Conference, Dr. Bhabha made his famous prediction about the future of thermonuclear energy, and since then the possibility of the peaceful use of this energy has attracted a rapidly growing interest in the newspapers and also in scientific literature. It is the purpose of this and the following sessions to give a survey of the present situation.

The way which we have to go towards thermonuclear energy differs in some respects from the way which has led to the release of fission energy. The latter was opened by research in nuclear physics carried out as pure science. The objective was to gain knowledge of the structure of matter and not to achieve any technological results. In fact, it was not until the discovery of fission that technological aspects entered this field in a serious way.

As a comparison, it is of interest to see what is needed before we could hope to release fusion energy for peaceful purposes. This possibility does not depend on nuclear physics because the processes which possibly are of importance are very well known in nuclear physics. The main difficulty is to heat a gas to a very high temperature and to keep this gas confined during a sufficiently long time. The field of physics which we must study in order to be able to do so is magnetohydrodynamics and the physics of hot ionized gases, which is usually referred to as plasma physics. Magnetohydrodynamics is a combination of electromagnetism and hydrodynamics, and this means that we are facing the difficulties of electromagnetism multiplied by the difficulties of hydrodynamics. But the thermonuclear problem furthermore requires a combination of magnetohydrodynamics with plasma physics, which means that we have to multiply once again by the difficulties of plasma physics. Hence we are entering an extremely complicated, but also extremely rich and fruitful field, which is well worth our investigations even from the purely scientific point of view.

Up to quite recently magnetohydrodynamics was studied mostly because of its interest to astrophysics, where it is of basic importance. In fact, contrary to what was believed up to recently, ordinary hydrodynamics is applicable only to very few astrophysical problems, because under most astrophysical conditions

the electrical conductivity is so great that there is generally a coupling between the mechanical motion and electromagnetic phenomena. In solar and stellar physics, the physics of interplanetary and interstellar space, and also in quite a few problems of the ionosphere and of the earth's interior, we must introduce magnetohydrodynamics. Thus, in the whole universe, we leave to ordinary hydrodynamics only the lower parts of the planetary atmospheres and the seven seas.

This means that the thermonuclear problem is closely coupled with modern astrophysics. Progress towards the solution of the thermonuclear problem will necessarily be fruitful for astrophysics and, *vice versa*, progress in the concerned fields of astrophysics may be important also from the thermonuclear point of view.

Thus, summarized in a brief statement, fission energy was released as a result of purely scientific research carried out in order to investigate the atom and find the microscopic structure of our world. The progress towards thermonuclear energy is connected with astrophysics, the study of the macroscopic structure of our world.

There are at present a number of big projects aiming at the production of the necessary conditions for the release of thermonuclear energy. They are, of course, very important because they have directed interest towards the field and demonstrated its difficulties. But I think that of equal importance is an advance on a very broad front in magnetohydrodynamics and plasma physics, and I believe that this may lead to a great number of new projects. The field is rich in new possibilities, and we cannot be sure that any of the present lines of approach will lead to the best solution.

Capture of a Plasma by a Magnetic Field

As an example of what I mean I shall discuss one of the basic principles in the field. In order to make a thermonuclear reactor it is essential to keep heated gas enclosed during a certain time. The first question which should be decided is whether we should

- (a) confine the gas at first and then heat it,
- (b) heat it at first and then confine it, or
- (c) heat and confine it at the same time.

* Department of Electronics, Royal Institute of Technology, Stockholm.

Most thermonuclear projects seem to have chosen the first or, in some cases, the third alternative, in spite of the fact that from a technological point of view it may be an advantage to heat the plasma at first and then shoot it into a region of confinement where it will be used.

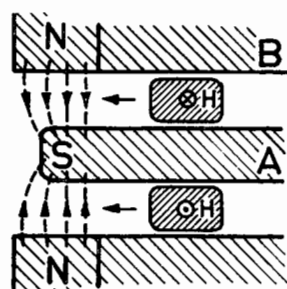
Let us discuss whether this could be an alternative to the present projects. This would be the case, for example, if we could shoot a heated plasma into a magnetic field in such a way that it is captured by the magnetic field. It has been argued that this is very difficult, if not impossible. A charged particle which is shot into a magnetic field from outside usually comes out of it again. Also, if we consider magnetic lines of force which are "frozen" into a plasma, we are tempted to conclude that the plasma could not be brought into a magnetic field from outside and captured by it. I think that we could learn from astrophysics that these objections are not watertight; for example, we can see what happens during magnetic storms and aurorae. This rather indicates that a magnetized plasma could be shot into and captured by the magnetic field. Certainly there are several conflicting theories about magnetic storms and aurorae but I think it is now fairly evident that a magnetic storm is caused by an extremely hot plasma which occupies parts of the geomagnetic field and is held captured by it during a considerable time. It

is also very likely that this plasma has been shot into the geomagnetic field from the sun, where it has been heated by "solar activity" and magnetized by the solar magnetic fields.

Hence, from a study of these phenomena, we could conclude that in principle it is possible to shoot a heated plasma into a magnetic field and capture it there. This might be checked by laboratory experiments in which we simulate the geophysical phenomena. However, there may also be other and simpler methods for achieving the same results.

A possible method of capturing a plasma which is shot into a magnetic field is illustrated by Fig. 1a. An electric field is applied between the cylindrical electrodes A and B and a magnetized plasma ring is produced which drifts in the axial direction due to the action of the electric and magnetic fields. In a region NS, a radial magnetic field is produced between a magnetic pole S inside the inner electrode and an annular pole N outside the outer electrode. If the plasma ring has a sufficiently high conductivity and density it may carry the magnetic lines of force with itself as illustrated by Fig. 1b. When the ring proceeds further the lines of force will disrupt so that we obtain a configuration as shown in Fig. 1c. The original field between the electrodes is almost restored and a magnetized plasma ring is produced.

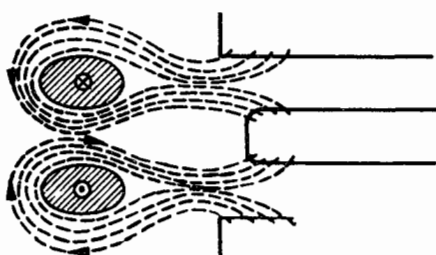
The ring is characterized by a longitudinal magnetic field which is wrapped up in field lines passing the hole of the ring. This structure is rather different from Bostick's plasmoids, but is similar to the field configuration in Zeta. The temperature of the plasma ring is a function of the kinetic and thermal energy of the initial plasma and of the strength of the radial magnetic field. The dependence is rather complicated.



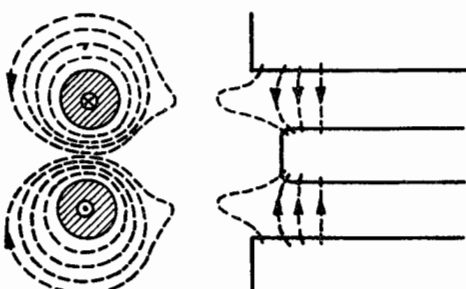
1a

Theoretical and Experimental Magnetohydrodynamics at Stockholm

Finally, I should mention, as an example of the different possibilities which there might be in this field, the theoretical and experimental work which is done in Stockholm. As will be reported later by Professors Ohlin and Siegbahn, interesting investigations in this field are also going on in Uppsala.



1b



1c

Figure 1. Capture of a plasma in a magnetic field

During the last decade the magnetohydrodynamic research in Stockholm has consisted of a series of theoretical and experimental investigations. The experiments have been carried out in conducting liquids, mercury or liquid sodium, which are placed in strong magnetic fields. But, of course, experimental investigations of hot plasmas seem to be an even more fascinating field. Following a long series of investigations of relatively cool plasmas, presented by Dr. Lehnert at this Conference, we have now started a series of experimental investigations on hot plasmas, concentrating on the fundamental properties of the plasma. We are also trying to analyze in a more or less systematic manner possible new ways of approach to the thermonuclear problem. As an example I should like to mention the theoretical and experimental investigations by Dr. Lehnert about whether a plasma could

be confined in a very simple way by the magnetic field of a circular loop of current. At first one would say "Yes, of course", but, on second thoughts, one would say "No, not at all, because the leads to the loop will disturb the conditions"; but I think it has been demonstrated that there is a fair chance that the effect of these leads to the loop can be eliminated by the magnetic fields of the leads themselves. We have also started experiments with discharges of a few hundred kiloamps to produce fast-moving magnetized plasmas,

in order to investigate whether they could be captured by magnetic fields in the way I have just discussed. Our investigations also cover other problems in magnetohydrodynamics with possible applications both to astrophysics and to the thermonuclear problem.

We have chosen this line of research because we believe that the way to solve the thermonuclear problem should be by an advance on a broad front in magnetohydrodynamics, plasma physics and astrophysics.

Research on Controlled Thermonuclear Reactions in the USSR

By L. A. Artsimovich *

The idea of developing controlled thermonuclear reactions for power generation by utilizing the synthesis of light atomic nuclei has been expressed time and again in the past. But it is only after many years of preparatory work, the results of which were not tangible earlier, that justified hopes have arisen as to the possibility of finding a successful solution to this problem. A rapid expansion of the scale of research followed, and now the development of ways of attaining controlled thermonuclear reactions has become the most important problem of atomic engineering.

The trend towards a lessening of the rigid bounds of secrecy that had earlier so fully isolated physicists working on this problem in the different countries from one another was important in speeding up scientific investigation. As long as the dominant doctrine in this field was that the danger of publishing one's scientific results cannot be balanced by the advantage to be gained through scientific information from abroad, research languished. A significant change occurred in 1956, when certain results of work done in the USSR were disclosed for the first time. This was followed by the gradual emergence of publications from England and the USA. It was then possible to begin some sort of an exchange of ideas and experience—a thing of incalculable value in attacking such a scientific and technical enigma as is the problem of controlled thermonuclear reactions.

In spite of the wide range of the investigations for controlled thermonuclear reactions, all of them are still in the stage of exploring various approaches to the problem. Not a single one of these approaches has been explored to such an extent as to permit one to say that success is assured. Apparently there is only one conviction that seems to be generally accepted, and that is that the solution of the problem must follow from a correct choice of the technique of magnetic confinement (thermoinsulation) for practical realization of the general idea of confining hot plasma by strong magnetic fields.

Methods of utilizing a magnetic field for the purpose of thermoinsulation and heating of plasmas may be divided into two basic groups. One group includes methods of accelerating the plasma by electrodynamic forces; the other, methods of obtaining equilibrium plasma configurations, that is, states in which the pressure of the plasma is balanced by magnetic pressure. The difference between these two groups of methods becomes more distinct if we express it in terms of magnetohydrodynamics, which deals with the general laws of behavior of a conducting fluid in a magnetic field. Under certain conditions which we shall assume fulfilled, the plasma may be regarded as analogous to a conducting fluid, macroscopically speaking. The equation describing the behavior of a plasma under the action of electrodynamic forces is

$$\rho \frac{d\mathbf{v}}{dt} = \frac{1}{c} \mathbf{j} \times \mathbf{H} - \nabla p \quad (1)$$

where $\mathbf{v}$ and ρ are respectively the velocity and density of an elementary volume of plasma moving under the action of electrodynamic forces and a pressure difference. The electrodynamic force acting on a unit volume of plasma is represented by the first term on the right-hand side of the equation. It is due to the interaction between the magnetic field and the currents flowing in the plasma ($\mathbf{H}$ is field strength and $\mathbf{j}$ is current density).

A glance at this equation is sufficient to see that two extreme cases are possible, each characterizing a large group of confinement methods.

If the gas kinetic pressure is relatively small, the electrodynamic force will be balanced by "inertial forces":

$$\rho \frac{d\mathbf{v}}{dt} = \frac{1}{c} \mathbf{j} \times \mathbf{H}.$$

Under these conditions, the plasma as a whole will acquire under the action of electrodynamic forces a directed velocity which may considerably exceed the random thermal velocity of the ions. The kinetic energy of directed motion due to acceleration of the plasma in the magnetic field may then be utilized for subsequent heating of the substance (in processes

Original language: Russian.

* Academy of Sciences of the USSR, Moscow.

of the implosive compression type, during impact of accelerated plasmoids on a target, etc.). Some concrete techniques of such utilization will be described below.

Characteristic of this type of plasma-magnetic field interaction is the short duration of the process. In order of magnitude it is equal to a/v , where a is the distance traversed by the plasma under the action of the accelerating forces, and v is the velocity attained. In cases of practical interest, the duration of this acceleration process should be of the order of 10^{-6} – 10^{-5} sec. Obviously, such momentary pulsed processes will be of considerable interest only if it is possible to utilize them as the first phase in heating the plasma. This phase should result in the transformation of kinetic energy into heat and in the transition to some quasi-stationary state in which the rapid inertial motions remaining after the first phase should damp out within a very short time.

An opposite case will occur if acceleration of the plasma is small and if the "inertial term" on the left-hand side of the equation may be disregarded compared with the pressure gradient. In this case the gas-kinetic and magnetic pressures balance each other for all times:

$$\nabla p = \frac{1}{c} \mathbf{j} \times \mathbf{H}.$$

One may imagine a multiplicity of ways for attaining such equilibrium plasma configurations characterizing a quasi-stationary state of plasma in a magnetic field. At present the following trends appear to be emerging in more or less clear outline:

(a) Development of methods for confinement and heating of plasma in systems with large closed discharge currents which are maintained by an external voltage and stabilized by an external magnetic field;

(b) Research into magnetic traps in which a high-temperature plasma is produced by accumulation of fast particles injected into the trap.

This classification of different approaches to the problem of obtaining and maintaining high temperatures in plasmas is of course very incomplete, but within the scope of this paper it is quite natural, since it corresponds to the principal directions of research on controlled thermonuclear reactions in the USSR.

Before describing some concrete results obtained in the theoretical and experimental studies carried out along the lines mentioned above, I should like to touch on some questions relating to the general properties of future thermonuclear reactors. Of course I fully realize that at the present stage of our knowledge any discussion of such questions can only rest on our faith in the ultimate triumph of human ingenuity.

First it is necessary to note that, irrespective of which concrete variant of magnetic system is proposed as a technical solution of the thermonuclear energy problem, the system must satisfy one fundamental condition: the energy released in nuclear fusion must cover by a wide margin the energy

consumed from other sources for maintenance of the high temperature in the plasma.

A simple analysis shows that this condition may be written in the form of the following relationship between the principal parameters determining the performance of a thermonuclear generator:

$$H^2 \tau > A(1 - \eta). \quad (2)$$

In this formula, H is the intensity of the plasma-confining magnetic field and τ is the time the high temperature is maintained in the plasma. The quantity η denotes that fraction of the thermal energy of the plasma which is converted into electrical energy at the end of the operation cycle. The constant A depends on the nuclear fuel selected. Under the most optimistic assumptions concerning the character of the processes occurring in a thermonuclear generator, this constant may be taken as equal to 10^{10} for pure deuterium and 10^8 for a mixture of D and T in equal proportion. These values of the constant are based on the supposition that thermonuclear reactions in the plasma proceed at an "optimum" temperature, which for deuterium amounts to 50 keV (5×10^8 °K) and for the D-T mixture to 15 keV (1.5×10^8 °K).

When applying the inequality (2) with the indicated values of the constant A one must bear in mind that strictly speaking it refers to the ideal case, where there is no escape of particles from the plasma in the high-temperature regime. This implies that the particle lifetime coincides with the time the high temperature is maintained in the plasma.

From condition (2) it follows that the shorter the time interval during which a high temperature is maintained, the higher the intensity of the magnetic field must be. In order to satisfy these requirements by means of present-day electrical facilities, one must have some method for confinement of the fast particles of the plasma during periods of time of the order of seconds or even tens of seconds. To illustrate, if τ is put equal to 10 seconds then the field strength in a generator with pure deuterium must be of the order of 30,000 gauss. This value is within limits technically feasible for stationary apparatus. However, it should be pointed out that in this case the power released per unit volume of the generator would be small, and for such a complex machine to be workable it would have to be of an enormous size.

We shall now examine the problem of direct conversion of thermonuclear energy into electric power. The energy released in a fusion reaction consists of two different parts that play quite unequal roles in the operation of a thermonuclear generator. The energy carried away by neutrons produces no effect on the processes occurring in a plasma. In the balance of electric energy produced by a generator, this part can participate with an efficiency that does not exceed 0.3 (which is the same as for the energy released in conventional nuclear power stations using the fission of heavy nuclei). The other part of the nuclear fusion energy is related to the charged particles produced in the high-energy reaction; it is

released directly in the plasma, producing the temperature rise and it may be converted into electrical energy with an efficiency η close to unity.

The possibility of a transformation of this nature stems from the fact that during magnetic thermo-insulation the high-temperature nuclear fuel is surrounded by a strong magnetic field which is similar to an elastic shell compressing the plasma. Upon expansion a plasma heated to a high temperature does work against the magnetic pressure at the expense of thermal energy which is thus converted into electromagnetic energy. If the maximum temperature of the plasma at the onset of expansion is equal to T_1 and the minimum temperature to which the plasma cools at the end of each work cycle of the thermonuclear generator is T_2 , the maximum value of η will be given by the familiar formula

$$\eta = 1 - \frac{T_2}{T_1}.$$

From the foregoing it follows that at least in principle it is possible to make η very close to unity, inasmuch as the upper temperature in the thermal cycle of the generator is very high and therefore the ratio T_2/T_1 can be made sufficiently small.

However, it should be borne in mind that a very considerable reduction of the temperature during expansion can be accomplished only at the expense of a large increase in the plasma volume. In the case of adiabatic expansion, the temperature is inversely proportional to $V^{2/3}$, where V is the plasma volume. For this reason, the volume the plasma occupies should be increased many-fold in order to convert a large part of the thermal energy of the plasma into electric power. This implies that, during the period of time the plasma is at maximum temperature and is a source of thermonuclear energy, it should occupy only a very insignificant part of volume of the vacuum chamber in the thermonuclear generator, the remaining volume being filled solely by the strong magnetic field. Since only a small part of the generator volume is utilized at η close to unity, some compromise value of η , probably not exceeding 0.75, must be chosen.

The principal advantage of direct conversion of thermal energy of the plasma into electrical energy consists in the fact that in this method utilization of thermal energy of the plasma is attended by much smaller irreversible thermal losses (the magnitude of which is proportional to $1-\eta$). Moreover, the thermal conditions in the system are also more favorable since the thermal load on the plasma chamber walls is lower.

The total amount of energy produced by a thermonuclear generator should not depend to any large extent on whether the thermal energy of the plasma is converted directly into electric energy or an intermediate heat-transfer agent is required. This can readily be seen if we recall that a large part of the energy released in nuclear synthesis is carried away by neutrons and cannot therefore be directly transformed into electric energy.

In most methods of exciting controlled thermonuclear reactions the energy source is either deuterium or a mixture of deuterium and tritium. Of these two types of nuclear fuel, the future will, apparently, belong to the D-T mixture. Its chief merit is the large value of the reaction cross section. In the temperature interval of practical interest the yield of a mixture with equal parts of D and T exceeds that of pure D by two orders of magnitude. Though the tritium obtained from conventional atomic reactors is at present a very expensive nuclear fuel, this can only be a temporary barrier to its use, for there exist methods that can easily compensate the tritium used up in a thermonuclear generator.

In each elementary D-T reaction event, a nucleus of tritium disappears and a fast 14.1 Mev neutron is produced that subsequently leaves the plasma. If a thermonuclear generator is surrounded with a sufficiently thick layer of a substance in which fast neutrons induce $(n, 2n)$ reactions, it should be possible to increase substantially the initial neutron flux. For neutron multiplication via $(n, 2n)$ reactions we may use either beryllium or such heavy elements as lead or bismuth. In these substances, the effective cross section of the $(n, 2n)$ reaction for 14.1 Mev neutrons exceeds that of other competing nuclear processes. If any one of these substances is used as the shell of the thermonuclear reactor the number of neutrons should increase roughly 1.5- to 2-fold. This enhanced neutron flux may be utilized for breeding tritium by disintegration of Li^6 . An analysis of data relating to $(n, 2n)$ reactions and tritium-producing reactions shows that even in very conservative estimates the breeding ratio of tritium in a thermonuclear reactor should easily be made to exceed 1.

Thus, the use of a D-T mixture should enable us to provide conditions for constant enhancement of the supply of tritium. Therefore, as long as there is no danger that the accessible supplies of Li^6 on the earth will be exhausted, thermonuclear reactors can be operated on a D-T mixture with regeneration of tritium. To this we may add that even if for some reason we are forced to give up regeneration of tritium and begin to use deuterium, the principal energy effect will nevertheless be obtained from tritium produced in the D-D reaction.

Now let us examine the theoretical and experimental material on controlled thermonuclear reactions amassed during the past few years in the USSR. Our attention will be directed chiefly to the results of recent work that have not yet been extensively discussed. I shall present the material in accordance with the earlier mentioned classification of research trends.

FAST PULSED PROCESSES

In studies of pulsed processes of short duration, the main efforts thus far have been concentrated on powerful pulsed discharges in deuterium at low pressure. The principal idea here is to obtain a high temperature and a high density in a compressed

plasma column over a short period of time. An investigation of pulsed discharges with a very high rate of current build-up (from 10^{10} to 10^{11} amp/sec) has shown that irrespective of whether such discharges occur in linear tubes or in toroidal chambers, the main role is played by acceleration of the plasma by electrodynamic forces. In the initial phase of the pulsed discharge the plasma is pinched to the axis of the discharge tube. This compression is the first stage of rapid oscillation of the plasma column. Maximum temperature and density is reached when the column radius is a minimum.

Theoretical

Two stages can be discerned in the theoretical analysis of phenomena occurring in plasma during pulsed discharges. At first a very simple model was used for a qualitative description of the dynamics of the plasma column. The motion of the ionized gas as a whole under the action of given electrodynamic forces was examined. At the next stage an attempt was made to carry out a more rigorous quantitative calculation of the contraction and pulsation of the plasma column that would take into account the formation and implosion of shock waves in the plasma and the time variation of the mass of the moving gas. This calculation was performed in the magneto-hydrodynamic approximation in which the plasma was regarded as a monatomic gas with constant conductivity. Even in this extremely idealized form the problem reduced to a complex system of partial differential equations which can be solved only by numerical integration in large electronic computers.

The solution of this problem yields data on the density, temperature and drift velocity for each part of the plasma column at different instants of time, and also data concerning the distribution of the current density and magnetic and electric fields in the plasma. The volume of this information is far in excess of what can be derived from experimental studies of the properties of plasmas. However, the reliability of theoretical results for a series of the more important special cases can be checked against experimental findings. This can be done for the distribution of current density and radial velocity of the plasma, since these characteristics of the process are accessible to experimental determination. A comparison of theory and experiment of this kind was performed in a number of particular cases, and as a whole resulted in satisfactory agreement, at least as regards the initial stage of the pulsed process up to the second phase of compression. For this reason, one may regard as justified the use of the results of theoretical calculations to estimate the magnitude of quantities which cannot readily be measured experimentally with sufficient accuracy. Most important of such characteristics of the plasma are the temperature at the instant of maximum implosion of the shock wave and the density distribution of the substance at this instant of time.

As numerical calculations have shown, the following equation may be used to evaluate the lower limit of the temperature at the first compression maximum:

$$T = 4.6 \times 10^{12} \frac{I^2}{N} \quad (3)$$

where I is the current in ka at this instant of time and N is the number of particles of a given sign per unit length of plasma column. This equation gives the mean plasma temperature (in kev) on the assumption of ideal heat exchange between ions and electrons. The temperature distribution along the radius of the column is plotted in Fig. 1. The dashed line is the mean value of T . Due to a cumulative effect created by the incident shock wave, the temperature at small distances from the axis rises sharply and exceeds the mean value several-fold. If in addition we note that under typical experimental conditions thermal equilibrium between ions and electrons is attained at the instant of maximum contraction only for $T < 0.1$ kev, it should be obvious that the foregoing equation serves as a most cautious estimate of ion temperatures in a real plasma column. We may therefore confidently use it to estimate the lower limit of the temperature achieved in the latest experiments on powerful pulsed discharges.

Figure 2 shows the density distribution of matter at the instant of implosion. The maximum density at this instant is from 30 to 40 times the original density of the gas. One of the interesting peculiarities of the theoretically determined density distribution is the low value in the neighbourhood of the column axis. This type of density variation is directly connected with the temperature rise near the axis (the pressure, proportional to ρT , should level out in this region).

Experimental

Experiments conducted during recent years on high-power pulsed discharges have continued the main trend of earlier research, the results of which were published in 1956 and are well known to our colleagues. In the recent experiments, much attention has been devoted to raising the parameters of the pulsed discharge with the aim of attaining higher temperatures.

Design refinements in the spark-gap system and electric power system have made it possible to raise considerably the voltage per unit length of the discharge tube and simultaneously reduce the spurious inductance of the circuit. This has resulted in a several-fold increase in the rate of current build-up and in a rise of the current during the first contraction up to 500 ka (in a discharge tube 50 cm long and 40 cm in diameter). Figure 3 shows the temperature in a plasma column as a function of the initial voltage applied to a discharge tube filled with deuterium at initial pressure of 0.05 mm Hg. The temperature was calculated from Eq. (3) which, as was mentioned above, yields values which are certainly too low. For this reason, we may assume with a high degree of confidence that in the experiments mentioned a temperature exceeding 3 to 4 million degrees was actually attained. One conclusion therefrom is that the neutron emission observed under these conditions is due in large measure to thermonuclear reactions.

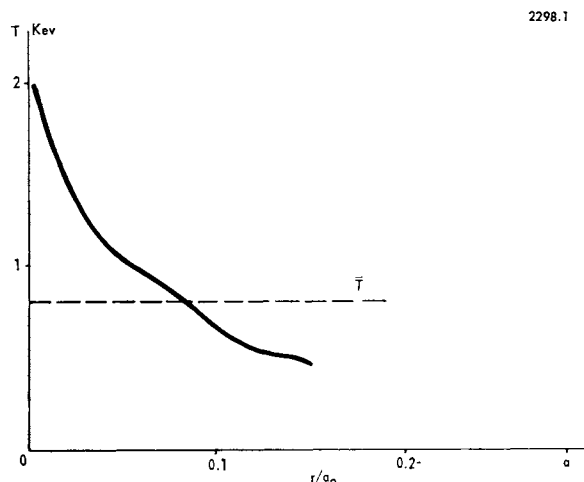


Figure 1. Radial distribution of temperature at the first compression maximum

In this connection, mention should be made of the fact that in such high-power processes neutrons appear immediately after the first phase of compression, that is, when both the temperature and density are at a maximum, and the neutron pulse is spread out over a period of the order of a microsecond.

At present, however, proof or disproof of the thermonuclear origin of a small burst of neutron emission in the pulse discharge is hardly of such importance as to warrant special attention in discussions on this subject. This is why I do not consider it necessary to insist that in the above-mentioned experiments thermonuclear reactions were actually observed.

The question of whether a given neutron belongs to the noble race of descendants of thermonuclear reactions or whether it is the dubious offspring of a shady acceleration process is something that may worry the pressmen but at the present stage in the development of our problem it should not ruffle the composure of the specialists. When the number of neutrons arising in a single discharge pulse reaches a value in excess of 10^{12} then all doubt as to the origin of this effect will vanish.

However, in order to achieve thermonuclear neutron emission of this magnitude in pulsed discharges, it will be necessary to conduct experiments with electric circuits of considerably higher parameters than have been involved heretofore. The chief difficulty in this direction is that further increase of the power of the discharge pulse is impeded by heating of the chamber walls. To a certain degree, this difficulty may be surmounted by using sectional metal-walled chambers. However, attempts should be made in other directions. For example, an interesting possibility would be magnetic protection of the walls and the use of vacuum chambers with local injection of directed streams of gas.

Besides the series of experiments with high-power discharges, a number of other investigations were carried out to study the different properties of the high-temperature plasma in a pulsed discharge.

Much headway has been made in spectroscopic studies of plasma. After a method had been developed for obtaining streak photographs of discharge spectras, it was found that at the instant of the first maximum compression a sharp flash of the continuum was observed throughout the whole range. This flash is particularly vivid in the photograph in Fig. 4, which is a streak photograph of the spectrum of a discharge in hydrogen at an initial pressure of 0.1 mm Hg and an initial voltage of 35 kv. The flash of the continuum is explained by the fact that at the instant of maximum compression there is a jump in the degree of ionization of the plasma and, consequently, in the concentration of free electrons, the result being an intensive bremsstrahlung and recombination glow. By measuring the intensity of the continuum in a given spectral range, it is possible to measure the density of the plasma with sufficient accuracy (on the assumption of total ionization).

As calculations show, due to a fortuitous play of numbers, the intensity of the emission is very insensitive to the magnitude of the electron temperature. For this reason, any arbitrariness in choice of the electron temperature produces practically no effect on the value of the density derived from measurements of the spectral continuum. Density measurements carried out in this way yield data that are in good agreement with the values obtained from magneto-hydrodynamic theory.

Spectral investigations likewise permit evaluation of the plasma temperature from Doppler broadening of impurity lines. As yet this method has been applied only to discharges under standard conditions (initial voltage 35 kilovolts, hydrogen pressure 0.05 mm Hg, tube length 90 cm). Under these conditions, the plasma temperature at the instant of maximum compression as measured from the width of the nitrogen line N IV 3479 works out at roughly 100 ev, as against 65 ev predicted by Eq. (3).

The aim of a number of studies was to determine the properties of hard radiation arising in the plasma and of the mechanism of production of this radiation. Cloud chambers were used to investigate the spectrum of electron produced by hard X rays from pulsed discharges. These studies confirmed the earlier

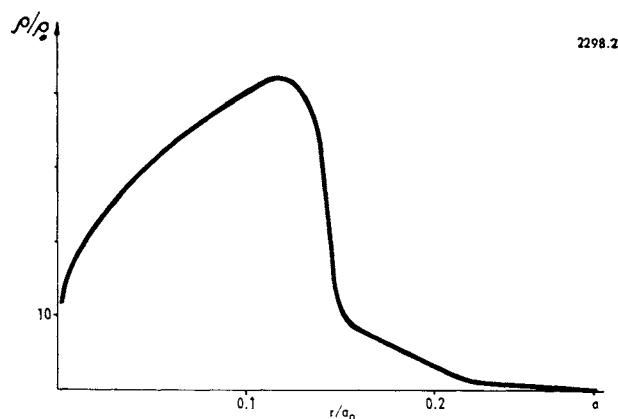


Figure 2. Radial distribution of density at the first compression maximum

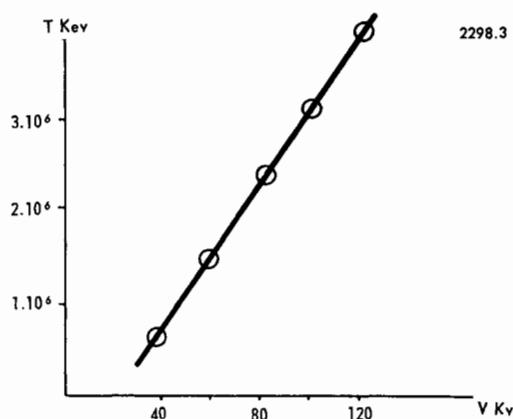


Figure 3. Temperature as a function of initial applied voltage

estimate of the maximum X-ray energy (about 350–400 keV).

Also successful was an attempt at mass-spectroscopic analysis of fast particles produced in the discharge. The parabola method was used to measure the value of e/m and the energies of ions extracted from the discharge chamber through openings in the side wall or in the electrode. It was found that the deuteron energy reaches 200 keV.

In discussions of the possible mechanism of processes that could lead to the appearance of hard radiations, a frequent opinion was that an essential role is played here by column instability of the "neck" type which enhances cumulation of the shock wave. To check this supposition, experiments were conducted with a discharge geometry which from the start created conditions that would make the compressing plasma assume an approximately spherical shape. These experiments yielded interesting results in that they proved that artificial creation of spherical implosion in the compressing plasma radically alters the emission of hard radiations.

Now let us briefly touch on the question of the future of thermonuclear reactions based on the utilization of powerful fast discharges. From the purely physical point of view, everything ultimately depends on two basic characteristics of the pulsed discharge:

- (a) the degree of compression of the plasma column, which is characterized by the minimum value of the column radius a ;
- (b) the duration of the compressed state τ .

It is easy to see that the energy efficiency of a powerful pulsed discharge viewed as a source of thermonuclear reactions is proportional to the square of the current and to the quantity τ/a^2 . Theoretical estimates of the numerical value of the quantity τ/a^2 in the case of very powerful pulsed devices vary more than two orders of magnitude, depending on the portion of optimism which either explicitly or implicitly enters into the initial conditions of the calculation. However, even with optimum prerequisites, it is found that the efficiency of a thermonuclear

reactor based on pulsed discharges can approach unity only if an enormous amount of energy is concentrated in the system (of the order of 10^{10} joules when operating on a D-T mixture). This energy is initially stored in the power sources and then for a brief moment of time it is converted, to a large extent, into the thermal and mechanical energy of the expanding plasma column. This stage of the process would have the nature of a powerful explosion, at the least estimate equivalent to that of 10 tons of TNT. At the present level of technical development we do not know of any way of rationally utilizing this explosive energy nor do we know of any means of protecting the unwieldy and expensive apparatus from the destruction that should occur after each pulse. Still we do believe that continued research on pulsed discharges at a technical level higher than so far achieved is of definite interest, with current build-up rates approaching 10^{13} amp/sec and in conditions where gases from the walls of the discharge chamber do not play any part at the initial stage of compression. Such experiments would not go beyond what is technically feasible at present. However, they may unexpectedly lead to the discovery of new facts which may effect in a fundamental way the general development of research into this problem. Studies of powerful fast discharges have been developing mainly in the Institute of Atomic Energy of the Academy of Sciences of the USSR. Certain questions in this domain have been the subject of investigations carried out at the Physics Department of the Moscow University. The Institute of Atomic Energy of the Academy of Sciences of the USSR, as well as the Ukrainian Physico-Technical Institute and the Sukhumi Institute of Electron Physics have also carried out experimental investigations of some other types of pulsed processes in which the plasma is acted on by strong magnetic fields.

Plasma acceleration by electrodynamic forces is achieved in its purest form in devices such as the electrodynamic gun. The simple principle underlying all such devices consists in the following. A plasmoid produced as a result of the passage of a large current through a fine wire or a gas cloud is accelerated along metallic rails under the action of a force F due to interaction of the current I with the magnetic field of the leads (or, in another version, with an external magnetic field). Experiments have shown that in systems of this type, plasma velocities up to 5×10^7



Figure 4. Streak photograph of the spectrum of a hydrogen discharge

cm/sec can be attained without special effort. Probably considerably greater velocities are attainable.

Another method of obtaining plasmoids has been investigated at the Institute of Atomic Energy. It consists in the following. First a circular plasma loop containing a current is produced in an alternating external magnetic field H_e , the lines of force of which are normal to the plane of the loop (Fig. 5a). The loop is created as the result of breakdown of the gas by the induced electric field during build-up of the H_e field. After a certain interval of time following the formation of the loop, the latter begins rapidly to contract to the axis and changes into a plasmoid (Fig. 5b). Experiments have shown that in this way

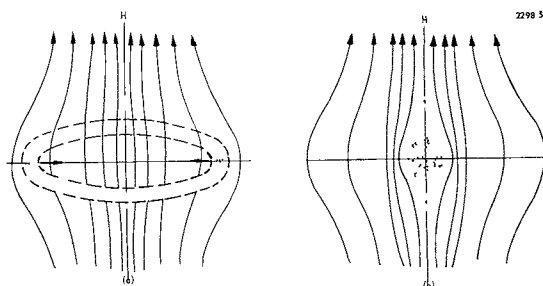


Figure 5. Plasma loop in an external magnetic field

it is possible to obtain plasmoids with particle concentrations up to 10^{16} cm $^{-3}$ and with an initial temperature of at least 100–200 ev. One of the imaginable practical consequences of these experiments may be the development of a method of injecting hot plasmas into magnetic traps.

A pulsed thermonuclear reaction may also be possible under conditions when a high temperature is reached during the compression and implosion produced not by electromagnetic forces, but by a charge of conventional explosives (such as TNT or something more powerful) surrounding a capsule of deuterium or a mixture of deuterium and tritium. Without going into details of the experiments, it should be mentioned that conditions have been found under which the generation of neutrons both in the $D + T$ and the $D + D$ reactions is detected with absolute reliability and reproducibility. In this case the recording apparatus is destroyed. However, the signal from the neutron pulse reached buildings located at a greater distance before the explosion has time to destroy the instruments. In experiments conducted in 1952 it was possible to record both fast neutrons that passed through the charge without any great loss of energy as well as neutrons that were slowed down in the explosive and entered the apparatus gradually, creating a pulse with a width of several tens of microseconds. In this case, obviously, the notorious question of whether these neutrons are thermonuclear or not is not present. We surely observed in this case neutrons produced as the result of the heating of matter to extremely high temperatures.

The chief difference between this process and electromagnetic compression is that the former

proceeds under conditions in which the density of the substance is very great, considerably exceeding that of solids. The brief duration of the heating process makes it possible to dispense with magnetic thermo-insulation. It is quite obvious that this process can be of economic interest only if the release of thermonuclear energy can balance the cost of the expensive explosives.

Just as in the case of large pulses of electric energy with power yields up to 10 tons of TNT (see above), difficulties connected with the explosive nature of the process are encountered on the way towards a practical utilization of implosive heating.

SLOW PULSED DISCHARGES IN TOROIDAL CHAMBERS

In cases where the current builds up at a slow rate the discharge conditions should be expected to differ fundamentally from those observed when the current rises at a fast rate. A quantitative criterion which may be used to differentiate between "slow" and "fast" discharges is the ratio of the current rise-time to the period of inertial radial oscillations of the plasma column. Hundreds of inertial oscillations may occur in rarefied gas discharges with peak currents of the order of 10^5 – 10^6 amp and durations of the first half-period of the order of 10^{-8} sec. Such discharges may be called "slow", in contrast to "fast" discharges, in which only two or three radial oscillations occur before the current reaches its peak value.

Theoretical

In slow discharges the gas-kinetic pressure of the plasma may be expected to balance the electrodynamic forces and the column temperature will be raised at the expense of Joule heat.

An equilibrium state of this type will be suitable for heating of the plasma to very high temperatures only if the following two conditions are satisfied:

(a) The plasma column should not be in contact with the walls.

(b) The state under consideration should not only be in equilibrium but be stable as well.

Progress in the theoretical investigation of equilibrium and stability conditions of plasma columns and of their laws of heating has been made in work carried out in the Institute of Atomic Energy of the Academy of Sciences of the USSR under the guidance of M. A. Leontovich. It was first shown in this work that stability of a plasma column with a more or less sharply defined boundary can be attained only if the discharge chamber is enclosed in a conducting sheath (which must be close to the chamber walls) and also only if a stabilizing magnetic field produced by external coils and directed along the column exists at the same time as the field produced by the plasma current. Two different stability regimes were inferred. One of these is realized when the contracting plasma column captures a large part of the magnetic flux of the longitudinal field which initially exists in the chamber ("paramagnetic

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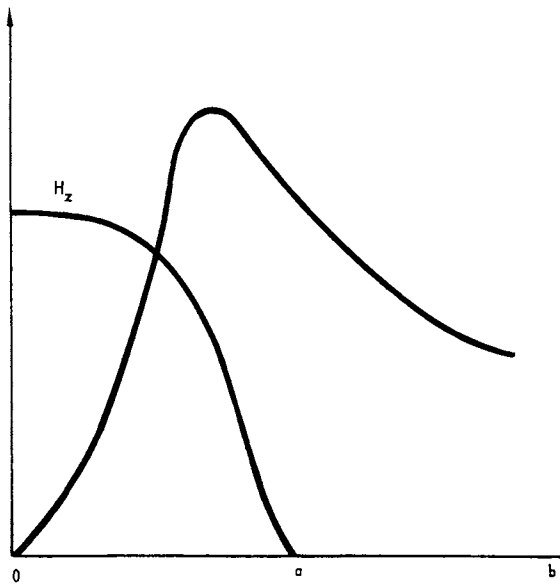


Figure 6. Radial distribution of longitudinal field H_z and current field H_ϕ (with maximum near a), in the "paramagnetic column"

column"). In this case the longitudinal field inside the column is approximately equal to $H_0 (b/a)^2$, where H_0 is the initial field strength, and b and a denote the inner radius of the discharge chamber and the radius of the plasma column, respectively.

The conditions considered above were investigated in detail by English physicists working with Zeta. Theoretically this approach seems to be the most promising as far as stabilization of the shape and size of the plasma column is concerned. In the magnetohydrodynamic approximation, the main condition for damping of all dangerous perturbations in highly conducting plasma columns is that H_z within the column should be sufficiently close to the field strength of the current on the boundary of the latter and should exceed several-fold the longitudinal field strength beyond this boundary (see Fig. 6, where the distribution of the longitudinal field H_z and current field H_ϕ are represented graphically). Another requirement is that the column radius a should not be too small compared to the radius of the chamber cross section, or else the stabilizing effect of the conducting sheath will vanish.

In order to realize the conditions discussed above a comparatively small initial value of the longitudinal field H_0 should suffice. Another advantage is that these conditions naturally arise during the development of pulsed discharges, which are initially spread over the whole cross section of the chamber and then, after H_ϕ becomes equal to H_0 , begin to contract, dragging along the lines of the longitudinal field. From a theoretical point of view, this possibility also possesses some weak points. Thus for a paramagnetic column to be stable, H_z and $H_\phi(a)$ should differ but slightly. Therefore only a minor part (0.3–0.4) of the electrodynamic pressure produced by the current

will be able to counteract the gas-kinetic pressure. This means that for equal values of I and N the temperature of the paramagnetic column will be several times smaller than if the longitudinal field were uniformly distributed over the cross section of the discharge chamber. It should also be added that in order to ensure satisfactory heat exchange between the electrons and ions under paramagnetic conditions the value of N should exceed at least by an order of magnitude the value corresponding to the case when the field is not captured.

The magnetohydrodynamic theory also predicts the existence of another stable discharge regime. In this case stabilization of the plasma column is attained with help of a strong longitudinal magnetic field H_z which everywhere is larger than H_ϕ . According to the theory, a necessary condition for stability in this case is

$$\frac{H_z}{H_\phi} > \frac{L}{2\pi a} \quad (4)$$

where L is the column length (equal to $2\pi R$ for an annular column in a toroidal chamber of radius R).

However, even if this requirement is met there still exist some types of unstabilized perturbations that tend to modify the shape of the column. The character of these perturbations is schematically depicted in Fig. 7. Apparently they should not be dangerous as they are not associated with displacement of the axis of the plasma column.

In the case under consideration the longitudinal magnetic field covers the whole cross section of the chamber and should not appreciably change near the column boundary (see Fig. 8). An advantage of this method of discharge stabilization is that the time during which the corresponding conditions are maintained does not depend on the skin-effect period but only on the time during which I and H_z persist. Another advantage is that a much higher ion temperature can be obtained than under "paramagnetic" conditions involving currents of the same magnitude. However, a high price must be paid for these possible advantages since very strong magnetic fields are required. Thus, magnetic fields of the order of 3×10^4 – 5×10^4 gauss will be needed in a large volume just to enable one to approach the threshold of temperatures of practical interest, and in order to produce such fields some very serious engineering

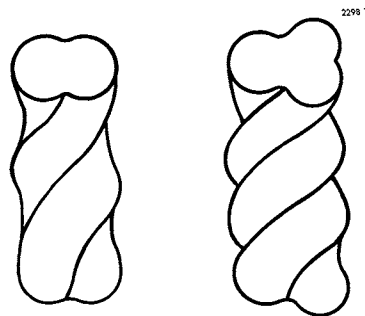


Figure 7. Unstable perturbations in the regime $H_z/H_\phi > L/2\pi a$

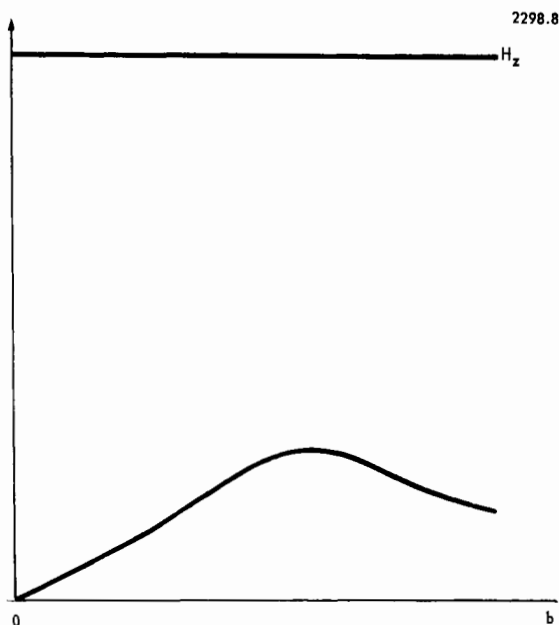


Figure 8. Radial distribution of longitudinal field H_z (upper curve), and current field H_ϕ (lower curve)

problems will have to be surmounted. Difficulties will also be encountered when establishing the necessary conditions in the system.

The results of magnetohydrodynamic theory briefly considered above nevertheless give good reason to believe that the approach suggested here should be a promising line of advance. It should be kept in mind that the preceding results were based on a simplification of the physical picture of the discharge and many important points may have escaped attention. The theory does not apply to displacements from the equilibrium position which possesses wave lengths smaller than the thickness of the skin-effect layer and which evidently cannot be damped by methods suitable for perturbations with larger wave lengths.

It should furthermore be noted that by its very essence magnetohydrodynamic theory cannot be employed to study, for example, such kinetic effects as transition of electrons to a state of continuous acceleration in an electric field. Due to the low density of matter and to resultant diminution in the retarding force which the ions exert on the electrons, such processes may occur near the boundary of the plasma column. They may also occur inside the column since in the electron energy spectrum there must be particles which possess energies considerably exceeding the mean energy, and for such particles acceleration may commence even at large values of N . It is well known that accelerated electron beams can excite various types of plasma oscillations and thus violate the normal course of the process. Other types of perturbations are conceivable which do not fit the simple magnetohydrodynamic picture and which are potentially dangerous as far as stability of the column is concerned.

Experimental

We now proceed to a discussion of the experimental results. In the early experiments all basic investigations were carried out with chambers made of insulating materials (glass, quartz and porcelain). Because of the gaseous efflux from the chamber walls reliable results could be obtained with these chambers only for relatively high gas densities in the chamber and for not too powerful discharges.

A study of the effect of a longitudinal magnetic field on the properties of slow discharges revealed that the paramagnetic effect earlier discovered by Soviet physicists in fast pulsed discharges could also be observed in discharges in which the plasma current build-up was slow. At low values of H_z the effect leads to contraction of the plasma column. The influence of the longitudinal magnetic field on plasma conductivity was found to be small. Irrespective of the initial pressure, the induced electric field strength and the value of H_z , the conductivity was approximately $1-3 \times 10^{14}$ esu which signifies a low plasma temperature in chambers with insulating walls. The assumption of pronounced instability of the discharge process at small values of the ratio H_z/H_ϕ is confirmed by studies in which the shape of the plasma column was recorded by ultra-high-speed cinematography and also by an analysis of oscillograms of the variation of the electric and magnetic parameters of the discharge. With growth of the external field the discharge becomes more stable but the conductivity does not exhibit an appreciable rise. In order to determine the stability conditions for a plasma column detached from the walls, experiments were performed with discharge tubes in which the discharge started near the axis and then began to expand towards the walls. These experiments qualitatively confirmed the theoretical conclusion that when inequality (4) is satisfied the column becomes stable. However, because of large losses at the electrodes there could be no hope of obtaining high plasma conductivities in these experiments.

The next step was the transition to chambers with metallic walls. It was hoped that in such chambers sufficient purity of the gas could be maintained during powerful discharges. Several large assemblies with

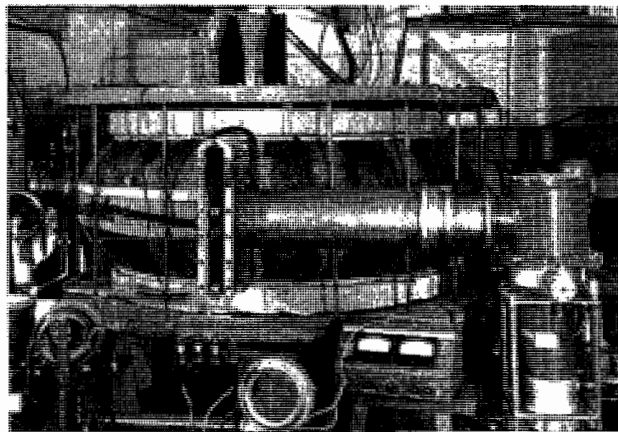


Figure 9. Apparatus for high-current toroidal discharges

metallic toroidal chambers were built at the Institute of Atomic Energy. A photograph of one of them designed for investigation of high-current discharges at various values of the external longitudinal field is shown in Fig. 9. A cutaway of the assembly is shown in Fig. 10. The discharge takes place in a closed toroidal chamber made of stainless steel 0.2 mm thick. This chamber was enclosed in a toroidal copper sheath 20 mm in thickness. Two insulated cuts in a plane parallel to the torus axis and an insulated cut along its generatrix are incorporated in the copper sheath. The inner thin-walled chamber and the space between it and the sheath are evacuated by separate vacuum systems. The diameter of the inner cross section of the discharge chamber is 0.5 m and the mean diameter of the torus 1.25 m. The chamber is the secondary coil of an air-core transformer. The primary coil is formed by 20 turns of a thick copper strip wound near the surface of the sheath. A copper shield which

encloses the sheath is arranged between the turns and the sheath and is employed to remove stray magnetic fields. The coils for producing the longitudinal field were wound directly on the surface of the copper sheath. The field can be increased up to 12,000 gauss. Electrical energy for the discharge circuit and longitudinal field coils is supplied by capacitor banks (at peak voltage the total amount of stored energy is 1.2 million joules). In the experiments carried out with the device described here the peak gas current was 400 ka (for a discharge voltage of 0.45 kv and longitudinal field strength of 12,000 gauss).

The main results obtained in the first stages of our experimental study of discharge processes in toroidal metallic chambers can be summarized as follows:

1. In hydrogen and deuterium, at initial pressures from 3×10^{-4} to 5×10^{-3} mm Hg, the peak current of a discharge whose first half-period lies between 300 μ sec and 1200 μ sec is approximately proportional

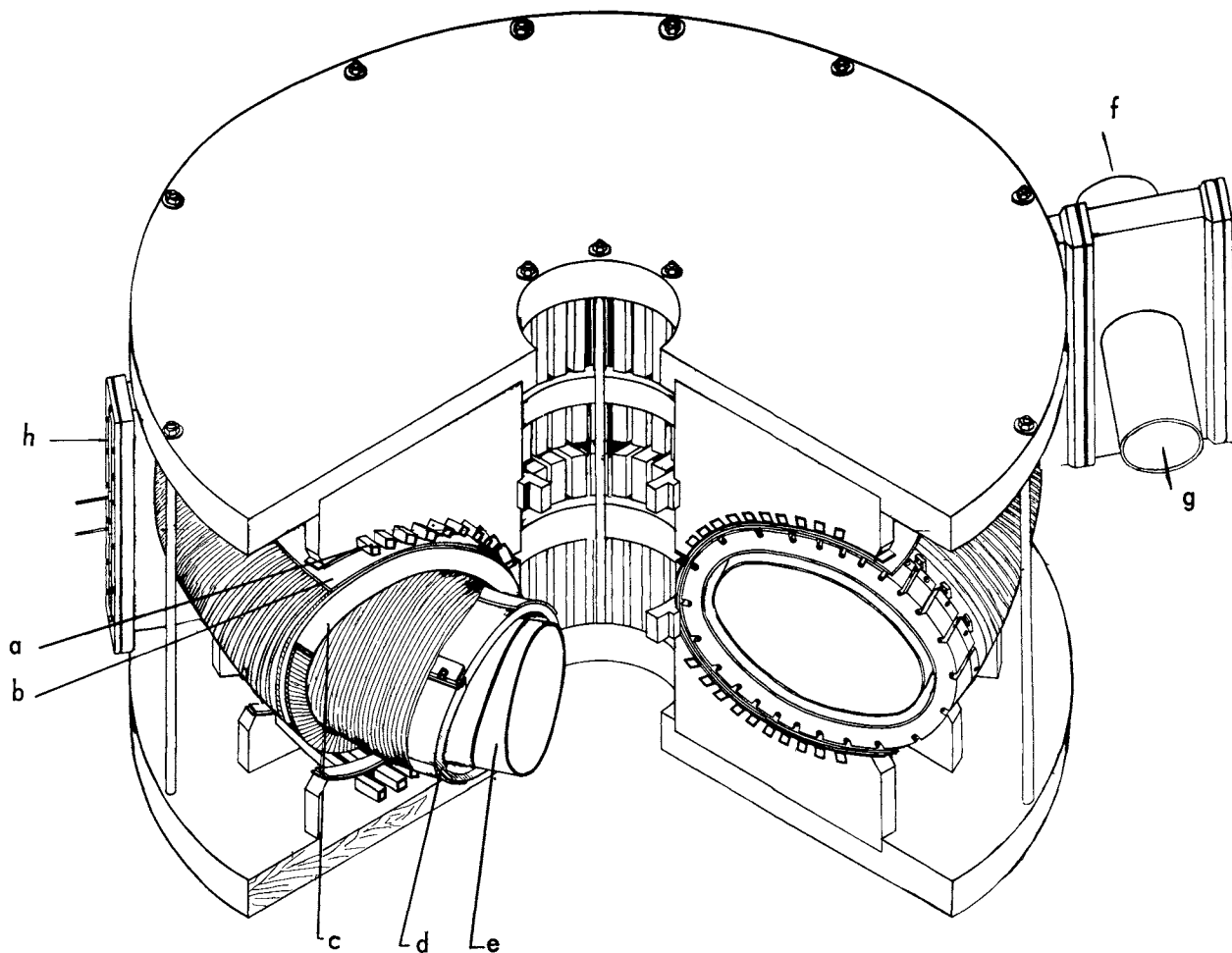


Figure 10. Cutaway of the apparatus of Fig. 9

- a Primary winding
- b Copper shield
- c Coils for axial magnetic field
- d Copper sheath (20 mm) surrounding discharge chamber
- e Stainless steel (0.2 mm) chamber with 0.5 m bore
- f, g, h Ports

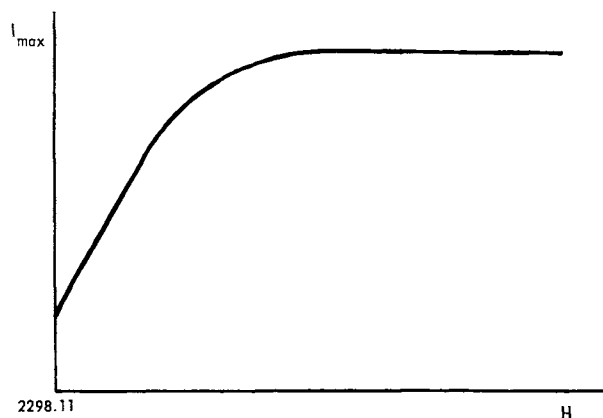


Figure 11. Dependence of $I_{\max}$ on H_z in the high-current toroidal discharge

to the initial discharge voltage and is weakly dependent on the gas density. With increasing H_z , the current first increases and then remains practically constant. The general character of the dependence of $I_{\max}$ on H_z is represented in Fig. 11.

2. At small values of H_z the plasma column separates from the walls, and its radius grows with increase of H_z . If $H_z \gg 0.2 I_{\max}/b$ where b is the radius of the chamber cross section, the plasma column will always be in contact with the walls.

3. Under conditions involving small values of H_z , when the column contracts and separates from the walls the plasma conductivity (computed on the assumption of uniform distribution of the current over the cross section of the column) approaches the value 1×10^{15} esu. In discharges located in strong longitudinal fields, when the plasma completely fills the cross section of the chamber, the conductivity drops to $1-2 \times 10^{14}$ esu.

The results presented above compel us to admit that in our experiments the electron temperature (deduced from the conductivity) probably did not exceed 15–25 eV.

This means that even under conditions in which one would expect the plasma column to be completely detached from the walls, energy losses are still very large and it is for just this reason that the plasma temperature does not reach the high value predicted by the theory. The mechanism of these energy losses is not clear at present. A possible explanation is that either the plasma column is not sufficiently stable or that impurity atoms emitted by the chamber walls contaminate the plasma.

These conclusions are of a preliminary nature since experimental investigations with metallic chambers are still in progress.

In the near future our main aim will be to create conditions which will significantly diminish the deleterious effect of impurities on plasma processes.

MAGNETIC TRAPS

Generally speaking, the term "magnetic trap" can be applied to any device used to obtain ultra-high temperatures and based on the principle of magnetic

confinement. However, it seems expedient to narrow the meaning of this term to denote a more definite type of system.

In the devices considered above, the plasma current maintained by external sources of voltage was the main factor in confining the heated substance. This implies that in such systems the particles appear to "hold on to each other" by means of the self-consistent magnetic field which they produce. External magnetic fields in this case play the subsidiary role of a medicine for combating the instability. In systems with large plasma currents the hydrodynamic pressure p is of the order of $H^2/8\pi$ and hence the plasma exerts a strong counterforce on the magnetic field, a consequence being the appearance of characteristic instabilities.

In contrast to devices of this type we shall apply the term magnetic traps to systems in which only external fields are employed to confine the plasma and in which plasma conduction currents do not play a decisive role. Therefore in magnetic traps $8\pi p/H^2$ may be small compared with unity, i.e., the rarefied plasma does not appreciably affect the external magnetic field.

Various methods can be employed to obtain highly heated matter in magnetic traps. One possibility is to fill the magnetic system with fast ions introduced, for example, by injection from a powerful accelerator. Another way would be to fill the trap with a plasma and then heat the plasma by dynamic magnetic fields or by a high-frequency electromagnetic field. Still another possibility is that of producing fast ions inside the trap itself, by accelerating ions from the plasma in the trap with the aid of a constant or variable electric field.

Theoretical

The most natural way of attacking the magnetic-trap problem would be to start with an analysis of the problem of the bounded motions of a single particle in an external field; after a satisfactory solution of this problem the next step would be to investigate the behavior of a large number of particles comprising a plasma. Despite obvious shortcomings in this procedure, it has the advantage of being graphic and, what is more, it permits one as a first step to bypass the difficulties inherent in an exact theory of the behavior of plasmas in fields of complex configuration.

Many Soviet physicists have been occupied with the problem of devising a magnetic trap which at least in principle could be realized practically. The work of Tamm¹ and Sakharov² stimulated discussion of this problem. In 1950 these authors proposed the first concrete model of a magnetic thermonuclear reactor. It was suggested that rarefied deuterium be ionized and heated in a toroidal chamber in which a strong longitudinal magnetic field is produced by a coil wound on the outer surface of the chamber.

As a prototype of a magnetic trap, however, this system is fundamentally defective. Each particle located in the inhomogeneous field inside the toroidal chamber would acquire a drift velocity perpendicular to the field lines; the drift would terminate only

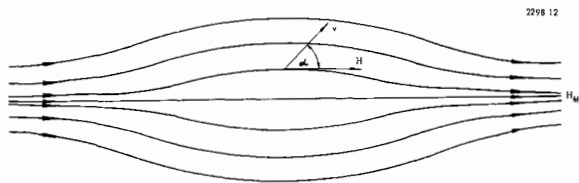


Figure 12. Axially symmetric magnetic trap utilizing "magnetic plugs"

after collision of the particle with the chamber wall. The authors of the toroidal generator model were aware of this circumstance and subsequently suggested that the drift could be suppressed by using the magnetic field of a plasma current.

Further work in this direction confirmed the fruitfulness of the original idea; several types of magnetic systems were found which could be used to confine the particles to a restricted region of space. As a result new horizons opened up for experimental work on different types of magnetic traps. However, the final goal, which is the invention of an ideal trap confining all particles regardless of direction of velocity, has not been reached and the question as to whether a system of this type can be devised at all still remains open.

The specific types of magnetic traps studied so far can be divided into two main classes:

1. Traps with magnetic plugs.
2. Traps with restricted drift.

A theoretical study of systems of the first type was started by G. I. Budker³ in 1953. In these systems the magnetic field increases along the lines of force on both sides of a median plane on which H is minimum (Fig. 12). Particles which possess velocities which form sufficiently large angles with the lines of force are reflected back when they approach the strong field region and hence are trapped in the magnetic system. For the sake of simplicity, the field in the central region will be considered to be uniform. Let H denote the strength of this field and H_m the maximum field strength at points of concentration of the lines of force (magnetic plugs). Then if the velocity vector of a particle located in the uniform field region lies within any of the cones defined by the condition

$$\sin^2 \alpha < \frac{H}{H_m},$$

during its subsequent motion it will pass in the vicinity of the maximum field and escape; if, however, the initial velocity lies outside these cones the particle will be trapped.

Suppose now that a thermonuclear generator is to be built on the basis of the magnetic trap considered above. A question which immediately arises in this case is that concerning the role of collisions between the particles. Evidently, one collision is sufficient for the velocity vector to enter the escape cone. The lifetime of particles of the plasma in the magnetic plug trap will equal in order of magnitude the mean time between two coulomb collisions of the ions. In

order that the energy released in a system of this type as a result of thermonuclear reactions be sufficient to balance energy losses due to fast particle escape as a consequence of coulomb collisions, the following condition must be satisfied:

$$\sigma_n W_n \sim \sigma_c kT.$$

Here, σ_n is the effective cross section for the nuclear reaction, σ_c is the effective cross section for coulomb collisions and W_n is the energy released in an elementary nuclear fusion event.

This condition can be used to estimate the minimum temperature at which the thermonuclear generator should begin to produce surplus energy. For D-T mixtures it is of the order of several tens of kilovolts, and for pure deuterium of the order of 1 Mev. Thus, for practical purposes pure deuterium evidently cannot be employed in such systems. This conclusion can be confirmed by more rigorous calculations in which other types of losses (bremsstrahlung and betatron radiation of fast electrons) are taken into account.

Theoretical studies of the stability of a plasma in traps with magnetic plugs carried out at the Institute of Atomic Energy predict the existence of perturbations of the equilibrium state which are unstable by themselves. Additional causes of instability in systems of the type considered here are anisotropy of the pressure in the plasma and the pronounced deviation of the component of the particle velocity (directed along the lines of force) from the Maxwellian distribution.

Some types of instability are closely related to the geometry of the magnetic field in the traps. It is therefore possible that a given type of disturbance can be stabilized for trap fields of some definite shape and will grow in traps of other types. Thus, for example, a dangerous type of instability can arise in systems in which the external boundary of the plasma has a negative curvature (Fig. 13). At points of negative curvature the plasma may leak out in the form of protuberances perpendicular to the lines of force. However, instability of this type should probably not be observed in systems with lines of force of positive curvature (Fig. 14).

The adverse effect of the chief types of instability should sharply increase with the ratio p/H^2 . Therefore, even if the practical value of traps with magnetic plugs turns out to be negligible such traps nevertheless may be used to demonstrate the physical feasibility of thermonuclear reactions at small values of $8\pi p/H^2$.

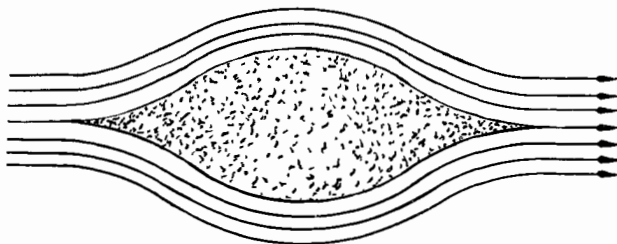


Figure 13. Theoretically unstable configuration

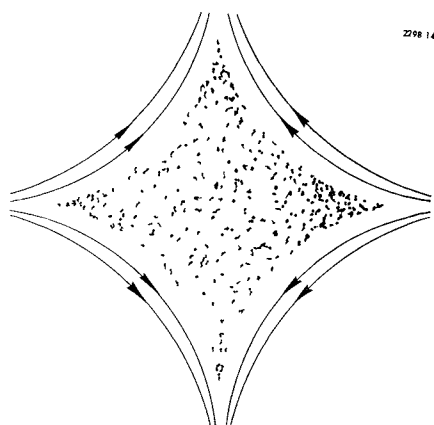


Figure 14. Theoretically stable configuration

Experimental

Experimental investigations of traps with magnetic plugs were initiated at the Institute of Atomic Energy several years ago. The first experimental system was built with the purpose of studying the storage of fast ions extracted from a cylindrical plasma column produced along the axis of the magnetic system (Fig. 15). Plasma columns with ion concentrations of the order of 10^{14} cm^{-3} were produced by arc discharges in a longitudinal field. A general view of the apparatus with which the chief experiments in the indicated direction are being carried out at present is shown in the photograph in Fig. 16. The length of the vacuum chamber is 2 m and the magnetic field strength is 8500 gauss in the central region and 12,000 gauss in the plugs. During operation of the ion source the pressure is maintained at 10^{-7} mm Hg by means of titanium pumps. A voltage of 40 kv between the plasma column and lateral wall of the chamber is applied for acceleration of the ions. The main aim of these experiments is the determination of the lifetime of fast ions for various ion concentrations in the magnetic system. The mean lifetime can be derived from the rate of decrease of the ion concentration after instantaneous switching off of the accelerating voltage.

Results of the first experiments permit one to conclude that in a system of the type considered plasmas containing fast ions exist during a period of the order of a few milliseconds. The lifetime increases with improvement of the vacuum and with increase of H .

In the summer of this year the construction of a large experimental assembly with magnetic plugs was completed at the Institute of Atomic Energy. The external injection of particles will be studied with this machine. For accumulation of D^+ ions, use is made of dissociation of D_2^+ molecular ions which are introduced from the injector into the central part of the trap. Dissociation of the molecular ions occurs either in collisions with the molecules and atoms of the residual gas or as a result of interaction with ions of the rarefied deuterium plasma previously produced inside the chamber. A schematic diagram of the appa-

ratus is depicted in Fig. 17. The distance between the centers of the plugs is 12 m; the inner diameter of the vacuum chamber is 1.4 m. The field strength in the central part can be raised to 5000 gauss and the field in the plugs up to 8000 gauss. An arc source with a transverse magnetic field is employed to produce the intense D_2^+ ion beam. The D_2^+ ions, accelerated by a voltage of 200 kv, are focused and injected into the magnetic trap by a system of electric and magnetic fields. It is expected that after adjustment of the apparatus it will be possible to raise the D_2^+ ion current entering the vacuum chamber to several hundred milliamperes. It is hoped that with injected ion beams of this intensity the concentration of the D^+ ions in the trap will reach 10^{12} .

An experimental study has also been started on systems possessing the field geometry shown in Fig. 14. This type of field is interesting in several respects. In particular, as mentioned above it is hoped that for some types of disturbances a higher degree of plasma stability will be attained.

Other types of traps

One may attempt to discover magnetic systems suitable for magnetic traps by applying a somewhat different method. The main problem is to ascertain under what conditions the particles do not escape from a restricted space region as a result of drift in an inhomogeneous field. A magnetic system which may be considered as illustrating one possible approach to this problem is represented in Fig. 18. In this system the magnetic field along the straight sections is "corrugated" (by a non-uniform winding on the chamber surface). A particle moving in such an inhomogeneous field will be subjected to rotational drift and its trajectory will therefore turn about the axis of the field. Thus, despite the presence of curvilinear crosspieces the particles should not approach the walls after a complete circuit, as rotation of the trajectories in the long linear parts should balance the effect of the directed drift in the crosspieces.

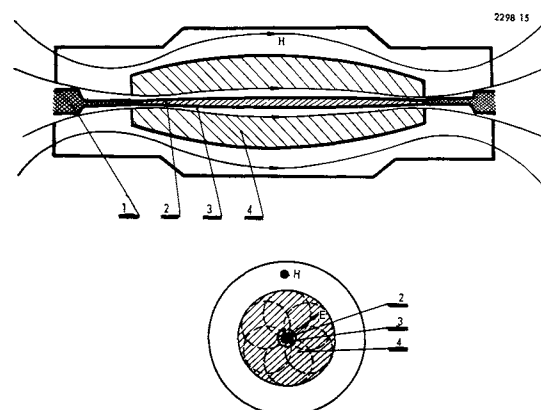


Figure 15. Experimental arrangement of magnetic trap experiment

- (1) Plasma source
- (2) Plasma beam
- (3) Region with strong electric field
- (4) Region of motion of fast ions

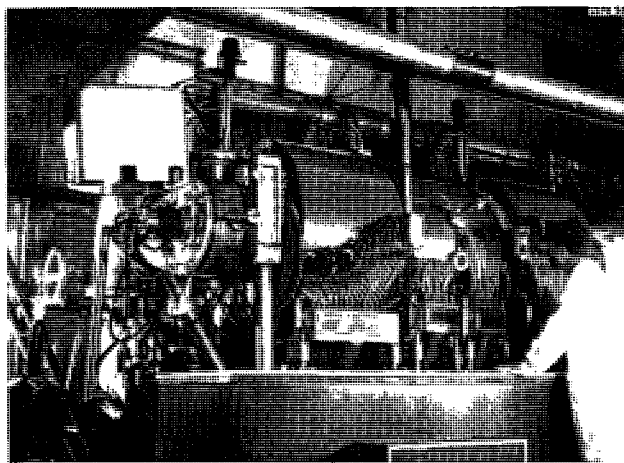


Figure 16. Experimental apparatus for magnetic trap experiment

It is easy to see that the system described above has the same defects as a trap with plugs. It is capable of confining only particles whose velocity vectors lie within an allowed range of directions. Any particle entering the curvilinear interval with a

small longitudinal velocity will drift to the wall before escaping from this interval. However, in all probability further development of the principle mentioned above should lead to systems in which the escape cone will be so narrow that the properties of these systems will approach those of ideal traps.

The methods described above for obtaining magnetic traps are based on the application of constant or slowly varying magnetic fields. Theoretical studies indicate broad possibilities of applying high-frequency electromagnetic fields for confinement and insulation of plasmas. Here, the term "high frequencies" means that the oscillation period of the field is much smaller than the periods characteristic of the behavior of the plasma. The respective frequencies ranged from tens of megacycles up. The plasma-confining force in this case is determined not by the instantaneous value of H but its mean-square-value.

Evidently, high frequency fields can be most effectively employed in conjunction with constant magnetic fields which perform the auxiliary task of eliminating energy losses due to corpuscular beams. For example, high-frequency electromagnetic fields can be used as "plugs" which trap particles escaping

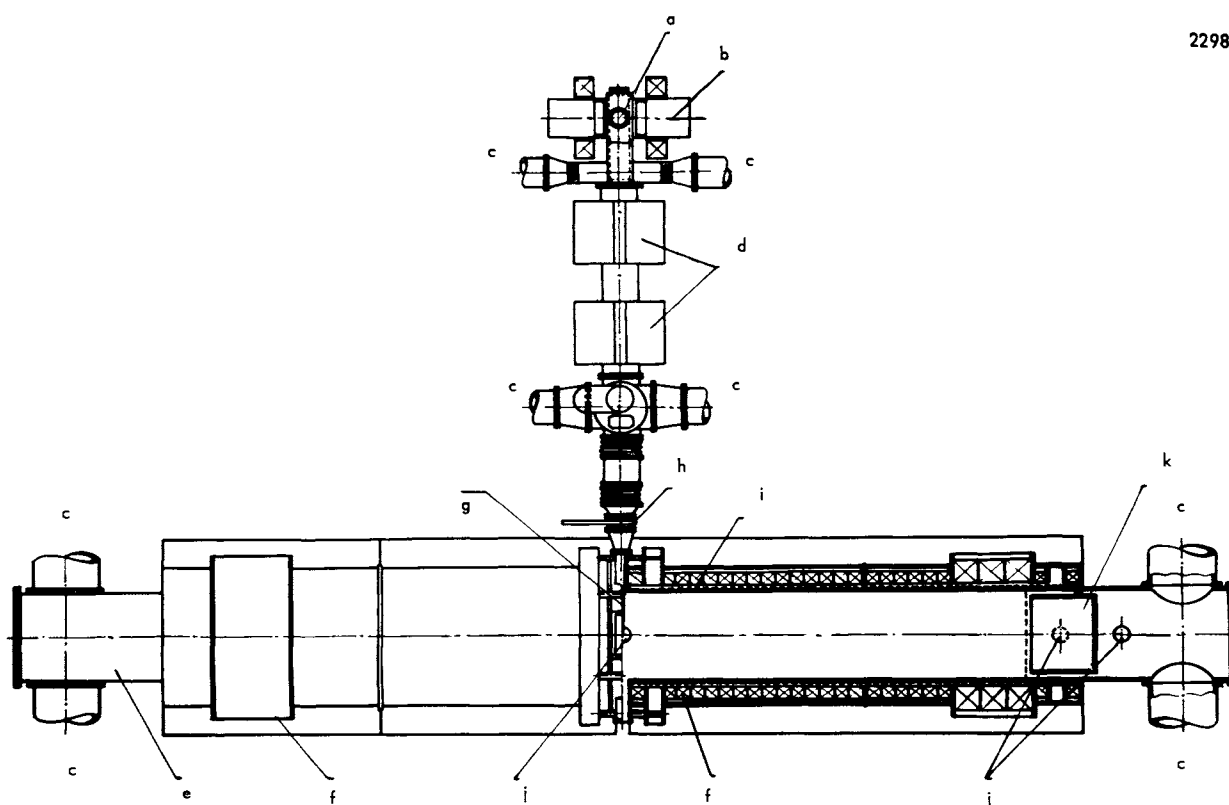


Figure 17. Magnetic trap with D_2 molecular ion injection

a 200 kv ion source	g Magnetic channel
b Source magnet	h Shutter
c Evacuation ports	i Chamber baking system
d Quadrupole lens	j Titanium evaporator
e Chamber	k Electrode
f Coil	

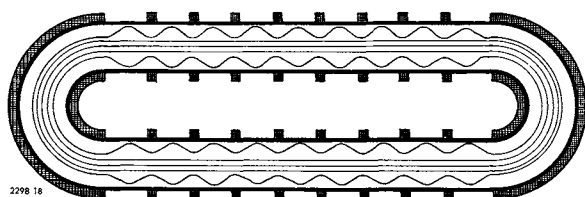


Figure 18. Magnetic trap with corrugated field in the straight sections to offset drift in the U-bends

along the lines of force from the strong constant field region.

The possibility of confining plasmas by means of field combinations of this type was experimentally confirmed with a small-size device at the Institute of Atomic Energy. Practically, the application of high-frequency fields for confinement of a heated plasma seems to be very unpromising because of the large amount of energy which will be required to maintain such fields. Nevertheless investigation of various possibilities of high-frequency thermal insulation deserves close attention.

High-frequency electromagnetic fields may also be used to heat the plasma. In this respect a study of the potentialities of ion cyclotron resonance would be of special interest. Investigations carried out in the USSR in this direction will be presented in a report from the Ukrainian Physico-Technical Institute.

CONCLUSION

In his speech at the First Geneva Conference on the Peaceful Uses of Atomic Energy, the President of the Conference, Dr. H. Bhabha said:

"The historical period we are just entering in which atomic energy released by the fission process will supply some of the power requirements of the world may well be regarded one day as the primitive period of the atomic age. It is well known that atomic energy can be obtained by fusion process as in the H-bomb, and there is no basic scientific knowledge in our possession today to show that it is impossible for us to obtain this energy from the fusion process in a controlled manner. The technical problems are formidable, but one should remember that it is not yet fifteen years since atomic energy was released in an atomic pile for the first time by Fermi. I venture to predict that a method will be found for liberating fusion energy in a controlled manner within the next two decades. When that happens, the energy problems of the world will truly have been solved forever for the fuel will be as plentiful as the heavy hydrogen in the oceans."

Three years have passed since this prophecy and now, before our eyes, there begins to emerge a rough outline of the scientific foundation on which the methods of solving the problem of controlled fusion reactions will probably rest. This foundation has been laid by the numerous experimental and theoretical

results obtained in recent years in Great Britain, the USA, the USSR and other countries. For the first time these results will be discussed on an international scale, and this is probably the most important step which has been made towards the solution of this problem. The importance of this fact is greater than that of the separate investigations, which as yet have not brought us very much nearer to our ultimate goal.

We do not wish to be pessimistic in appraising the future of our work, yet we must not underestimate the difficulties which will have to be overcome before we learn to master thermonuclear fusion. In the long run, the main difficulty lies in the fact that in such a light substance as rarefied plasma, any manifestation of instability develops at an enormous rate. The creation of an automatic monitor that could quickly damp various deviations from the equilibrium state is no simple task. It would therefore seem that the most radical solution of this problem would be to create a system in which all types of instability are removed beforehand.

Most likely the solution of this problem will be simpler when the reaction of the plasma on the magnetic field is small, that is for $\beta \ll H^2/8\pi$.

Returning now to a more general appraisal of the present state of affairs, it may be asserted that so far not a single one of the ideas regarding controlled fusion reactions is decisively superior to any other. Therefore, investigations in the near future should be carried out in diverse directions.

A most important factor in ensuring success in these investigations is the continuation and further development of the international cooperation initiated by our conference. The solution of the problem of thermonuclear fusion will require a maximum concentration of intellectual effort and the mobilization of very appreciable material facilities and complex apparatus.

This problem seems to have been created especially for the purpose of developing close cooperation between the scientists and engineers of various countries, working at this problem according to a common plan, and continuously exchanging the results of their calculations, experiments and engineering developments.

The combining of efforts on an international scale in the field of controlled fusion reaction investigation will undoubtedly shorten the time needed for us to arrive at our ultimate goal.

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Recent Work on Controlled Thermonuclear Fusion in Germany (Federal Republic)

By L. Biermann *

Before presenting a summary of the work on controlled thermonuclear fusion in the Federal Republic of Germany, I should like to make some general comments about the scope and background of this work

Soon after the first Geneva Conference on the Peaceful Uses of Atomic Energy, physicists in a number of scientific institutions in Western Germany began to give serious attention to the potential possibilities of controlled fusion. The advantages of thermonuclear devices and of using magnetic fields for containing the hot charged particles were realized very early. It was natural, therefore, that both experimental physicists with experience in the physics of gaseous discharges, plasmas and theoretical astrophysicists with their acquaintance with magnetohydrodynamics and with the propagation of cosmic rays in cosmic magnetic fields, started to think about how a plasma with the required high temperature could be controlled

I may say that the closest connections to our work are to be found, perhaps, in the developments at Princeton and at Harwell. Experiments on high current discharges at Gottingen[†] and Aachen[‡] are as yet on a fairly small scale as compared with some of the other experiments described at this conference, although experiments with toroidal discharges, suitable exterior magnetic fields, and large diameters ranging between 40 cm and 1 m, have also been considered at Aachen and Gottingen

In addition to the work on toroidal discharges, high current linear discharges are or will be used to investigate controlled fusion problems at Aachen, Munich, Hanover, Kiel and Stuttgart. The work at Kiel and Hanover is especially related to spectroscopic problems. The Aachen group has also studied

the magnetic compression of plasmas by means of fast electrodeless discharges

The theoretical work, most of which has been done in Gottingen, was started there in 1956 in advance of the experiments and pertains to a number of separate subjects. The subject of stability has also been considered by the Aachen group

EXPERIMENTS

Linear Discharges

In the Munich experiments,¹ a cylindrical glass vessel 20 cm in diameter and 50-cm long, filled with very pure deuterium at some 10^{-2} mm Hg, is used. The electrodes are formed by copper plates at the ends. A concentric copper cylinder conducts the current at the exterior of the glass cylinder. The capacitor bank stores approximately 30,000 J when charged to 40 kV. Measurements show that the current rises to some 100,000 amp in 1–2 μ sec and then fluctuates during several μ sec. Beginning with the second maximum of the current, neutrons are emitted for 2 or 3 μ sec in a continuous manner, their integrated number being of the order of 10^6 to 10^7 per discharge with considerable differences between the separate experiments. The fluctuations of the current are apparently produced by compressions of the plasma column (by a factor of ~ 25 in the density) owing to the pinch effect. The maximum associated radial mass velocities are of the order of 10^7 cm/sec. High energy quanta (in the range 50 to some 100 kV) were looked for, but none was observed. The number of neutrons depended critically on the purity of the deuterium. From the collective evidence mentioned it is concluded that approximate thermal equilibrium and a temperature of the order of several million degrees is present in the later phase of the discharge. In the Aachen experiments,^{||} currents of the order of a million amperes and temperatures of the order of 10^6 °K appear to have been reached, but further

Original language English

* Gottingen

[†] The Gottingen Max Planck Institute has just been transferred to Munich where facilities will allow work on a much larger scale than was possible in Gottingen. The experiments were carried out in the Physics Division of the Institute whereas the theoretical work was performed in the Astrophysics Division

[‡] The Aachen group will move shortly to a Fusion Institute at the Nuclear Center now under construction near Jülich

[§] E. Funfer *et al*, Laboratorium für techn. Physik, Techn. Hochschule

^{||} W. Fucks and H. L. Jordan *et al*, Physikalisches Institut, Techn. Hochschule

details have not been given. In the Stuttgart work,[¶] the corresponding values are 10,000 amp and 10^{50} K.

The investigations in Hanover^{**} and in Kiel^{††} are related especially to the spectroscopic aspects of the problem, and pertain up to now mainly to the temperature range around or below $100,000^\circ\text{K}$. Those in Kiel, where special attention is given to transition probabilities, may be characterized as "experimental astrophysics" related to the physics of stellar atmospheres. In the experiments in Hanover, the exploding wire technique has also been used; methods of measuring the temperature in the plasma column have been studied.

Toroidal Discharges

The present experiments in Göttingen^{‡‡} are performed with a pyrex torus of 100-cm major diameter and 4.5-cm bore. The coaxial electric field, of the order of several v/cm, is induced by means of a set of copper wires along the surface; a large number of ring-shaped magnetic cores serve to improve the coupling with the plasma column. The capacitor bank stores up to 10^4 j when charged to 0.1–30 kv, the potential used depending on the pressure. A number of coils are distributed around the torus in order to provide a coaxial magnetic field up to 20,000 gauss. The initial pressure of the gas is of the order 10^{-4} – 10^{-2} mm Hg and the pre-ionization is effected by RF excitation.

The measurements, which are still in progress, show that during approximately 100 μsec , current densities of the order of some 10^3 amp/cm are reached. The temperature should be of the order of 10^{60} K. Spectroscopic measurements and photographic and X-ray emission observations are being carried out. Experiments with larger toruses of 30-cm bore and much larger energies are being prepared, but these can be made only after the transfer of the institute to Munich.

THEORY

Basic Considerations

The Göttingen work originated in 1956 in connection with researches in theoretical astrophysics, in particular on astrophysical applications of plasma physics and magnetohydrodynamics and on the origin of cosmic radiation.²

The dependence of the ionization state of added heavier elements on the physical conditions is of interest in connection with experimental techniques of measuring the temperature in a discharge by observing the contours of spectral-line ions, e.g., of oxygen (to be added in a small amount or present as impurities), and the state of ionization of the heavier elements. Given sufficient time, for temperatures around 10^{60} K and electron densities below 10^{18} cm^{-3} ,

the ionization depends only on the temperature, precisely as in the solar corona,^{3,4} since radiative recombination is balanced by collisional ionization (in contrast to the conditions in perfect thermal equilibrium). It has been evaluated for oxygen by Knorr⁵ who found that at (and above) $kT = 50$ ev ($\approx 500,000^\circ\text{K}$), there should be $< 0.1\%$ of O^{V} left, the prevailing state of ionization being O^{VII} at that temperature. The results depend critically on the cross sections in question, for which the earlier work of Elwert⁴ was used. The time scale in which stationary conditions are approximately reached is of the order of a fraction of a millisecond for this temperature and an electron density of 10^{14} cm^{-3} , but much shorter for higher temperatures. The radiative energy losses by impurities of heavy elements in that temperature range are due mainly to the excitation of discrete levels.⁵ At lower temperatures other mechanisms dominate,⁶ e.g., excitation of molecules around $20,000^\circ\text{K}$, where the energy losses have a first maximum, and recombination radiation and loss of non-ionized particles around 10^5 to 10^{60} K.

The results of the work of Knorr⁵ are compatible with the more recent experiments with ZETA⁷ if it is assumed that the electron temperature is much lower than the value of some 10^{60} K which apparently pertains to the ions. The radiation losses due to impurities were found to be relatively large in the temperature range 10^5 – 10^{60} K even for small amounts of heavier elements.

The work on the equilibrium of dissociation and ionization and the energy losses of a high temperature discharge in pure hydrogen, begun by Schmidt and Stodiek, has been extended by Schmidt.⁸ The equilibrium of dissociation and ionization has been calculated by studying the different elementary processes as a function of the electron temperature for a density interval 10^{12} to 10^{15} H/cm³. It is found that there are deviations from thermodynamic equilibrium, since the photo processes are not important because of the small optical thickness of a typical gas discharge. Molecular recombination on the walls and the cooling of the neutral particles by impacts with the walls reinforce these deviations. (The interactions of the charged particles with the wall, which might be important under certain conditions, have not been included in this discussion because of the assumption of strong magnetic fields.) The losses of energy by radiation and by the transport of the kinetic energy of neutral particles to the walls due to recombination and charge exchange have been estimated. The maximum losses occur around $15,000^\circ\text{K}$.

Methods of Heating

Three methods have been studied in some detail: Joule heating by means of azimuthal currents induced in the plasma ring; high frequency radiation, in particular near the eigenfrequencies of the plasma; and, finally, forced oscillations of the magnetic field.

The possibilities of the mechanism first mentioned have been discussed by Grossmann-Doerth,⁹ who

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reached the conclusion that it should be effective with reasonable technical means up to temperatures of the order of one million degrees. Heating by high frequency radiation, and in particular, the dependence of the refractive index on the parameters involved, has been investigated by Körper.¹⁰ He found that its effectiveness depends critically on the physical parameters which enter. The third mechanism was recognized to be particularly effective if the oscillation frequency of the magnetic field is near to or larger than the collision frequency of the ions. This heating effect is due to the constancy of the magnetic moment between collisions and to the redistribution of the energy by collisions ("gyro-relaxation");¹¹ it has been studied in some detail by numerical integrations,¹² in which attention was paid also to the energy losses of electrons.

The equations for gyro-relaxation given by Schlüter have been integrated by Schmidt¹² for a simple model taking into account the free-free radiation of the electrons. Periodic variations of the magnetic field with a frequency of the order of the ion collision frequency lead to an increase of the temperature with the two-thirds power of the time. Finally, one can reach temperatures of the order of 10^9 °K asymptotically if the amplitudes are large enough. Here the time scale for the increase is somewhat larger than the collision time of the final state.

Körper¹³ has investigated the problem of heating by high frequency radiation in more detail. From the behavior of the refractive index it follows that one can use frequencies below the geometric mean of the gyro-frequency of the ions and the electrons. The efficiency depends very critically on the phase of the oscillations of the plasma at the surface. Since the wave-length in the plasma is very small compared to the linear dimensions of a normal plasma cylinder, a small variation of the radius, of the frequency, or of the physical parameters will change the phase drastically. Thus, averages over all phases have to be taken into account. The input impedance of the heating coils varies in the same manner with the physical parameters and therefore it is not possible to use the customary high frequency methods for matching the generator to the coil. Variational methods are to be used to find the optimum matching.

Configurations in Magnetohydrostatic Equilibrium

The aim of these investigations is to find the properties of exact solutions, of appropriate general character, of the equation of magnetohydrostatic equilibrium. This equation expresses the exact balance of the gas pressure gradient (or pressure difference) by the magnetic volume (or surface) force given by the vector product of the electric current density and the magnetic field intensity. The majority of those investigations pertain to the case of surface currents and to ring-shaped (toroidal) configurations of the plasma surrounded by a vacuum. The special case of axial symmetry and circular cross section was discussed with potential theory methods in 1957:¹⁴ the properties of the solution with ex-

clusively azimuthal surface currents, i.e., parallel to the generating circle, and the geometry of the magnetic field of exterior origin, which have to be superposed, were given in detail. Jörgens¹⁵ extended the analysis to the more general case of non-circular meridional cross section, using a stream function related to the magnetic flux, and indicated the locations of the singularities, outside the torus, which correspond to electric currents in exterior conductors. Kippenhahn¹⁶ established necessary and sufficient differential geometrical conditions for the surface of a plasma which must be fulfilled if the plasma is to be in equilibrium with a surrounding magnetic vacuum field and discussed, in particular, the case of azimuth-dependent circular cross sections and currents spiraling around the torus surface.

In all these cases it was found that a component of the electric current parallel to the generating circle is necessary for equilibrium. An attempt, made by Meyer and Schmidt,¹⁷ to prove this property to be true for still more general cases, revealed, however, that, in fact, solutions can be constructed, of torus-like geometry but with neither axial symmetry nor circular cross section, which are characterized by electric currents flowing everywhere with equal surface density along closed lines of meridional character, but not in meridional planes. The cross section appears to have at least two maxima along the generating circle. If there are only a few maxima, the relative variation of the cross section along the generating circle appears to be fairly large; if, in contrast, there are many extrema, the surface must have a rippled structure at least on the "inner" side (as seen from the centre of the aperture of the torus). The magnetic lines of force, in this case, are everywhere perpendicular to the current if there is no magnetic field in the plasma.

Axially symmetric solutions with volume currents have also been considered, using the stream-function method.¹⁸ For cylindrical symmetry, without dependence on z , the general solution has been given for the case of vanishing volume-forces; i.e., "force-free" magnetic fields.¹⁹

The investigations of Meyer and Schmidt¹⁷ on configurations of torus-like character with surface currents with vanishing line integral of the total current across closed meridional lines have been completed. In addition to the analytic treatment by means of a special coordinate system, a simple model of the surface currents and fields has been constructed, using paper strips of suitable shapes, folded and joined together in a special way; this model is a true analogue to the physical situation for the case of vanishing magnetic field in the plasma, and it allows one to survey existing solutions and their general properties rather easily.

Stability Problems

In 1957, general conditions for the stability of hydromagnetic configurations were investigated in Göttingen, by Hain, Lüst and Schlüter,²⁰ on lines

largely analogous to those along which the simultaneous work in Princeton has been done. For the special case of cylindrical symmetry the stability of certain specific configurations has been treated by Hain *et al.*²¹ by studying the various possible modes of disturbances. For $j \sim (r^2 + a^2)^{-2}$ and homogeneous coaxial magnetic fields, instability was found, the rates of growth decreasing with increasing coaxial magnetic field. More general cases for the current distributions and for the coaxial magnetic field are still being investigated. Since the emphasis in this work is to compare the properties of various current distributions, no regard is being taken of the possible stabilizing influence of external conductors. The stability of a linear discharge in an exterior longitudinal magnetic field with surface currents has been investigated theoretically by Jordan²² in Aachen.

Of the work done by Hain, Lüst and de Vries,²¹ a first paper by Hain and Lüst²³ has been completed which contains the discussion of the more special case.

To study the problem of the so-called exchange stability, Meyer²⁴ has extended an investigation by Kruskal and Schwarzschild²⁴ in such a manner that a plasma with gravity and horizontal magnetic field is supported by a vacuum magnetic field, which is also horizontal but is skew to the inner one. With the approximation used by Kruskal and Schwarzschild the unstable perturbations found by them can be stabilized. The stabilization depends on the angle between the two fields and on their relative strengths. The calculations are extended to the case of large wave-length. It is shown that here unstable perturbations with very large wave-length exist for arbitrary angles and strengths of the magnetic field.

Particle Losses by Drift Motions

Since the macroscopic equations of plasma physics give information only on the net balance of mass, momentum and energy, their discussion must be supplemented by studying, from the microscopic point of view, the motion and, in particular, the containment of individual charged particles in suitable electromagnetic fields. This can be done with the help of methods analogous to those developed earlier in connection with the theory of aurorae and of cosmic radiation.

For axially symmetric magnetic fields, Lüst and Schlüter²⁵ derived an integral for the azimuthal component of the velocity which shows that this component depends only on the meridional component of the magnetic field. It may be seen, in addition, that it is not affected by a meridional electric field (as produced, e.g., by pressure gradients). Also, the influence of an azimuthal electric field (as, e.g., generated by means of a transformer) can be included. This leads (by the energy integral) to the existence of so-called forbidden regions which show that all particles within a certain range of initial conditions can, indeed, be contained in such fields with a suitable

meridional component of the magnetic field; that is to say, with azimuthal electric currents. For the simplest cases of toroidal symmetry, the structure of the forbidden regions and the geometrical properties of a number of individual trajectories have been studied by Fisser and Kippenhahn.²⁶ The more special case of the motion in an azimuthal curl-free magnetic field of cylindrical symmetry (without dependence on the z coordinate) has been discussed in detail by Hertweck.²⁷ The particles show, in general, a drift parallel to the z axis. The magnitude of this drift has been calculated using the approximation method of Alfvén and by numerical solutions of the equations of motion. A comparison of the results indicates the range of validity of the approximation. The constancy ("adiabatic invariance") of the magnetic moment M or of the angular momentum of the gyrating motion of a charged particle in a magnetic field has been the subject of an investigation of Hertweck and Schlüter,²⁸ who established that, for a homogeneous field varying in time, the relative variation of M goes to zero at least exponentially with the rate of change of the magnetic field strength.

Fisser and Kippenhahn²⁶ have developed methods by which the properties of given trajectories in relation to the geometry of the allowed and the forbidden regions can be surveyed easily. These methods can also be used to discuss problems of particle loss. The authors have applied these methods to the special case of the magnetic field of an ideal circular ring current.

Magnetohydrodynamic Shock Waves

In connection with the theory of controlled fusion, as well as in that of the acceleration of cosmic rays in cosmic magnetic fields, the structure of hydromagnetic shock waves, in particular their thickness as compared to the kinetic mean path and the gyro-radius, is of importance. For this reason, the special case of a stationary plane wave, propagating in the absence of collisions with unchanging form and speed perpendicular to a magnetic field, in a completely ionized gas (plasma) of "zero" temperature, has been investigated rigorously by Davis, Lüst and Schlüter.²⁹ In spite of these severe restrictions (in addition, the Debye length has been assumed to be small compared with the gyro-radius), the results seem to make it possible to draw certain general conclusions regarding the structure of hydromagnetic shocks under more general circumstances.

The macroscopic equations for this case are identical with the microscopic ones and both ions and electrons have the same velocity $U(x)$ and the same particle flux F in the direction of propagation x . Furthermore, as in the hydromagnetic analogue to the Rankine-Hugoniot equations of ordinary gas dynamics,^{30, 31} $FU(x) + B^2(x)/8\pi$ is an integral of the motion. If one includes in the equations of motion the terms arising from the inertia of the current-carrying particles, one finds that $B(x)$ is determined by the second order differential equation of a one-dimensional,

undamped, anharmonic oscillator. Thus, the oscillatory character of the solutions is obvious, and the wave-length turns out to be of the order of the geometric mean of the gyro-radii of the ions and the electrons when moving with the velocity of the wave. The assumption that $U(x)$ is everywhere positive on all trajectories limits the Mach number $\bar{U}/B(4\pi\rho)^{-1/2}$, ρ being the mean mass density, to values below 2. Numerical integration of the differential equation has been carried out for a systematic array of values of the parameters. For the limiting case of an oscillatory wave train when the frequency goes to zero, one gets solitary waves with Mach numbers between 1 and 2. In the complete absence of collisions, all particle trajectories are symmetrical about their extrema and no stationary shock waves can be found. But if a few collisions are admitted, both the trajectories and the variation of B with x become asymmetrical. It appears that a shock will start with a wave that is almost a solitary wave followed by some kind of wave train, and that after the particles have travelled a distance of the order of a mean-free path there will be a fairly smooth flow of gas of higher temperature. Thus the total shock thickness will be of the order of a few mean-free paths while its detailed structure should be related to the waves found here. In the limit in which the time between collisions goes to infinity, the only solutions connecting regions of different properties must be non-stationary solutions; thus the stationary waves treated in this work are completely inadequate for the discussion of such cases.

The case of non-zero temperature, both with isotropic and non-isotropic pressure in the plane perpendicular to the magnetic field, has been studied by the same authors and Hain,³² using mainly the macro-

scopic formulations of the equations. For the first case, the results do not seem to differ essentially from those obtained for the case of zero pressure. The second case is still being studied; for this, new equations have been derived by discussing suitable moments of the Boltzmann equation.

The investigations of Davis, Hain, Lüst and Schlüter³³ on the structure of hydromagnetic shock waves with non-zero pressure (isotropic perpendicular but not parallel to the magnetic field) have been completed. The results are very similar to the case of zero pressure. To get shock waves, dissipative mechanisms such as viscosity, heat conduction, electrical resistivity and gyro-relaxation were taken into account. A discussion showed that the last two are the most important mechanisms. The dissipation has the consequence that a solitary wave, which one obtains under certain initial conditions and which is symmetrical without dissipation, is somewhat asymmetrical. This solitary wave is followed by a train of waves with decreasing amplitude until finally the homogeneous subsonic solution is reached. The whole thickness of this region will be of the order of the mean-free path and may be very large compared to the thickness of the solitary wave.

That concludes my very incomplete and brief survey of the work that has been done, mainly in the last two years, in the Federal Republic of Germany. You will have noticed that hitherto there has been very much duplication of effort, but I should like to say that I share very much the satisfaction that has been expressed by earlier speakers that now the period of duplication and non-communication has apparently come to an end and that international co-operation gives better promise for the future of physics.

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Peaceful Uses of Fusion

By Edward Teller *

INTRODUCTORY REMARKS

It is a great pleasure to discuss with you the topic of the day and I should like to begin by expressing the same sentiments with which the previous speaker finished his paper. It is wonderful that over a large and important area of research we can now all talk and work together freely. I hope that this spirit of cooperation will endure, that it will be generally exercised throughout the world in this field and that it will be extended also to other fields.

It is remarkable how closely parallel the developments in the different countries are and this, of course, is due to the fact that we all live in the same world and obey the same laws of nature. I was particularly impressed by the wealth of detail given by the previous speaker. I should like to tell you a few generalities and, perhaps, a few little details which might help you or might amuse you. Additional details can, of course, be found in other papers submitted to this conference.

FUSION AND FISSION

The problem of how to release fusion energy in an explosion process was solved several years ago. How to release this energy in a slow and controlled manner has proved to be a much harder question. This situation is in sharp contrast with the history of energy release from fission. One year of intensive work had sufficed to produce the first nuclear reactor in the early winter of 1942, while several more years were needed to perform the first successful nuclear detonation.

Among the reasons for this difference I should like to mention two which are simple and important.

Fission energy is released by the mechanism of a chain reaction. The chain-carrying neutrons can be slowed easily. This facilitates a controlled reaction not only because the time that the process takes becomes longer but also because slow neutrons are more easily controlled by specific absorbers and other means. On the other hand, release of fusion energy is accomplished by a heat explosion or, to use the technical term, a thermonuclear reaction. The par-

ticles involved in that explosion will necessarily move with high velocity.

The second and more important difference is the following: The fission chain reaction is regulated by assembling the correct amounts of active materials and absorbers. If the correct amount is exceeded, the energy release rises in a rather slow manner due to the action of delayed neutrons. In contrast, the main regulating feature in a fusion reaction is the temperature. When a certain ignition temperature has been exceeded, the reaction itself raises the temperature and this leads to an exceedingly rapid acceleration of further energy release. Slow energy release is possible only at low densities; but this raises further problems.

HISTORY

The problem of controlled fusion is difficult but not necessarily insoluble. Some of us discussed it in a rather detailed manner during the war in Los Alamos. This group included Enrico Fermi, John von Neumann, James Tuck and Luis Alvarez.

Some elementary general facts were recognized at that time: That deuterium gas could react [†] above an ignition point of approximately 35 kilovolts; that deuterium-tritium mixtures could react [‡] at a considerably lower temperature of a few kilovolts; that the gas should be introduced at an exceedingly low density of approximately 10^{14} to 10^{15} particles/cm³ in order to make the reaction rates sufficiently slow and in order to avoid excessively high pressures; that at these high temperatures the gas will be completely ionized (such an ionized gas is called a plasma); that with the presence of magnetic fields, the ions in this plasma follow spiral paths and that by appropriate arrangement of the magnetic fields the losses to the walls can be reduced; that the pressure of the plasma on the field leads to a thermal expansion of the plasma which tends to stabilize the reaction; that equilibrium with radiation is not established and that the energy emitted as bremsstrahlung should be treated as a loss; that for this reason atoms other than

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[†] The reactions are $D + D \rightarrow T + H + 4 \text{ Mev}$ and $D + D \rightarrow He^3 + n + 3.25 \text{ Mev}$.

[‡] The reaction is $D + T \rightarrow He^3 + n + 18 \text{ Mev}$.

hydrogen isotopes must be eliminated as completely as possible; and that even the equilibrium between electron and positive ion energies will not be complete at the highest temperatures. Very particularly, it was also noticed that in a simple closed field along a torus, the particles will not continue to spiral indefinitely around the same magnetic line of force but that they will drift in a direction perpendicular to both the magnetic field and the field gradient. This leads to smaller but nevertheless prohibitive wall losses. For the time being, no experimental work was undertaken for the specific purpose of carrying the theoretical considerations into practice.

In 1952 and 1953, under the stimulus of a successful man-made release of thermonuclear energy, such an experimental approach was started. To a very great extent this was due to the initiative and confidence of the chairman of the Atomic Energy Commission, Lewis L. Strauss. It was recognized that the problems of heating and containing the plasma would be extremely difficult. For this reason, several parallel approaches were undertaken which, between them, amounted to an attack on plasma research on a broad front.

THE PRESENT APPROACHES

Two of these approaches, the pinch and the stellarator, were based on the idea of containing the plasma in an annular region. The main difference between these two approaches is that in the pinch an axial current is established with magnetic lines of force encircling the current, while in the stellarator the roles of current and magnetic lines are reversed with the principal currents established outside the reacting region. In the United States, the work on the pinch effect was started by James Tuck in Los Alamos. Later, similar work was undertaken in the University of California Radiation Laboratory by W. R. Baker and S. A. Colgate and still later at other sites. This approach has also been very strongly emphasized by workers in the United Kingdom and in the USSR. Nicholas C. Christofilos proposed an ingenious steady-state arrangement, the Astron, which bears some similarity to the pinch effect. The stellarator was proposed by Lyman Spitzer at the very beginning of the thermonuclear program and work on it was carried out in Princeton.

Taking an entirely different approach, the advantages of a confinement geometry based on externally generated, cylindrically symmetric magnetic fields were recognized by Richard Post and Herbert York. Starting with these concepts, Post evolved an approach which has come to be called the "Mirror Machine", which uses various applications of the magnetic mirror principle in its operation. In the magnetic mirror machine the ions spiral along magnetic lines between two regions in which the magnetic field is more intense and where the ions are reflected. This kind of machine is now under study at the Radiation Laboratory at Livermore, at the Oak Ridge National

Laboratory, at the Los Alamos Scientific Laboratory and at the Naval Research Laboratory.

A COMPARISON OF THE APPROACHES

Each of these proposals has its peculiar advantages and its peculiar difficulties. The pinch effect is based on an earlier idea of Willard Bennett¹, who derived the relations according to which a violent gas discharge will contract into a narrow filament under the influence of its own magnetic field. The great virtue of this scheme lies in the fact that starting from a rather moderate plasma density one can push the system to very much higher densities and pressures and at the same time expose the containing system to much more moderate pressures. One price that one has to pay is that the high densities are obtained only for short periods of time. In other words, we are dealing with a pulsed machine. In order to avoid instability and breakup of the pinch, the original simplicity of the concept had to be compromised by introducing magnetic fields parallel to the direction of the plasma current. The detailed theory of the pinch contraction, as well as the theory of pinch stabilization, was worked out by M. Rosenbluth at Los Alamos.

Rather early in the game it was noticed that the pinch effect in deuterium gives rise to copious neutron emission. The first public announcement of this fact was made by I. V. Kurchatov.² Unfortunately, these neutrons do not appear to be of thermonuclear origin. Considerable progress has been made by S. A. Colgate in explaining the mechanism by which these neutrons are produced. A general difficulty in the pinch effect seems to arise from the circumstance that during the formation of the high-current filament most of the original energy output is lost to the walls. It seems to be of the greatest importance to understand and prevent these losses.

The Astron is based on the following observation: In contrast to the instability of the pinch current, it is expected that a current of relativistic electrons will be stable. If this proves to be correct then one can confine a plasma in the field which encircles this relativistic current for a longer period than in an unstable pinch. This arrangement is under development at Livermore.

The thorough theoretical studies of the stellarator, carried out by L. Spitzer, M. Kruskal, E. Frieman and others, led to the recognition that the simplest arrangements of magnetic lines are unstable in this case also. In order to prevent these instabilities, additional magnetic fields are being introduced in a direction perpendicular to the original fields. Thus the present pinch and stellarator designs are no longer as sharply different as they used to be. However, the stellarator continues to be a device in which steady-state operation will be attempted and in which the main magnetic fields will originate in a region outside the plasma. The problem of heating the plasma in the stellarator has turned out to be a rather difficult one. The first step in the heating process is by

ohmic current, but soon the conductivity of the plasma becomes high and, instead of uniform heating, runaway electrons are produced with velocities greatly exceeding the Maxwellian values. From this point on, heating is accomplished by transferring energy from magnetohydrodynamic motions to the particles in the plasma. This so-called "magnetic pumping" process bears a similarity to the acceleration of cosmic rays as described by Fermi. Unfortunately, in the stellarator, as in the pinch, a very considerable fraction of the energy is lost during the heating process. The heating derived from hydromagnetic shocks has been investigated by H. Grad at New York University, M. Rosenbluth at Los Alamos and others.

In the evolution of the mirror machine the basic simplicity of the original design has been maintained. In addition to this advantage, the straight geometry makes it possible to operate with a relatively small volume. An obvious disadvantage of the magnetic mirror machine is the loss of particles through the mirrors. In fact, all particles get lost whose motion includes a sufficiently small angle with the magnetic lines. In the course of time, more and more particles will be scattered into these orbits and eventually all of the plasma escapes. Detailed calculations have shown that escape through the mirror is a serious consideration but that it does not rule out the mirror machine as an effective thermonuclear energy generator. Furthermore, the end losses can in principle be reduced by inducing a rapid rotation of the plasma. The open ends of the machine also lead to an advantage: a simple possibility of injection. If one can introduce a sufficient quantity of high-velocity, ionized material through these ends, then, in contrast with the other machines, one can circumvent the problem of heating a plasma which is already present. The open ends have another advantage: they allow one to study the particles which escape and, therefore, to determine what energy particles must be added to the plasma. In injecting fast ion beams, difficulties have appeared due to space-charge limitation in the injection mechanism.

As has been pointed out by the representative of the USSR, we are particularly anxious to inject at high energy because then the leak is not so serious. A radically different system of injection is being tried at the Oak Ridge National Laboratory. This system consists in bombarding the interior of the mirror machine with D_2^+ ions. Inside the mirror machine the molecular ions are disassociated in a carbon arc (but we could use a hydrogen arc) and the more strongly curved D^+ ions are trapped. Considerable progress has been made in developing this technique and it may lead to the establishment of a steady-state plasma which is suitable for detailed experimental investigation. In any case, a completely satisfactory density and temperature need not be obtained at once by the injection into the mirror machine because simply strengthening the magnetic fields or moving the mirrors closer together will lead to an adiabatic compression of the plasma which will increase both the density and the temperature.

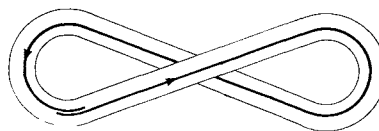
The homopolar, conceived by William Baker at Berkeley, is another machine based upon mirror containment, but with an enhanced containment due to the large induced rotation of the plasma. Similar work on rotating plasmas (Ixion) is being done by Keith Boyer at Los Alamos. Generally speaking, there are many designs involving a rotating plasma trapped in a mirror machine. This is accomplished by using crossed electric and magnetic fields to induce a rotation, with consequent enhancement of the mirror action.

There is one additional design which has received serious consideration but on which little experimental work has proceeded. The name of this design is the picket fence or cusp geometry considered by J. Tuck and H. Grad. It consists of a series of magnetic field systems so arranged that the magnetic fields curve at each point away from the space which is enclosed and which contains the plasma. At the juncture of any two of these systems, the magnetic fields have a cusp through which the plasma can escape. The difficulties connected with the rate of this escape have been the reason for not pursuing this particular design any further. It is by no means clear, however, that such a design could not be successful.

In later sessions a more detailed discussion will be given of the several designs. Here I should like to mention a few general difficulties and to discuss the way and extent to which these difficulties have been overcome.

SINGLE PARTICLE CONFINEMENT

One may consider the plasma from the point of view of the individual charged particles in a fixed field. The first and simplest question is whether or not a field of magnetic lines can retain ions on their orbits for a sufficiently long time. Since the ions slowly drift in a direction perpendicular to the magnetic lines of force and to the gradient of these magnetic lines, it is not clear that an effective confinement is actually possible. In a simple torus with a longitudinal magnetic field, the charged particles spiral around closed magnetic lines of force. The inhomogeneity of the field causes them to wander off steadily to the walls. The stellarator is designed to avoid the difficulties associated with the drift of particles out of a simple torus. One such method, for instance, is to twist the torus:



This was proposed as early as 1951.

In the stellarator the lines get twisted, with the result that a line no longer closes but instead returns, after a circuit, into the neighborhood of its starting point. If many circuits are executed, a single magnetic line will have closely covered a toroidal surface. Now an ionized particle, in following this more com-

plicated line of force, will be subject to displacements due to the inhomogeneity of the magnetic field. One can show, however, that these displacements will average to zero in good approximation.

In the mirror machine, an ion spiraling around a magnetic line of force performs a quasi-periodic motion due to oscillations between reflecting points. The periodicity of the orbit is not strict because the guiding center drifts to neighboring lines of force, as has been mentioned above. In a cylindrically symmetric case, it is evident that the drift perpendicular to the line of force will map out a surface of cylindrical symmetry. But it is not evident where the particles will drift if there exists a non-symmetric perturbation. The following consideration shows, however, that despite such a perturbation the particles do return to their original line of force with a high degree of precision.

During the periodic motion in a static magnetic field the energy will be strictly conserved. Two "adiabatic invariants" are also very accurately preserved, even though they vary to a small extent. These are the magnetic moment of the spiraling particle in a direction along the magnetic line and the action $\oint p_{\parallel} dq$. Here $p_{\parallel}$ is the momentum of the ion (or the electron) along the magnetic line of force and dq is an element along the magnetic line on which the guiding center of the spiraling particle happens to move. The essential character of the orbit of a particle is determined if its energy and magnetic moment are known. From this, the action $\oint p_{\parallel} dq$ follows uniquely. If now the particle were to shift to an arbitrary neighboring line of force, conserving its energy and magnetic moment, the action integral would in general be different. However, the action integral must be conserved also. Therefore, a shift to an arbitrary neighboring line is not possible, but the guiding center of the particle must remain on a sheet consisting of a single continuous set of magnetic lines of force. In this way, the particle will shift around within the machine, remaining always on a surface on which the action integral is conserved. When the particle arrives back in the original neighborhood it will, as a general rule, actually embrace the same magnetic line from which it had started. While this statement is not absolutely correct, it holds with a very high precision for small Larmor radii. Unfortunately, the conservation of the adiabatic invariants is not valid if the field varies with a sufficiently high frequency. Under these conditions, of course, the energy is not conserved either.

INSTABILITIES

Let us consider a time-independent distribution of magnetic lines of force in space, and let us also consider a static distribution of a plasma with vanishing electrical resistivity. The magnetic lines need not be curl-free because we may assume that there are electric currents in the plasma to account for the curl. No electric fields will be present: if they were parallel to the magnetic fields they would give rise to a change

in the electric current; while if they were perpendicular to the magnetic lines they would cause the plasma to move, which is contrary to our assumption of a static situation.

Now the stability of this situation may be decided by investigating infinitesimal virtual displacements. In such a displacement, the material of the plasma may slide along magnetic lines of force but, in any motion transverse to the lines, the plasma and the lines must move together. Furthermore, any expansion or compression of the plasma accompanying the displacement must be considered as adiabatic. If some of these displacements lead to a linear change of the energy of the plasma then the system is not in equilibrium. If all changes depend quadratically on the displacement and if all infinitesimal displacements are connected with positive energy changes, then the plasma is stable. If some displacements are connected with negative quadratic terms we expect an unstable equilibrium. Many such instabilities have been discussed, and the discussion has been carried to a detailed calculation of the way the particle distribution, magnetic fields and currents change during the displacement. However, the hydromagnetic consideration given above appears to be a simple and general description which in many cases is satisfactory.

The first models considered for the pinch machine and for the stellarator turned out to be unstable. Appropriate modifications of the field distribution have given rise to reasonable hopes for circumventing various troubles. It appears, in a great measure, to be a question of whether these instabilities are of a hydromagnetic origin or are connected with plasma vibrations excited by runaway processes. We are more doubtful about the plasma vibrations, which may be a very real and more lasting stumbling block. In this kind of motion, it turns out that the faster the electron the longer is the wavelength of the plasma vibration that can be excited, and the longer the wavelength the more energy it can take up. To summarize the situation very briefly: it has been suspected and partly found that these plasma vibrations have the disagreeable consequence that the plasma can rapidly diffuse and there is a leak in the magnetic confinement. These are some of the difficulties in the stellarator and pinch devices. Work is proceeding and there are methods, partly hoped for and partly realized, which will be discussed in other sessions.

The general question of whether a moving, steady-state, hydromagnetic system is stable or not has been discussed in some interesting special cases but has not been treated along similarly simple lines as has the corresponding static problem. These dynamic effects probably play a very important role during the constriction of the electric current in the pinch process and during the heating of the plasma in the stellarator. At the same time, the runaway electrons mentioned above probably give rise to plasma vibrations, which also might arise by coupling with the hydromagnetic motions. All these motions can generate enough disorder to destroy conservation of energy

and conservation of the adiabatic invariants for the single particles which are supposed to be confined in the system. As a result, both plasma and energy can be lost to the wall.

We see, therefore, that heating, plasma injection and plasma compression do represent sizeable additional problems. These problems are, however, closely connected with the more general question of confinement. If the latter were solved completely, then we could take time with the other processes and probably execute them sufficiently carefully so that the plasma could be brought to the appropriate conditions. The actual fact is, however, that the problem of plasma confinement is only partially solved and, therefore, this problem and that of the establishment of the right temperatures and densities have to be handled together.

EFFECTS OF IMPURITIES

Another difficulty that is quite generally encountered is connected with impurities. The presence of impurities leads to increased bremsstrahlung radiation and the consequent rapid loss of energy from the plasma. At temperatures low compared with the ignition temperature, these losses are often aggravated by radiation from incompletely ionized atoms. These partially ionized systems are usually not in Saha equilibrium, since of the four possible processes, collisional ionization, radiative electron capture, recombination by triple collisions and ionization by photons, only the first two occur. This somewhat involved problem has been thoroughly investigated by R. F. Post and has led to the recognition that heating must be carried out rapidly, particularly in the low temperature regions.

If neutral atoms are knocked off the walls, these cause an energy loss by an additional mechanism: They can neutralize fast plasma ions by charge exchange and these fast neutral particles can cross the magnetic barrier and reach the wall. Here they can knock out more neutral particles and a loss mechanism can be built up.

EXPERIMENTAL DEVELOPMENT

The above enumeration of difficulties and problems is by no means complete. Neither does this theoretical discussion do justice to the experimental problems of machine design and to the diagnostic work which has been carried out. The development of extremely large, pulsed capacitor banks for creating the large volume, high magnetic fields required, and the operation of large "baked out" vacuum systems for high plasma purity at pressures as low as 10^{-10} mm Hg are major achievements of the experimentalists. The observation of gross current and electric field behavior and the determination of magnetic field and plasma pressure by means of probes introduced into a system have given valuable data. Plasma densities have been measured by experiments involving absorption and transmission of electromagnetic waves in the radar range. Spectroscopic observations have been

carried out and the appearance and behavior of neutrons have led to additional knowledge about the state of the plasma. Thus, our knowledge of plasma physics is increasing considerably with each passing year. In this way, important features of plasma theory have been experimentally verified. The interaction between magnetic pressure and hydrodynamic motions have been observed and transport phenomena in the plasma have been investigated. These are greatly complicated by the effects of collective motions but, when properly attacked, have yielded satisfactory verification of our expectations.

Hydromagnetic motions have been extensively studied, most particularly in observing the formation of a pinch. In connection with the same experiment, the evolution of instabilities has also been investigated. In the stellarator, the mirror machine and the pinch discharge, the importance of impurities has been demonstrated by studies of the influence that the wall materials have on the observed phenomena. In the case of the stellarator, a very ingenious device, the divertor, has been developed which harmlessly eliminates that part of the plasma which otherwise would touch the wall and would give rise to the emission of disturbing atomic species. The use of the divertor has greatly improved the operation of the machine.

After proper precautions, temperatures of 100 electron volts or more have been reached in the various machines. This should be compared with the few thousand electron volts needed for the effective burning of a deuterium-tritium mixture. More than 100 electron volts have been reached in the stellarator and still higher temperatures in the mirror machine. Proved temperatures in pinch devices are a little lower but, on the other hand, greater densities have been obtained.

Perhaps the most important observable variable is the containment time. In the pinch effect, the time during which a high density prevails is only a few tens of microseconds. On the other hand, the operation of this machine is in least need of protracted confinement times. In the stellarator, confinement times of a millisecond or more have been established. In the mirror machine, confinement times of several milliseconds are not unusual. This is particularly pleasing because of the small size (a few cubic centimeters) and high electron temperature (more than 10 kev) of the plasma. Furthermore, in this machine no difficulties have arisen in connection with instabilities or disordered motion. I would therefore term this quite a hopeful approach. These times have to be compared with a period of approximately one second in which a successful thermonuclear reaction would double the original input energy.

NEUTRONS

Throughout the development, it has been believed that production of true thermonuclear neutrons would be a milestone in the development of a controlled thermonuclear reactor. The observation of such neutrons would not be merely encouraging but would,

at the same time, give us a sensitive method of measuring temperature distributions within the machine: the slightest rise in temperature would give an enormous rise of neutron production. In "thermonuclear machines" neutrons have been repeatedly observed. Unfortunately, these neutrons may be due to some form of organized motion rather than to the random thermal agitation. This shows in a most direct manner that we are still very far from a successful thermonuclear machine. Furthermore, even when true thermonuclear neutrons are obtained, their effects could at first be obscured by the presence of the neutrons due to collective motions.

THE FUTURE OF THERMONUCLEAR POWER

I should now like to ask: Where are we going? I believe that thermonuclear energy generation is possible. Whether it will be in precisely seventeen years, as our Chairman predicted three years ago, or at some other time is a matter that I think he will not argue with me and I shall not argue with him. However, I will say this: The problem is not quite easy. I will also say that on the path there may be some little flowers to be picked. Plasma physics has importance in the cosmic arrangement of things, as we heard in Professor Alfvén's paper during this session. It may have important technical applications other than energy production. I wish I could know and tell you more about this, and I hope suggestions will come forward; because this needs imagination and I am lacking in imagination. If we want to shoot for the jackpot, for energy production, I think that it can be done, but do not believe that in this century it will be a thing of practical importance. This view may be much too conservative and may be proved wrong within the coming decade, but I shall tell you why I believe what I believe.

It is likely that we shall be dealing with an intricate machine which is inaccessible to human hands because of radiation and on which all control and maintenance must proceed by remote control. The irradiation of materials by neutrons and gamma rays will cause the properties of these materials to change. Surfaces bombarded by the bremsstrahlung radiation will get heated more fiercely than is the case in any portion of our present nuclear reactors. You can operate the machine to the extent that this one surface can be cooled, the rest of the machine being at a relatively low temperature. These and other difficulties are likely to make the released energy so costly that an economic exploitation of controlled thermonuclear reactions may not turn out to be possible before the end of the 20th century.

Nevertheless, the ultimate goals toward which we are working are apt to be highly rewarding. When economic thermonuclear energy production becomes feasible we shall reap a number of important benefits. The fuel of the thermonuclear reactor is cheap and practically inexhaustible so that, if I may put it this way, we have deuterium to burn. Thermonuclear reactors produce less dangerous radioactive materials

and, when once brought under control, are not likely to be subjected to dangerous excursions. Therefore, they can be operated more safely than fission reactors. Finally, the interaction of a hot plasma with magnetic fields opens up the way to the direct production of electrical energy. This may be of great practical advantage since high-temperature heat exchangers and many moving parts could be eliminated.

Now I have a question: Can all this be done? I think we are at a stage similar to the stage at which flying was about one hundred years ago. There are some wise people now, as there were at that time, who have proved that it cannot be done. I should like to say that those people were perhaps better off because at least they saw the birds. All we can see are the sun and the stars. The sun produces thermonuclear energy by brute force or, what is worse, by sheer inertia. Other people will say that the sun does it with the help of infinite patience. I do not think any physicist wants to go along either of those directions.

PLOWSHARE

This survey should not be concluded without mentioning another way in which thermonuclear energy can be made to serve peaceful and constructive purposes. Our recent investigations have led to the increasing expectation that thermonuclear explosions can be brought sufficiently under control to be of help in earth-moving, in mining and also, conceivably, even in the production of energy. We can also undertake projects which I should like to designate as geographical engineering. We can dig canals, we can make harbors, we can open up water reservoirs for better water supply of arid or semi-arid regions. In this field, people from the USSR have made very early statements which I believe raise very early hope, hope which I believe, in principle, to be correct.

Recently, an underground detonation of a fission device in Nevada has demonstrated the feasibility of the confinement of the energy of the device. Furthermore, this experiment has shown that the bulk of the radioactive fission products were entrapped in the glass formed by the molten rock. The energy in this case was primarily dissipated at a low temperature in vaporizing the relatively large water content of the rock but, in addition, a large quantity of rock was crushed. With the equivalent of one ton of high explosive we can pulverize more than one hundred tons of rock. This could be useful in mining and for the enhancement of the oil flow in large low-specific-yield formations. Therefore, as has been the case so often in history, a method for destruction has turned in a direction where it can be made useful for the most important and, in the long run, cooperative effort between the nations.

In the peaceful use of nuclear explosives, the fusion reaction is likely to play a prominent role for two reasons. One is that the fusion fuel is exceedingly cheap. The other reason is that fusion reactions give rise to less radioactive contaminations than do

fission reactions. In fact, a careful planning of the fusion explosion can reduce the activity to a point where its effects are quite small compared with the effects from an equivalent fission reaction. This is particularly important in earth-moving applications designed to build harbors, canals, change the distribution of water or lay bare mineral deposits.

There is no difficulty, in principle, of releasing large amounts of fusion energy underground, in the conversion of this energy into high pressure steam and in the use of this steam for the production of electricity. It is, however, important to find a cheap way of confining the explosion in order that the economical exploitation of explosive fusion energy for the production of electricity may become possible. It is not unreasonable to expect that explosive fusion energy can be harnessed for the production of electricity before the same feat can be accomplished in the controlled fusion process. It will also be noticed that both the cheapness of the fuel and the reduction of radioactive contamination are the same factors which also play an important role in our future expectations concerning controlled fusion reactions.

Even though the first application of thermonuclear power may be by using controlled explosions—and this would be a very great achievement indeed—in

the long run, the inherent advantages of controlled fusion in a steady-state system will probably dominate our planning for the far future. At any rate, in the direction of controlled explosions, as well as in the great field of plasma physics, I see a very fruitful field of research for physicists, for groups of physicists, for scientists in a whole country and for scientists in the whole world. I also believe that one thing which science and technology can do for us is to draw us all closer together into mutual help and into better understanding if we are guided by a search for the things that can be accomplished by active work, which we communicate to one another and in which we cooperate. In any case, I am convinced that the extraordinary and abundant power of fusion energy can be used in peaceful pursuits for the benefit of mankind. When this has been accomplished it may well turn out to be one of the most important advances of our age.

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Controlled Thermonuclear Research in the United Kingdom

By P. C. Thonemann*

HISTORICAL

The possibility of utilising the energy released in nuclear reactions between the light elements was probably first discussed at the Cavendish Laboratory, Cambridge, in 1932. It was at this time that Lord Rutherford and his colleagues demonstrated that artificial nuclear reactions could be produced in the laboratory and that energy was released in each nuclear event. These reactions were produced by accelerating positive ions or nuclei to high velocities, using potential differences of hundreds of kilovolts, and firing them at a target containing the other elements. At low bombarding energies these reactions were found to be most probable with the light elements.

No direct application of their discoveries for power generation was foreseen; in fact, Lord Rutherford described such speculations as "moonshine". At best, only 1 in 10^5 of the bombarding particles reacted with a target nucleus, the remainder being brought to rest by unproductive energy losses to the target atoms.

It is the intention of controlled thermonuclear research to find an efficient way of utilising the energy produced when light elements react together to form heavier ones. Such a process occurs at the centre of the sun and in other stars under conditions of immense pressures and extreme temperatures. It is altogether impossible to reproduce similar conditions in the laboratory. Fortunately, there are nuclear reactions which occur much more easily than those responsible for the sun's energy, but even then extremely high temperatures are believed to be necessary.

The concept of thermonuclear reactions providing the energy radiated by the stars was clearly put forward by Atkinson and Houtermans¹ in 1928, but no suggestions for achieving a similar energy source on a terrestrial scale were seriously considered in the United Kingdom until 1946.

It was clear that the high temperatures necessary could not be reached unless some means was devised for keeping the hot gas away from the walls of the containing vessel. Experimental investigation of the confinement of a hot, electrically-conducting gas by

magnetic fields was started by Sir George Thomson in London in 1947 and by myself in Oxford in 1948. These independent programmes attracted Government support in 1948 and continued in the two Universities until 1951.

Several possible magnetic field configurations were studied in relation to toroidal tubes but confinement by the self-magnetic field of a current flowing in a gas appeared the most promising. Some of the results of this early work were published in 1951.²⁻⁴ Figures 1 and 2 show the type of apparatus employed at that time. In 1951 the London group moved to the Research Laboratory of Associated Electrical Industries, Aldermaston, and the Oxford group to Harwell. A small team at the Atomic Weapons Research Establishment began in 1956 to pay special attention to the interesting features of the dynamic pinch.

It will be appreciated that this subject involves almost the whole range of classical physics included in the electromagnetic spectrum—from steady electrical currents to X-rays. Measurements of ionisation and excitation cross-section of multiply charged ions are now required together with further development in the basic theory of particle transport processes and in the new subject of magnetohydrodynamics. Perhaps it is not surprising that progress is slower than one would wish.

We have now reached the stage of producing deuterium plasmas at temperatures in excess of 10^6 °K for appreciable intervals of time. At these temperatures we can study some of the phenomena which are likely to occur in a fusion reactor. Whilst other approaches to the problem of producing and maintaining a very high temperature gas will undoubtedly be described to this Conference, research in the United Kingdom has been devoted to a study of gas confinement by the "pinch effect"; that is, confinement by the magnetic field which accompanies an axial flow of current through the gas. Many other possible methods have been studied at earlier times but none appeared so promising.

The size of the apparatus employed depends to a large extent on the time for which the electrical currents are required to pass through the gas. Thus the group at the Atomic Weapons Establishment are interested in extremely short times and a rapid increase in discharge current. A recent form of their

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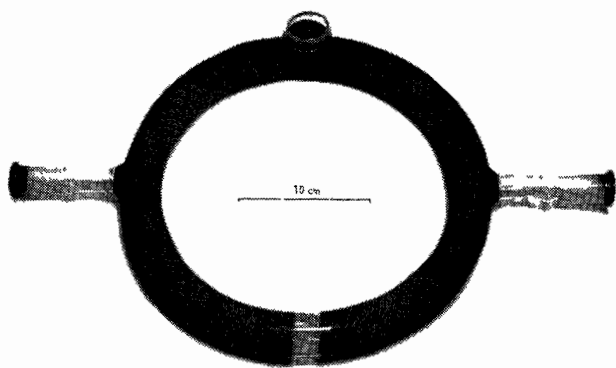


Figure 1. Early glass torus (1948) used by Cousins and Ware in their demonstration of the "pinch effect"

apparatus is shown in Fig. 3. The essential feature of the equipment lies in the use of a large number of separate switches (spark gaps), synchronously triggered so as to connect in a parallel arrangement the large number of capacitors that form the total bank. This reduces the inductance of the circuit external to the discharge to a very low value. At the other end of the time scale is the Zeta apparatus at Harwell. The discharge in the gas takes place in a toroidal vessel for longer periods of time and at much lower gas densities. Figure 4 shows a general view of the arrangement. In between these two extremes is the Sceptre device at the Research Laboratory of Associated Electrical Industries which is similar to Zeta but which has a more rapid rate of current rise. This apparatus is shown in Fig. 5.

THE BASIC FUSION PROCESS

I now turn to a short discussion of the basic fusion process. What is fusion and where does the energy come from? In order that a nuclear reaction takes place between two charged nuclei these particles must be brought close together against the Coulomb repulsion of like charges. As an example, consider the fusion of two nuclei of deuterium. When the nuclei approach within the range of the attracting nuclear forces, about 10^{-13} cm, we can speak of a compound nucleus of two neutrons and two protons. This nucleus is similar to that of ordinary helium.

The compound nucleus can break up in the three ways described by the following equations



In the first two cases, the new nuclei Helium 3 and Tritium are formed. The third case corresponds to simple elastic scattering and no energy is released.

Because three of the attracting nuclear particles come much closer together, forming a new stable configuration, the fourth is ejected with the energy difference. This energy difference ΔE may be calculated from the mass difference ΔM between the initial and final nuclei taking part, according to the well-known equation $\Delta E = \Delta M c^2$.

At the particle energies of interest in thermonuclear research the probability of elastic nuclear scattering is much greater than the probability of a nuclear reaction. On the average, therefore, a fuel nucleus will make many elastic collisions with other nuclei before a reaction occurs and any ordered motion initially present will be completely disorganised before appreciable energy is released. For this reason, a fusion power source will almost inevitably be a thermonuclear process since nuclear reactions take place as a result of impacts between particles with thermal or random motion.

To calculate the minimum temperatures required for a net energy gain, the conditions under which a high temperature gas can be confined must be discussed in more detail. It is essential that there be an equal concentration of negative and positive charges, otherwise the resultant electric fields would be so enormous that confinement would be impossible. The presence of electrons leads to electromagnetic radiation due to the acceleration of electrons as they are deflected by the positive nuclei. This radiation escapes without absorption in the tenuous gas and no radiative equilibrium is possible. The energy radiated is many orders of magnitude less than that from a black body at the same temperature.

Consider a cycle in which electrical energy is used to heat the gas and maintain it at a high temperature. Energy must be supplied continuously to make up the radiation loss. This energy can be recovered only as heat, some 70% of which is lost in its conversion back to electrical energy. To make up the deficit, the energy released by nuclear fusion must be at least twice the energy radiated. Both the energy radiated and the nuclear energy released can be calculated. The minimum temperature turns out to be about 3×10^7 °K for a deuterium-tritium mixture and 3×10^8 °K for pure deuterium fuel. At these temperatures the fuel nuclei have energies corresponding to about 3 kev and 30 kev. This is small compared to that attainable in modern particle accelerators. The difference lies in the much higher particle densities

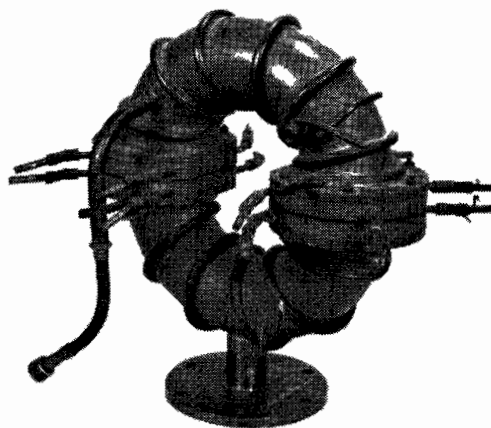


Figure 2. Metal torus (1949) used by Thonemann and Cowhig to study "pinch effect" produced by 100 kilocycle continuous excitation. The iron core which linked the torus is not shown

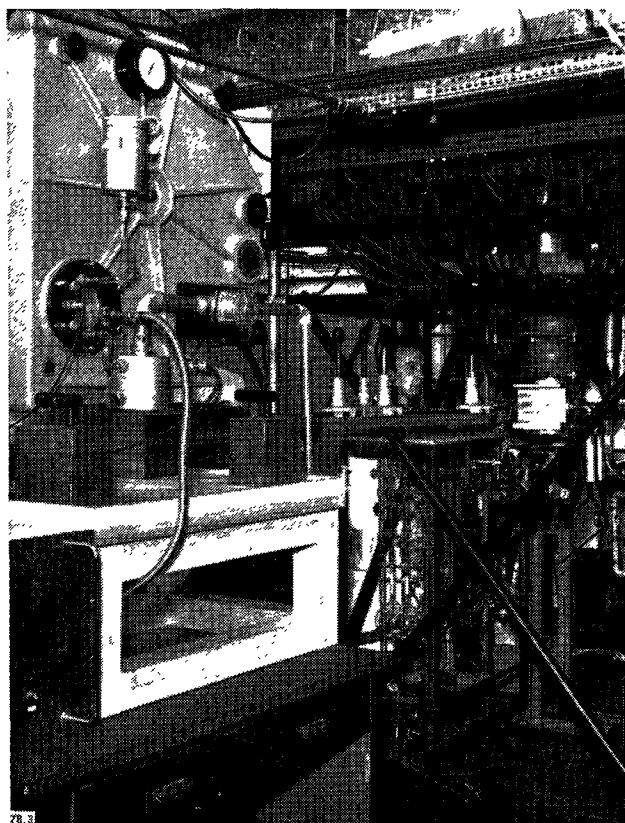


Figure 3. One of the dynamic pinch arrangements used at the Atomic Weapons Research Establishment

necessary and the requirement of confining the particles in spite of repeated collision.

The margin between success and failure is narrow. Under the best conditions, the rate at which deuterium gas produces nuclear energy is only about ten times the rate of radiation from the same gas. For a tritium-deuterium mixture the factor is a hundred and it is likely that the first successful fusion reactor will employ this fuel. At the present time the use of fuels other than the hydrogen isotopes appears to be excluded, first by the enhanced radiation due to the higher nuclear charge and secondly by the slower reaction rates.

The previous discussion has assumed that the charged particles, nuclei of deuterium and electrons, are perfectly confined and only electromagnetic radiation reaches the walls of the surrounding vessel. It is the problem of particle containment which is the principal stumbling block at the present time. Gravitational fields are not available and the next best thing is a magnetic field. Whilst it is possible, in principle, to confine a high temperature plasma by rapidly oscillating electric fields,⁵ all methods studied experimentally in the United Kingdom have employed strong magnetic fields.

A high temperature gas is a good electrical conductor because of the free electrons. The force acting on an element of the gas, carrying an electric current normal to a magnetic field, can be used to balance the kinetic pressure of the gas, $p = nkT$. A static balance of pressure is ensured if $\nabla p = \mathbf{J} \times \mathbf{B}$

where $\mathbf{J}$ is the current density and $\mathbf{B}$ the magnetic field.

The current density $\mathbf{J}$ in the plasma may be produced by an external emf or it may result from the slow expansion or diffusion of the charged particles across the magnetic lines of force. In the latter case, new energetic particles must be introduced to maintain a steady state. There are several possible solutions to this equation, but all must satisfy two conditions. First, the net force acting on the gas as a whole is zero, and second, any displacement of the gas from a position of equilibrium must lead to a restoring force.

This restoring force must be produced by the interaction of electrical currents in the plasma and currents in external conductors, because any force due to particles striking the walls is to be made zero. The first condition is satisfied by the magnetic field associated with the flow of current in a cylindrical conductor and gives rise to confinement of the plasma by the pinch effect. The second condition is satisfied by enclosing the current channel in a cylindrical conductor. Any displacement of the current channel away from the axis of the cylinder results in currents in the outer conductor which, by Lenz's law, oppose the motion. Both the Zeta and Sceptre experiments have employed this principle.

INSTABILITY

The greatest difficulties we have experienced in reaching high temperatures is the instability of a magnetically-confined plasma. Satisfying the above conditions only ensures that the plasma as a whole is in stable equilibrium but tells us nothing about the behaviour of a local deformation. Trying to stabilise a plasma against these deformations is rather like trying to balance a ball on the end of a stick. To be successful, you have to be either very experienced or very clever. The lowest state of "stored energy" is undoubtedly on the ground or, in the case of a plasma, at the walls. The best that can be done is to delay the transition for as long as possible.



Figure 4. General view of Zeta apparatus at the Atomic Energy Research Establishment

Charged particles moving in a static magnetic field can only cross lines of force between regions of different magnetic fields strength as a result of collision or longer range electric fields. Collision with other particles are unavoidable and the rate of diffusion to the tube walls is not prohibitive in most cases of practical interest. The effect of longer range electric fields is much more serious. I do not refer to those electric fields which are responsible for ordered current flow in the plasma but to local electric fields arising from charge separation or by magnetic induction. If an electric field exists normal to the confining magnetic field, plasma may flow across the lines of force at a velocity given by $v = Ec/B$. Thus, a random fluctuating electric field will allow a plasma element to make a random walk with the above speed and it will drift steadily away from its starting point across the lines of force.

Many magnetic field configurations for achieving stability have been suggested and the search is still in progress. The theoretical work of Shafranov,⁶ Tayler,⁷ Bickerton⁸ and Rosenbluth⁹ on the pinched discharge stabilised by an axial magnetic field has guided the experiments in the United Kingdom but I cannot say that we have achieved the configuration demanded by the theoreticians nor can they say they understand the stability achieved in the Zeta and Sceptre experiments.

THE UK CONTRIBUTION

The principal UK contributions to this Conference are concerned with the results of the Zeta and Sceptre experiments and the fast pinch experiments in straight tubes at the Atomic Weapons Research Establishment.

Preliminary results obtained at Harwell and the Laboratories of the AEI[†] have already been reported.¹⁰ Both experiments are essentially high current discharges in toroidal metal vacuum vessels in which an axial magnetic field is used to provide stability. Since the earlier reports, efforts have been made to understand the manner in which the current channel is formed and to explain the stability. Measurement of ion energies, X rays and neutron intensities have been extended, and the role played by impurities investigated. These results, together with details of the construction, will be reported in later sessions. A full account of the experimental methods used to obtain these results will also be given.

The AWRE group has studied the effect of the rate of current rise on the discharge properties as well as the effect of an axial magnetic field and impurities. They have observed some new phenomena which will be reported in detail in a technical session.

The theoretical investigations to be reported are concerned, firstly, with the structure of hydro-magnetic shock waves in a fully ionised gas and, secondly, with the stability of a magnetically-confined plasma. The first subject is of importance in fusion research because shock waves provide a powerful

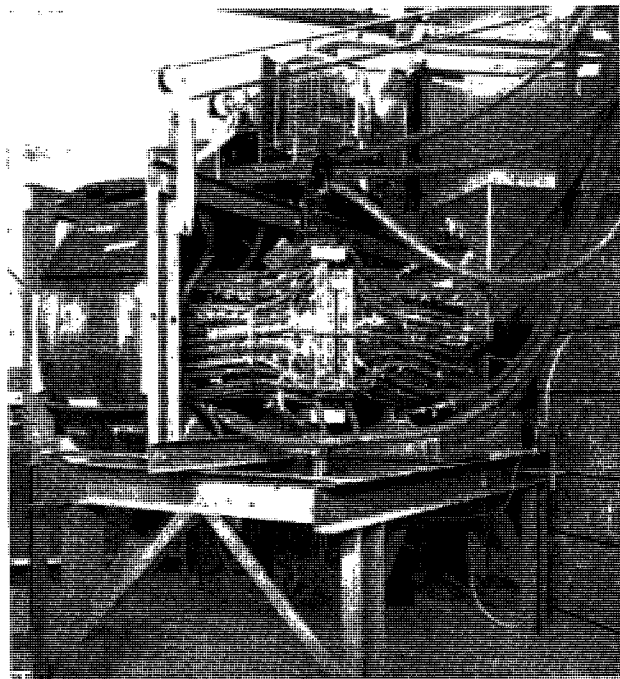


Figure 5. View of Sceptre apparatus at the Research Laboratory of Associated Electrical Industries Limited

mechanism for communicating energy to positive ions, even when collisions are rare. It is very likely that the transfer of energy by binary collisions between electrons and positive ions will be too slow to reach fusion temperatures in the times required. The importance of the second object has already been pointed out. The principal theoretical objective is to devise an ideal model in which particles cannot reach the vessel walls because of instabilities. This does not mean that instabilities should be completely suppressed but that their amplitude must be limited. Further details along these lines will be presented, together with a discussion of the relevance of the results to the performance of Zeta and Sceptre.

The emission of impurities from the walls of the containing vessel plays an important role, both in the amount of energy radiated by the gas and in determining the magnetic field configuration. When the walls are conducting, the formation of arc spots and vaporisation of wall material is a potent source of impurities. The mechanism of arc formation has been studied as well as the motion of arc spots in magnetic fields. Investigations on both these aspects will be reported.

FUTURE TRENDS

I now turn to the future of research in the controlled fusion field. To my mind, the problem of stability is of paramount importance. Unless the rate at which charged particles cross magnetic lines of force can be reduced to that given by classical diffusion theory, the loss of energy to the walls will prevent fusion reactions from becoming a practical power source. A decrease in the energy flux to the walls by

[†] Associated Electrical Industries Limited, Aldermaston, Berkshire.

particle bombardment also reduces the amount of impurities thrown into the plasma and this in turn reduces the energy loss by radiation.

Assuming that the stability problem is solved, it is estimated that currents of about 10 million amperes will be required in a tritium-deuterium mixture before a net balance of power is achieved using self-magnetic confinement. In a tube of reasonable dimensions, these currents must continue for at least one-tenth of a second. Achieving such current amplitudes clearly

presents difficult technological problems. I think that the papers to be presented at this Conference, and the discussions which follow them, will show that it is still impossible to answer the question, "Can electrical power be generated using the light elements as fuel by themselves?" I believe that this question will be answered in the next decade. If the answer is yes, a further ten years will be required to answer the next question, "Is such a power source economically valuable?"

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Record of Proceedings of Session 4

Possibility of Controlled Fusion

TUESDAY AFTERNOON, 2 SEPTEMBER 1958

Chairman: Mr. H. J. Bhabha (India)

Scientific Secretaries: Messrs. H. Sethna and T. Coor

PROGRAMME

- P/145 Magnetohydrodynamics and the thermonuclear problem H. Alfvén
P/78 Controlled thermonuclear research in the United Kingdom P. C. Thonemann

DISCUSSION

- P/2298 Research on controlled thermonuclear reactions in the USSR L. A. Artsimovich
 (Presented by Mr. E. I. Dobrokhotov)
P/2410 Peaceful uses of fusion E. Teller

DISCUSSION

- P/1056 Recent work on controlled thermonuclear fusion in Germany (Federal
 Republic) L. Biermann

DISCUSSION

The CHAIRMAN: We come now to one of the sessions which is going to raise the greatest interest. As all of you who were at the 1955 Geneva Conference will recall, there was no mention of fusion in that Conference except for a very passing and brief remark which I made in my presidential address. Since then a very large amount of work has been declassified on this subject and we all know that a tremendous amount of work is going on in many countries. Briefly, the work in this line takes two general directions: one of accelerating plasma by electromagnetic forces; the other of equilibrium containment, in which the pressure of the plasma is balanced by magnetic pressure.

As regards the first approach, it is very easy to see the orders of magnitude. The time in which compression takes place is roughly the size of the vessel or the distance which the plasma has to travel divided by its velocity, and this is of the order of 10^{-5} to 10^{-6} seconds. After that you have to do your best to contain the plasma while it is hot and get all the energy you can out of it.

As regards the second method there are two approaches. One is the approach of plasma containment and heating by closed discharge currents main-

tained through external electromagnetic forces. The second one is what you might describe as a magnetic trap in which the plasma is trapped and then heated in some continuous way, as by injecting particles into it.

In this connexion one might mention that as regards the D-D reaction the temperatures to be achieved are of the order of about 500 million volts, and for the D-T reaction smaller by a factor of about 3.

The one other point I should like to mention is the interesting question as to whether the fusion energy can be turned directly into electrical energy, or whether it must necessarily take its path through heat. Here there is only one remark I should like to make. In the course of fusion reactions a certain and considerable amount of energy always escapes as neutrons which are therefore lost from the direct production of electricity. These neutrons, however, are not entirely lost in the sense that one can think of methods by which they might be utilized for making some of the products that one wants. As I have just mentioned, the D-T reaction is much more favorable than the D-D reaction and one could imagine that the neutrons might be used for producing

tritium in some envelope of suitable material, like lithium, which surrounds the reactor. However, since the aim of fusion is really in the long run to produce a virtually limitless source of energy, it is clear that it would have to be based on the D-D reaction rather than the D-T reaction, and for this purpose one would naturally like not to base it on the availability of such materials as tritium.

DISCUSSION OF P/145 AND P/78

There was no discussion of these papers.

DISCUSSION OF P/2298 AND P/2410

Mr. TELLER: I should be grateful, Mr. Dobrokhotov, if I could get information as to the status of the machine which has been described to us and which, I believe, is called the "Ogra" machine. In what state of operation is it at the present time? What are its stability properties? *

Mr. DOBROKHOTOV (USSR): The apparatus of which I spoke, and which is displayed in the Scientific Exhibition, was constructed very recently and the experiments that will give the data asked for have not been performed yet. The machine is operating and data will soon be obtained.

Mr. HOYAUX (Belgium): Dr. Teller, can the stabilized toroidal pinch of the Zeta type on the one hand, and the toroidal stellarator with auxiliary coils twisting the lines of force on the other hand, be considered theoretically as one and the same apparatus, differing only in the relative amount of auto-confinement and external confinement?

Mr. TELLER: Thank you very much for the question; it gives me a chance to emphasize something that I discussed briefly in my paper. The two machines are not the same. The main differences are partly the thing that is pointed out in the question: namely, the predominance of the auto-confinement in the case of the pinch, but the other part, which is connected with it, is the following: the pinch is designed as a pulsed machine which produces, nevertheless, a considerable number of thermonuclear reactions

hopefully, because there is, due to the outer confinement, a great concentration of matter in the pinch; and the rate at which a single deuteron reacts is proportional to the density of the other deuterons about it. This compression is an advantage absent in the stellarator. On the other hand, it has to be balanced against the disadvantage that these high densities are not maintained for a long time. It is in principle a pulsed machine, whereas the stellarator in principle is a stable machine—a steady machine in which the steadiness does not last forever. If we could make it last for a second we would be happy; that is what we are shooting for. But we are not shooting for making the pinch last a second, so there are various directions in which these various machines are looking.

Mr. SPITZER (USA): I should like to ask Dr. Dobrokhotov, what is the status in the USSR of experimental and theoretical confinement in magnetic traps with the figure-8 geometry?

Mr. DOBROKHOTOV (USSR): These theoretical proposals have been studied only qualitatively. We have not yet set up an actual trap with this geometry.

Mr. SINGWI (India): I have a question for Professor Teller on his remarks about the diagnostic importance of the "noble" neutrons. Could I ask whether any experiments have been done regarding the anisotropic distribution of these neutrons?

Mr. TELLER: The answer is yes. I think you will find details in our papers and also in the description of the exhibits. There have been investigations in every case. Various cases have shown great anisotropies. Such great anisotropies, for instance, have been found in the pinch but not in the mirror machine, which did produce neutrons—not in the steady state but when you apparently collapsed it. This is the machine called Scylla. In this machine there is also a trace of anisotropy but it is an anisotropy which is quite small, so that in this case we seem to be getting closer to the point where we might hope to call these things mostly thermonuclear and where there might therefore be a diagnostic value, but I should not like to make any claim.

DISCUSSION OF P/1056

There was no discussion of this paper.

* *Ed. Note*: "Ogra" stands for Artsimovich-Golovin.

Session A-5

THEORETICAL ASPECTS OF PLASMA PHYSICS

LIST OF PAPERS

	<i>Page</i>
P/2500 High-temperature pinches S. I. Braginsky and V. D. Shafranov	43
P/2292 Run away electrons H. Dreicer	57
P/383 Individual particle motion and the effect of scatter- ing in an axially symmetric magnetic field A. Garren <i>et al.</i>	65
P/1799 Effect of distant collisions on magnetic autostriction in an arc M. Hoyaux <i>et al.</i>	72
P/1861 The static pinch C. L. Longmire	84
P/347 Stability and heating in the pinch effect M. N. Rosenbluth	85
P/2213 Plasma radiation in a magnetic field B. A. Trubnikov and V. S. Kudryavtsev	93
P/2300 High frequency plasma oscillations A. Akhiezer <i>et al.</i>	99
P/357 Heating of a confined plasma by oscillating electro- magnetic fields J. M. Berger <i>et al.</i>	112
P/2215 Absorption of high-frequency electromagnetic energy in a high temperature plasma R. S. Sagdeyev and V. D. Shafranov	118
P/361 Generation and thermalization of plasma waves T. H. Stix	125
P/1307 Plasma oscillations in a stationary external magnetic field S. Ciulli and M. Micu	134
P/365 Stability of plasma in static equilibrium..... M. D. Kruskal and C. R. Oberman	137
P/349 Theoretical structure of plasma equations M. N. Rosenbluth and N. Rostoker	144
P/2214 Dynamics of a rarefied plasma in a magnetic field R. S. Sagdeyev <i>et al.</i>	151
P/354 Stability of a linear pinch B. R. Suydam	157
P/33 Stability of a constricted gas discharge R. J. Tayler	160
P/1538 Cusped geometries J. Berkowitz <i>et al.</i>	171
P/376 Magnetohydrodynamic stability..... J. Berkowitz <i>et al.</i>	177
P/386 Hydromagnetic equilibria and force-free fields.....H. Grad and H. Rubin	190
P/1875 Some stable hydromagnetic equilibria J. L. Johnson <i>et al.</i>	198
P/1876 Equilibrium of a magnetically confined plasma in a toroid..... M. D. Kruskal and R. M. Kulsrud	213

LIST OF PAPERS

(Continued)

		<i>Page</i>
P/1	Hydromagnetic disturbances of large amplitude in a plasma J. H. Adlam and J. E. Allen	221
P/2509	Simple waves and shock waves in magnetohydrodynamics A. I. Akhiezer <i>et al.</i>	225
P/374	Hydromagnetic shock waves in high-temperature plasmas C. S. Gardner <i>et al.</i>	230
P/1368	Possibilities of achieving fusion reaction with a 4π focusing system G. Barsanti <i>et al.</i>	238
P/2532	Slowing down of charged particles in a plasma G. Boulègue <i>et al.</i>	242
P/1183	Mechanization of non-linear calculations in fusion reactor theory P. Braffort and M. Chaigne	247
P/2358	Engineering aspects of a thermonuclear power plant L. Geller <i>et al.</i>	257
P/1765	Thermonuclear reaction rate in a discharge H. Grad	263
P/2430	Transient temperature variations during the self-heating of a plasma by thermonuclear reactions H. D. Greyber	267
P/1941	Fusion chain reaction M. Gryziński	270
P/1184	Theory of the energy development in a thermonuclear plasma of deuterium and of deuterium-tritium D. Magnac-Valette <i>et al.</i>	275

High-temperature Pinches

By S. I. Braginsky and V. D. Shafranov *

It is well known that one of the means of containing and thermo-insulating a hot plasma is based on the use of the pinch effect. When a sufficiently high current flows through the plasma it contracts under the effect of electrodynamic forces into a pinch separated from the chamber walls. The plasma is isolated from the walls by a vacuum (to be more exact, by a region with a very low density), which ensures its heat insulation. The confinement and thermo-isolation can be maintained for a limited period of time. A simple plasma pinch turns out to be unstable. One possible way to make it stable is to use a longitudinal magnetic field. The behaviour of a plasma filament both in the absence and presence of a longitudinal magnetic field has been studied theoretically¹ and later (1953) by the authors. The stabilizing effect of a longitudinal magnetic field was examined theoretically by Shafranov in 1953,² also by Tayler³ and by Rosenbluth.⁴ The first experimental evidence for the stabilizing action of a longitudinal magnetic field was obtained in 1956.⁵

In this work some systems with plasma pinch stabilization by a longitudinal magnetic field are examined from the standpoint of the possibility of heating the plasma and obtaining thermonuclear reactions. It is assumed that the plasma pinch is toroidal so that there are no losses due to electrodes as there are in straight tube discharges. The stretching forces due to the circular current and the gas pressure may be compensated in various ways; for instance, with the aid of an external magnetic field perpendicular to the plane of the ring or by placing the plasma ring in a well-conducting tube (Leontovich 1951, Polievktov-Nikoladze, 1952).[‡] In both these cases a current flowing along the axis is required for equilibrium. For systems with axial symmetry, the need for such a current may be shown generally.⁶ Other methods of stabilizing a plasma pinch are possible; for instance, by the use of guarding conductors.⁷

Stabilization by a longitudinal field can be realized, in principle, by various methods.

One possible method is to use the longitudinal magnetic field trapped in the pinch, together with well-conducting walls. Calculations made for a current flowing in the thin surface layer of the pinch show that if the longitudinal field outside the pinch is small and the pinch radius is not too small compared with the large torus radius, a complete stabilization can be obtained. This method of stabilization is used in the well-known ZETA system.

Another method is the stabilization by a longitudinal field distribution more or less uniformly over the entire volume of the chamber (both inside and outside the plasma). If the current flows in a thin surface layer, complete stability of the plasma pinch over a limited length may be achieved. However, if the current is uniformly distributed through the cross section, not all the types of disturbances are stabilized. The most dangerous types of disturbances—"necking-in" and "bending"—can be stabilized with a sufficiently large but practicable longitudinal magnetic field. The experimental data available⁸ justify the hope that a plasma pinch isolated from the walls may be realized under these conditions. The plasma pressure can be opposed in this method of stabilization not only by the compressive action of the current field but also by the pressure of the longitudinal magnetic field, when it is greater outside than inside the pinch. At the extreme, the pressure of the current field may be small but an axial current is still necessary to keep the ring in equilibrium.

In both cases the plasma is heated by the Joule effect and compression of the plasma caused by the constricting currents. No other heating mechanism, such as heating by external high-frequency fields, is considered.

The heating calculations were made on the assumption that the plasma pinch is "laminar" and using expressions for electric conductivity, heat conductivity and other processes in a fully ionized plasma corresponding to ordinary Coulomb collisions.^{9, 10} In reality, some other processes of dissipation are possible, connected with the partial instability and the development of oscillations in the plasma pinch. These "turbulent" processes have not yet been adequately investigated, so that this work is, of necessity, incomplete and offers only a first approach to the problem.

Original language: Russian.

* USSR Academy of Sciences, Moscow.

‡ References of this form are to articles in *Fizika Plazmy i Problema Upravlyaemykh Termoiadernykh Reaktsii*, AN SSSR (M. A. Leontovich, ed.) Moscow, 1958.

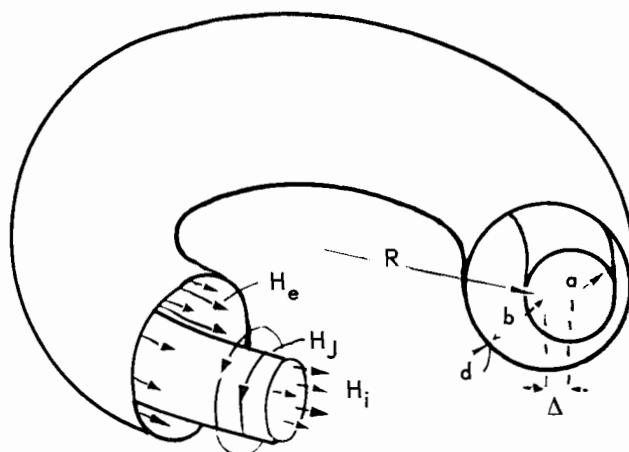


Figure 1. Illustration of principal symbols

EQUILIBRIUM

Cross-sectional Equilibrium

The condition for the equilibrium in the cross section of the plasma pinch (equilibrium along the small radius a) is

$$8\pi\bar{p} + \bar{H}_i^2 = H_J^2 + H_e^2, \quad (1.1)$$

where the bar denotes averaging over the cross section, and the symbols are defined in Table 1 and illustrated in Fig. 1. Taking into account the relations $\pi\bar{p}a^2 = N(T_e + T_i)$ and $H_J = 2J/ca$, Eq. (1.1) can be written (omitting the dash over H_i^2)

$$2c^2N(T_e + T_i) = J^2 + (H_e^2 - H_i^2)a^2c^2/4 = \beta J^2 \quad (1.2)$$

or

$$3.2 \times 10^{-13}N(T_e + T_i) = \beta J^2 \quad (1.2a)$$

where the temperature is given in kev and the current in ka. Also

$$\beta = 8\pi\bar{p}/H_J^2 = 1 - (H_i^2 - H_e^2)/H_J^2 \quad (1.3)$$

where β is the coefficient characterizing the use of current to hold the plasma. The coefficient β , generally speaking, may be both very small (with considerable "paramagnetism", when the longitudinal magnetic field inside is such that $H_i^2 = H_e^2 + H_J^2$) and very large (with "diamagnetism", when $H_e^2 - H_i^2 \gg H_J^2$). Under the conditions close to the stationary ones the magnetic fields within and outside the pinch have the tendency to equalize; but, as was pointed out by the authors in 1953, anisotropy of the electric conductivity of the plasma causes the internal field in the stationary case to be larger than that outside, so that $\beta < 1$. With H_e/H_J changing from 0 to ∞ , β changes from 1 to 0.5. In general, however, the value of β depends upon the development of the process in time, upon the heat balance in the pinch and upon the degree to which the number of magnetic lines of force is increased by the plasma.

Toroidal Equilibrium

This term is used to describe equilibrium along the "large" radius R . The element of the plasma ring of unit length (see Fig. 2) is influenced in the direction

Table 1

a	= small radius of the pinch
b	= small radius of toroidal conducting tube and discharge chamber
R	= large radius of torus
J	= current along the tube axis
m	= electron mass
M	= ion mass
n	= number of ions (electrons) in a unit volume
N	= $\pi a^2 \bar{n}$ = number of ions (electrons) in unit length of the pinch
p	= plasma pressure
T_e	= temperature of electrons
T_i	= temperature of ions
T	= $\frac{1}{2}(T_i + T_e)$ = average temperature
H_J	= $2J/ca$ = current field near the pinch surface
H_i	= longitudinal magnetic field within the pinch
H_e	= longitudinal magnetic field outside the pinch
σ	= plasma conductivity
τ	= effective time between electron collisions
τ_i	= effective time between ion collisions
β	= $8\pi\bar{p}/H_J^2 = 2c^2N(T_e + T_i)/J^2$
β_H	= $8\pi\bar{p}/H^2$.

† Ed. note: For notational uniformity we have introduced the symbol β where the authors had η .

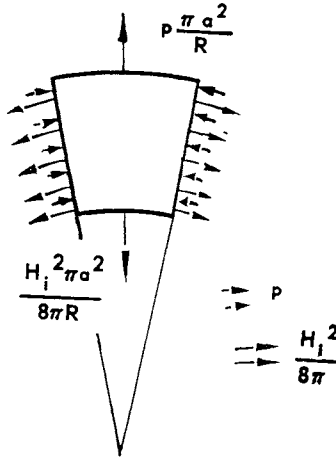


Figure 2

of R by the force of gaseous pressure equal to $p\pi a^2/R$. The internal longitudinal field creates a longitudinal tension leading to a force equal to $-(H_i^2/8\pi)\pi a^2/R$. The external longitudinal field provides a force in the opposite direction $(H_e^2/8\pi)\pi a^2/R$ so that with $H_e = H_i$ these forces cancel each other. With a current flowing along the ring a stretching force arises equal, per length unit, to

$$(2\pi R)^{-1}(J^2/2c^2)dL/dR = (H_J^2/8\pi)2\Lambda(\pi a^2/R)$$

where $L = 4\pi R\Lambda$ is the inductance of the ring, ($\Lambda = \ln b/a$ in the presence of the walls). To compensate the stretching force, a magnetic field is required with a component $H_\perp$, perpendicular to the plane of the ring and therefore resulting in the contracting force $(1/c)H_\perp J = (a/2)H_\perp H_J$. Thus the condition of equilibrium along R may be written

$$H_\perp = \frac{a}{2R} H_J \left\{ \Lambda + \frac{4\pi p}{H_J^2} + \frac{H_e^2 - H_i^2}{2H_J^2} \right\} \quad (1.4)$$

or, excluding the pressure from the equilibrium condition (1.1)

$$\begin{aligned} H_\perp &= \frac{a}{2R} H_J \left(\Lambda + \frac{1}{2} + \frac{H_e^2 - H_i^2}{H_J^2} \right) \\ &= \frac{a}{2R} H_J (\Lambda + \beta - \frac{1}{2}). \end{aligned} \quad (1.5)$$

The field $H_\perp$ can be created, in principle, by currents in special external conductors.

If the pinch is placed in a well-conducting tube, the field $H_\perp$ is created automatically through eddy currents forming in the walls. Taking the pinch, for the purpose of estimation, to be thin $a^2/b^2 \ll 1$ and ignoring the "toroidality" of the tube, it is easy to find $H_\perp$ as the field of the "reflection current". With the pinch displaced from the centre of the tube by a distance Δ , the "reflection current" is at the distance $(b^2/\Delta) - \Delta$ from the pinch and creates a field $H_\perp = 2J\Delta/c(b^2 - \Delta^2) = H_J a \Delta / (b^2 - a^2)$. This leads to the following expression for the displacement

of the equilibrium of the pinch in an ideally conductive tube:

$$\frac{\Delta}{b - \Delta} = \frac{b + \Delta}{2R} (\Lambda + \beta - \frac{1}{2}). \quad (1.6)$$

In reality the tube has a limited conductivity and the displacement will be greater.

It may be seen from (1.6) that if $\beta > R/b$ the pinch approaches very close to the walls so that, practically, the value of β cannot be very high. This is due to the radial decrease of the toroidal field. The value of β may exceed 1 only if the longitudinal magnetic field inside the pinch is smaller than that outside, but such a "diamagnetic" plasma is pushed away from the non-uniform field and at $\beta > R/b$ the force of expulsion is so great that the reflection current in the walls cannot effectively compensate for it. If the field $H_\perp$, which compensates for the toroidal repulsion, is created by special external currents and can be made sufficiently large, the value of β can, in principle, be made as large as desirable.

The action of ideally conductive walls may be graphically explained as follows. With the pinch displaced outwardly, the magnetic lines of force between the pinch and the walls become denser on the outer side and more rarefied on the inner side. The resulting magnetic pressure creates an equilibrium force. In reality, the walls have finite conductivity, so that with the displacement of the pinch the magnetic lines of force partly penetrate through the walls and their compression is weaker. The force of balance steadily diminishes, leading to systematic displacement of the pinch.

The "image current" which creates the field $H_\perp$ is actually realized in the form of currents distributed over the tube. To maintain it, the magnetic lines of force of the pinch current must constantly cross the walls at some speed. For all the changes occurring during the period $t \ll t_e$, where

$$t_e = 2\pi b d \sigma_w / c^2 \quad (1.7)$$

(σ_w is the electric conductivity of the walls, and d their thickness), the walls may be regarded as being ideally conductive, and the pinch displacement from the position of equilibrium may be neglected. If $t \gtrsim t_e$ the displacement must be taken into account. With the pinch moving at the rate $d\Delta/dt$ an electric field $E = (2J/c^2 b)(d\Delta/dt) \cos \phi$ is produced in the tube, inducing currents with a surface density of $i = \sigma_w E d$. The magnetic field of these currents has a dipole character, being homogeneous inside the tube, and is determined by the relation $(4\pi/c)i = 2H_\perp \cos \phi$ so that

$$H_\perp = H_J (a/b) (t_e d\Delta / b dt). \quad (1.8)$$

This expression presupposes that $\Delta \ll b$ and the current in the pinch is constant. Substituting (1.8) in the equilibrium condition (1.5) we can estimate the characteristic time for the tube action, $t_w \sim \frac{1}{2} b (d\Delta/dt)^{-1}$:

$$t_w = t_e (R/b) (\Lambda + \beta - \frac{1}{2})^{-1}. \quad (1.9)$$

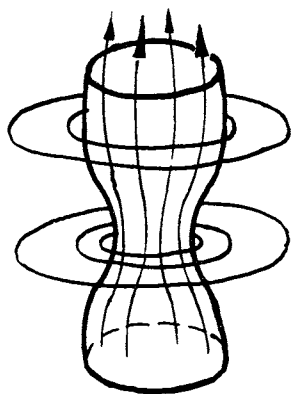


Figure 3

The presence of one or more insulating gaps increases the resistance to these induced currents in the tube and hence reduces the characteristic time of the tube action. If there are no special shields ("labyrinths") the gaps distort the field as well (Leontovich, 1951).

Toroidal equilibrium may be established also with the use of an iron core which becomes magnetized by the pinch current and creates the necessary $H_{\perp}$. The position of equilibrium resulting from the interaction of the current and the core is unstable; however, if the point of equilibrium is inside the chamber, then, with the displacement of the pinch, the uncompensated radial force is $\sim R/b$ times smaller than the force of repulsion without the core (Budker, 1951). Joint use of a conducting tube and an iron core therefore makes it possible to prolong the action of the walls $\sim (R/b)$ times as much as given by Eq. (1.9). In the presence of a conducting tube the influence of the iron core begins to make itself felt in $\sim t_c < t_w$ after the beginning of the process, when the magnetic field penetrates the walls and weakens the force of repulsion. The radial motion of the pinch is then retarded and may even be reversed.

STABILITY

A plasma pinch with a current is known to be unstable. Any slight accidental constrictions rapidly increase (Trubnikov, 1952). A plasma pinch, like any flexible conductor with a current, is unstable to other deformations also (Leontovich, Shafranov, 1952). The pinch can be stabilized, however, by the use of a longitudinal magnetic field and a well-conducting wall.

The stabilization mechanism can be explained as follows. With good conductivity of the plasma the magnetic lines of force are as if frozen into the substance, so that any deformation of the pinch causes a corresponding deformation of the magnetic field. If the stabilizing field is so chosen that the resulting magnetic force tends to cancel the deformation, the pinch will be stable.

Neck Stability

Suppose that a neck has been formed in the pinch (Fig. 3). In this case the magnetic field of the

current, $H_J = 2J/ca$ grows in the constricted place, being proportional to J/a , and tends to increase the constriction. However, the longitudinal magnetic field grows faster, as $1/a$, and by its pressure hampers the further constriction. The increase of the longitudinal field pressure is

$$(H_1^2/8\pi)\{[(a+\Delta)/a]^4 - 1\} \approx (H_1^2/8\pi)4\Delta/a$$

while the increase in the pressure of the pinch field is

$$(H_J^2/8\pi)\{[(a+\Delta)/a]^2 - 1\} \approx (H_J^2/8\pi)2\Delta/a$$

where Δ is the change of the radius. For stability, the first value must be larger than the second; i.e.,

$$H_1^2 > H_J^2/2. \quad (2.1)$$

This criterion is valid irrespective of whether there is or is not a longitudinal magnetic field outside the pinch. It changes slightly, however, if the current flows not only through the pinch surface but if, for instance, the current is uniformly distributed over the cross section, then the neck stability condition appears as follows (Volkov, 1953)

$$H_1^2 > 0.69 H_J^2. \quad (2.1')$$

Bending Stability

With a bending of the pinch (Fig. 4) the magnetic field of the current grows on the concave side and decreases on the convex side, causing a force similar to that of toroidal repulsion and tending to increase the distortion. The forces of the gas kinetic pressure in the plasma act in the same direction. As a result the plasma pinch turns out to be unstable for such distortion, like a compressed spring. If, however, there is a longitudinal magnetic field within the pinch, its lines of force are elongated by distortion and, acting like tight rubber bands, cause a force that tends to straighten out the pinch. The stabilizing force is apparently the greater, the larger the pinch curvature radius. Calculations⁴ show that if the internal longitudinal field H_1 approximates H_J then all disturbances with a "wavelength" $\lambda < 14a$ are stabilized (λ is the period of disturbance along the

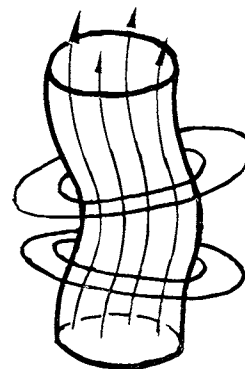


Figure 4

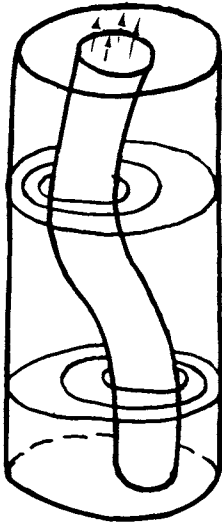


Figure 5

axis). With H_i having a smaller value the short-wave disturbances alone will be stabilized. According to the condition of equilibrium (1.1), in the absence of a longitudinal magnetic field H_i is always less than H_J , it is therefore clear that a "frozen-in" field alone cannot provide a complete stabilization of the pinch. Fortunately, the conducting walls which prevent the radial stretch of the toroidal pinch exercise at the same time a stabilizing effect upon the long-wave bends. Indeed, when the pinch is displaced from the position of equilibrium it approaches the conducting wall, compressing the magnetic lines of force between the pinch and the walls, which results in the appearance of the restoring force (Fig. 5). The stabilizing action of the containing tube depends upon the relation between the length of the disturbance wave and the distance to the walls. For a very short wavelength the change of the magnetic field outside gradually fades away at a short distance from the pinch, and the tube fails to exert a stabilizing effect. A very thin pinch will therefore be unstable for disturbances with such intermediate wavelengths as are no longer stabilized by a frozen-in field and are not yet stabilized by the conducting walls. To ensure stability over the entire range of wavelengths the ratio between the tube radius and the pinch radius must not be too large. Calculations show, for instance, that with $H_i = H_J$ the pinch will be stable at $b/a < 5$, and with $H_i/H_J = 0.9$; 0.8 ; 0.7 , respectively, at $b/a = 4.0$; 3.2 ; 2.5 .

Supposing now that, in addition to the longitudinal field inside the pinch, there is a considerable longitudinal field outside, H_e . Then the magnetic lines of force outside the pinch will be not rings but spirals, with the ratio between their pitch and the circumference $2\pi a$ being equal to H_e/H_J . A distortion of the pinch causing it to form a spiral with the same pitch as that of the magnetic lines of force is not stabilized by the conducting tube, since with such deformations it is as if the pinch "moves apart" the

lines of force and disturbs the outer magnetic field only slightly. The long-wave bends thus turn out to be unstable.

Figure 6 shows the regions of stability for various values of $\epsilon = H_e/H_i$ and κ , the ratio of the magnetic flux inside the pinch to the whole flux inside the tube, as well as the respective values of β as functions of a/b . Even a small field outside the pinch weakens the stability. For example if the longitudinal field outside, H_e , is equal to 0.1 of the longitudinal field inside, H_i , the pinch will be stable at $b/a < 4$ (instead of $b/a < 5$ at $H_e = 0$); if $H_e = 0.5 H_i$ then the region of stability lies within $b/a < 1.7$. With an increase of the outside field, stability for the long-wave disturbances disappears.

If the longitudinal field is large, another method of stabilization is possible: the pinch may be stabilized against bending if it is shorter than the helical pitch of the magnetic lines of force, as in this case no dangerous curls along the lines of force are possible. The length of the toroidal pinch is $2\pi R$ so that the condition of stability, $2\pi R < 2\pi a H_e/H_J$ assumes the form

$$\frac{H_e}{H_J} > \frac{R}{a}, \quad \text{or} \quad \frac{J}{\pi a^2} < \frac{cH}{2\pi R}. \quad (2.2)$$

Here the longitudinal field considerably exceeds the current field, but this does not contravene the con-

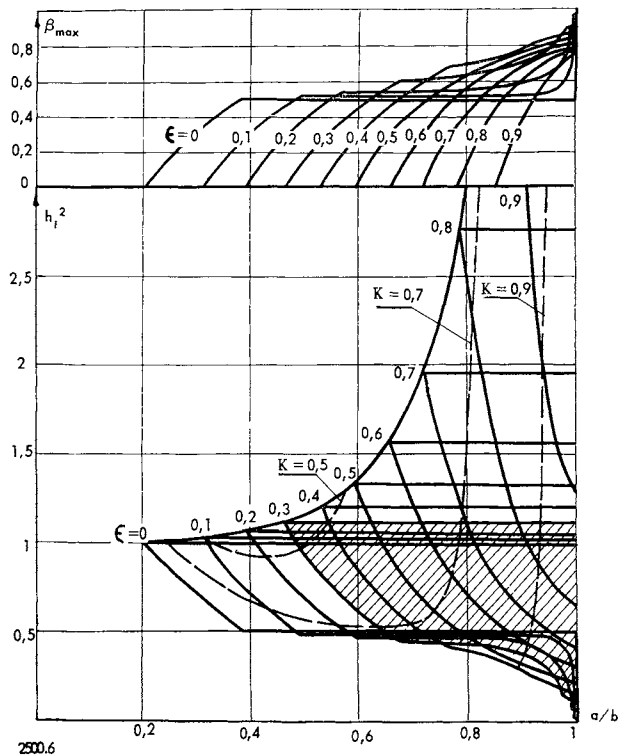


Figure 6. Conditions for stabilization by a weak magnetic field:

$$\epsilon = \frac{H_e}{H_i}; \quad h_i^2 = \frac{H_i}{H_J^2}; \quad \kappa = \frac{H_i \pi a^2}{H_i \pi a^2 + H_e \pi (b^2 - a^2)}$$

The shaded section represents the stability region for $E = 0.3$

dition for equilibrium since the longitudinal field exists both inside and outside the pinch. Condition (2.2) being observed, there still remain unstabilized some types of disturbances of more complex form, with which, however, no displacement of the pinch axis takes place. It may be hoped that such disturbances will not be dangerous.

Two methods of stabilization by a longitudinal field are thus possible: stabilization by a frozen-in field and a conducting tube, in which case the external magnetic field must be small (Fig. 7), and stabilization by a strong longitudinal field (Fig. 8) which is more or less homogeneous over the whole cross section of the chamber.

HEATING, ENERGY BALANCE

Plasma Heating Conditions

A flow of current through the plasma generates Joule heat. In addition, the constriction of the pinch also causes the plasma to be heated by the work performed by the compressing forces. These two types of heating may be approximately accounted for by expressing the heat balance equation in the following form:

$$3N \frac{dT}{dt} = Q_J - \frac{2NT}{a^2} \frac{da^2}{dt}. \quad (3.1)$$

It is assumed here that no heat losses exist. If this is not the case, an additional member, Q_{loss} , appears on the right-hand side. The Joule heat generated is given by

$$Q_J = J^2 / \pi a^2 \sigma \quad (3.2)$$

where $\sigma = e^2 n \tau / m$ is the electrical conductivity of the plasma across the magnetic field (the "Coulomb logarithm" is taken as 15):

$$\sigma = \sigma_1 T_e^{-3/2}, \quad \sigma_1 = 1.83 \times 10^{17} \text{ sec}^{-1} \text{ kev}^{-3/2}; \quad (3.3)$$

l is a dimensionless coefficient, taking into account the skin-effect and anisotropy of the conductivity, as well as the heat generated by azimuthal currents. For simplicity's sake we shall assume throughout that the ratio between the temperatures of the electrons and ions is independent of time.

Let us introduce a characteristic magnitude for the plasma column, viz., a "skin-time" s :

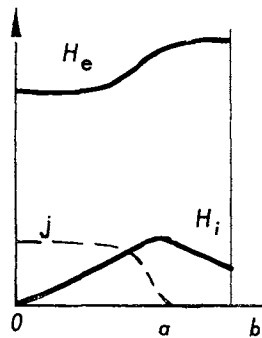


Figure 7

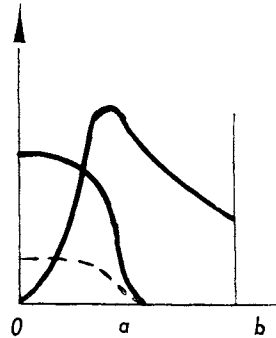


Figure 8

$$s = \pi a^2 \sigma / c^2 \quad (3.4)$$

or, for ordinary Coulomb collisions when (3.3) is valid,

$$s = s_1 a^2 T_e^{3/2}, \quad s_1 = 6.4 \times 10^{-4} \text{ sec cm kev}^{-3/2}. \quad (3.4')$$

The quantity s determines the characteristic skin-effect time; I/s , apart from a factor c^2 , coincides with the pinch resistance per unit length; $\ln s$, apart from a constant, is the entropy of the plasma pinch per unit length.

The time between particle collisions is simply expressed in terms of s . For electron and ion (deuteron) collisions:

$$\tau = \frac{mc^2 s}{e^2 N} = \frac{3.6 \times 10^{12}}{N} s;$$

$$\tau_2 = \frac{(2mM)^{1/2} c^2 s}{e^2 N} \left(\frac{T_e}{T_i} \right)^{3/2} = \frac{3.1 \times 10^{14} s}{N} \left(\frac{T_e}{T_i} \right)^{3/2}. \quad (3.5)$$

Equation (3.3) may be rewritten, using Eqs. (1.2) and (3.2), in a form convenient for estimations:

$$ds/dt = 2f/\beta, \quad (3.6)$$

or, if there are heat losses,

$$ds/dt = 2f/\beta - Q_{\text{loss}}/2NT. \quad (3.6')$$

In the absence of heat losses the value of s grows monotonically with time, $s = s_0 + \int (2f/\beta) dt$, and for an approximate evaluation it is possible to assume (for $t \gg S_0$) §

$$s = \frac{2ft}{\beta} = \frac{J^2 ft}{c^2 N (T_e + T_i)}. \quad (3.7)$$

Substituting here β and s from (1.2) and (3.4) we obtain an expression for the ion and electron temperatures:

$$\begin{aligned} T_i &= \left(\frac{2}{1+x} \right)^{2/5} x T_0; \\ T_e &= \left(\frac{2}{1+x} \right)^{2/5} T_0; \\ T_0 &= \left(\frac{f}{2\pi\sigma_1} \frac{J^2 t}{Na^2} \right)^{5/2} = 1.88 \left(\frac{10^{15} f J^2 t}{Na^2} \right) \text{ kev } (J \text{ in ka}), \\ x &= T_i/T_e, \end{aligned} \quad (3.8)$$

where T_0 is the temperature that would be attained at $T_e = T_i$. It should be taken into account that this expression may include, in addition, a coefficient ~ 1 , depending on the laws of change of current

§ It should be noted that formula (3.7) can be written approximately as

$$\frac{2}{3} N (T_e + T_i) = \frac{2}{3} Q_J t; \quad Q_J = J^2 f / \pi a^2 \sigma = J^2 f / c^2 s. \quad (3.7')$$

The energy balance expressed in this way accounts for the heating due to compression, in addition to the Joule heat.

This expression, as well as (3.7), is valid in order of magnitude also in cases when the conductivity differs from that given by (3.3).

and radius in time (since f/β may vary in time). If $J(t)$, $a(t)$ are known then, taking f to be constant and integrating (3.1) we get in (3.8) J^2/a should be replaced by $(5/3)a^{1/3}\int J^2a^{1/3}dt$. At $J \sim t$, $a \sim 1/t$, for instance, this coefficient will be equal to 1.

The factor f may be larger with a strong skin-effect. Supposing that the current flows through a narrow ring with a thickness δ , the Joule heat and the corresponding coefficient f are approximately

$$Q_J = \frac{J^2 f}{\sigma 2\pi a \delta}; \quad \delta = a \left(\frac{t}{s} \right)^{1/2}; \quad f = f' \frac{a}{2\delta} = \frac{f'}{2} \left(\frac{s}{t} \right)^{1/2}. \quad (3.9)$$

Coefficient f' takes into account the heating by azimuthal currents j_ϕ . Using these estimates we obtain for a strong skin-effect $\delta \ll a$ (practically, for $s/t > 4$)

$$T_i = \left(\frac{2}{1+x} \right)^{4/7} x T_0; \quad T_e = \left(\frac{2}{1+x} \right)^{4/7} T_0; \quad (3.10)$$

instead of (3.8), where $x = T_i/T_e$ and $T_0 = 0.76 (10^{14} J^2 t^{1/2} / a N)^{4/7}$.

From expressions (3.7) and (3.9) we get a simple relation, valuable for making estimates:

$$f = (f')^2 / 2\beta. \quad (3.9')$$

Heating the Ions

In Joule heating the energy goes directly to the electrons which then convey part of it to the ions by means of collisions:

$$\Delta Q = \frac{3mN}{M\tau} (T_e - T_i) = \frac{3\pi N}{s} (T_e - T_i). \quad (3.11)$$

Here it is assumed that the density of the plasma is constant over the pinch cross section. If the density is distributed differently, a corrective coefficient must be introduced; for instance, for parabolic distribution, the coefficient equals 4/3.

The dimensionless quantity Π depends upon the number of particles per length unit in the pinch:

$$\Pi = e^2 N / M c^2. \quad (3.12)$$

For hydrogen, $\Pi = 1.54 \times 10^{-16} N$; for deuterium, $\Pi = 0.77 \times 10^{-16} N$.

Ion heating is described by an equation similar to (3.1)

$$\frac{3}{2} N \frac{dT_i}{dt} + \frac{2NT_i}{a} \frac{da}{dt} = \Delta Q = \frac{3\Pi N}{s} (T_e - T_i). \quad (3.13)$$

The heating due to compression (the term in da/dt), as distinct from the Joule mechanism, is equally effective for both ions and electrons.

If for the sake of simplicity we assume the ratio between the temperatures of the ions and electrons to be constant, we get

$$\frac{T_e - T_i}{T_i} = \frac{1}{3\Pi} \frac{ds}{dt} = \frac{2}{3\Pi} \frac{f}{\beta} \quad (3.14)$$

or approximately

$$\frac{T_e - T_i}{T_i} = \frac{1}{3\Pi} \frac{s}{t} = 2.13 \times 10^{-4} \frac{a^2 T_e^{3/2}}{\Pi t} \quad (3.15)$$

when T is in kev. With a sufficiently large value of Π the temperatures of the ions and electrons are close to one another. However, the increase of Π leads to a drop in the plasma temperature, so that the temperature of the ions is the highest at some optimum value of Π corresponding to some not-very-large difference between ion and electron temperatures: $T_e - T_i \sim T_i$; i.e., $\Pi_{\text{opt}} \sim \frac{1}{3} ds/dt = \frac{2}{3} f/\beta$. If we make use, for instance, of the condition (3.7) for heating the plasma and of the condition (3.15) for heating the ions and find the maximum value of T_i with a , J and t fixed, the following optimal values result: for a weak skin-effect

$$(T_i/T_e)_{\text{opt}} = 0.72; \quad \Pi_{\text{opt}} = 0.86 s/t;$$

for a strong skin-effect

$$(T_i/T_e)_{\text{opt}} = 0.65; \quad \Pi_{\text{opt}} = 0.62 s/t.$$

However, if we seek the maximum at fixed values of s/t and β we get $(T_i/T_e)_{\text{opt}} = 0.58$; $\Pi_{\text{opt}} = 0.46 s/t$. With very high temperatures (dozens of kev) the optimal value of Π , from the standpoint of an increase of the number of reactions, is much higher (Kogan, 1953). The temperatures of the ions and electrons will be the same in this case.

Constriction of the Pinch

The change of the pinch radius in time is determined by the condition of the pinch equilibrium (1.2) and the heat balance in it (3.4), as well as by the change of the magnetic flux in the pinch, described by the Maxwell equations. Neglecting the anisotropy of conductivity, we get $\dot{\phi} = -2\pi a J \phi(a)/c\sigma$. For estimations we may take $\dot{\phi} = 2(f_1/s)(H_e - H_i)\pi a^2$, where f_1 is a coefficient equal to about 1. Differentiating (1.2) with respect to time and substituting dT/dt from the heat balance for a pinch separated from the walls (3.4), we get

$$\begin{aligned} \frac{1}{a} \frac{da}{dt} \left[\frac{2}{3} \frac{4c^2 NT}{J^2} + \frac{H_e^2 + H_i^2}{H J^2} \right] \\ = \frac{2}{3} \frac{\bar{f}}{s} - \frac{1}{J} \frac{dJ}{dt} \frac{H_e}{H J^2} \frac{dH_e}{dt}, \end{aligned} \quad (3.16)$$

where

$$\bar{f} = f + \frac{2}{3} f_1 (\beta - 1) 2H_e / (H_e + H_i).$$

If heat losses from the pinch occur, then the term $-\frac{2}{3} c^2 Q_{\text{loss}}/J^2$ appears on the right side of (3.16).

For the plasma pinch to separate from the walls and begin to contract it is necessary for the right side of (3.16) to be negative. The point of this condition is that the outward diffusion of the plasma must be smaller than the drift inward under the action of the axial and azimuthal electric fields.

From (3.16) it is evident that for large $H^2/8\pi p$ the pinch radius changes very slowly, the characteristic period of change being of the order of $s(H^2/8\pi p)$.

Role of Heat Conductivity

In a plasma pinch detached from the chamber walls and separated from them by vacuum, the role of heat conductivity consists only in equalizing the temperature over the pinch cross section. The main contribution to the heat conductivity of "magnetized" plasma across the magnetic field is made by the ions. Their heat conductivity is

$$\kappa = \frac{2nT_i}{M\omega_i^2\tau_i} = \frac{2nT_iMc^2}{c^2H^2\tau_i} = \left(\frac{2M}{m}\right)^{\frac{1}{2}} \frac{n^2Tc^2}{\sigma H^2}. \quad (3.17)$$

The temperature difference between various points of the pinch cross section may be estimated, by order of values, by comparing the heat flow with Q_J thus: $2\pi a\kappa(\Delta T/a) \sim J^2/\pi a^2\sigma$. This leads to

$$\frac{\Delta T}{T} \sim \left(\frac{m}{M}\right)^{\frac{1}{2}} \frac{1}{\beta\beta_H}. \quad (3.18)$$

It may thus be taken that the temperature is constant with a fair degree of accuracy over the pinch cross section.¹ All additional heat conduction mechanisms only improve this equalization of temperature.

Heat Losses Due to Radiation

With a rise in the temperature of the electrons, heat begins to be lost through both bremsstrahlung and magnetic radiation.

Bremsstrahlung consists of comparatively high energy quanta for which the plasma is transparent, so that the energy losses (per electron) do not depend upon the form and size of the plasma and are approximately

$$Q_{br} = 5.4 \times 10^{-24} n T_e^{\frac{1}{2}} \text{ erg sec}^{-1} \text{ cm}^{-3}, \quad (3.19)$$

for T in kev. The ratio between the losses due to bremsstrahlung and Joule heat is

$$\frac{Q_{br}}{Q_J} = \frac{\beta^2}{f} \left(\frac{2T_e}{T_e + T_i} \right)^2 \frac{J^2}{J_u^2} \quad (3.20)$$

where $J_u \simeq 2 \times 10^6$ amperes. This shows that bremsstrahlung becomes essential in the pinch heat balance only at high currents. For instance, let $N = 10^{17}$ and $T = 10$ kev; then $\beta J^2/J_u^2 = 0.16$ so that the bremsstrahlung is not yet appreciable.

Magnetic radiation was studied by Galitsky and Migdal (1951) and Trubnikov.¹¹ It is substantial only with temperatures of scores of kev. It is known that a charge revolving in a magnetic field radiates at a cyclotron frequency ω_c and its harmonics. The intensity of this radiation is proportional to H^2T_e . If it came out freely from the plasma the losses caused by it would surpass the bremsstrahlung at $T_e > 5$ kev (if $H^2/8\pi = 2nT_e$). In the range of low frequencies, however, the magnetic radiation is strongly absorbed by the plasma and its intensity does not depend upon the volume but is

determined by the Rayleigh-Jeans law for black-body radiation. The resultant energy losses here are proportional to T/ω^{*3} where ω^* is the limiting frequency of the effective radiation spectrum. If the temperature is not very high, practically the entire radiation occurs at the fundamental frequency, $\omega^* \sim \omega_c$, but if the electron velocity approximates that of light then the radiation in the harmonics increases and the maximum emission intensity shifts towards high frequencies. For a Maxwell velocity distribution, the electrons with an energy several times higher than the average one radiate effectively, so that even with $T_e \sim 30-50$ kev, ω^* may exceed ω_c almost by an order of magnitude.

The expression for energy losses (per electron) from a flat layer of plasma of thickness l obtained by Trubnikov is

$$Q_H' = \frac{4}{3} (l^4/m^3c^5) T_e H^2 F(\alpha, T_e) \text{ erg/sec} \quad (3.21)$$

where α is a coefficient (less than 1) taking absorption into account, $\alpha = 4\pi enl/H$. For the plasma pinch we may take $H^2 = 16NT/a^2\beta_H^2$, $\alpha = e^2N\beta_H^2/T$. The losses via magnetic radiation (with a fixed N) are in this case proportional to $1/a^2$, i.e., the same as the losses on bremsstrahlung and generation of the nuclear reaction energy. Hence, there are no critical dimensions for the plasma pinch connected with magnetic radiation.

The numerical values of F obtained¹¹ for $\alpha = 10^3, 10^4$ and for $T_e = 10, 25, 50$ kev are approximated with sufficient accuracy by the function $F = 1.15 \times 10^{-2} T_e^{3/2} \alpha^{-1/2}$ so that the losses per unit length of the pinch are

$$Q_H = 2 \times 10^5 \frac{(10^{-17}N)^{1/4} T_e^{3/2} \langle T_e^{3/4} \rangle}{\beta_H^{1/4} a^2} \text{ erg sec}^{-1} \text{ cm}^{-1}, \quad (3.22)$$

when the temperatures T and T_e are given in kev.

Thermonuclear Reactions and Energy Balance

The number of thermonuclear reactions per length unit of the pinch is

$$N_n = N^2 (t_{eff}/\pi a^2) \langle \sigma_n v \rangle, \quad (3.23)$$

where σ_n is the effective cross section for the reactions, the angle brackets indicate averaging over the distribution of the ion velocities and t_{eff} is the effective period of time for which the reactions take place. Since $\langle \sigma_n v \rangle$ grows with the temperature, the rate of reaction is greater near the maximum temperature, but already at $T \geq 1$ kev we may, for estimation, evaluate $\langle \sigma_n v \rangle$ for the maximum temperature, and take t_{eff} of the order of the characteristic time of the process (at $T = 1$ kev, $t_{eff} \sim 0.5$ quarter-period of the current; (Kogan, 1953)). We shall henceforth assume $t_{eff} = t/2$.

If N is fixed, then the total number of reactions per impulse is determined by t/a^2 . Since heating time is proportional to a^2 and, according to the conditions of stability, $a \sim a_0$, the number of reactions per impulse in similar systems of various

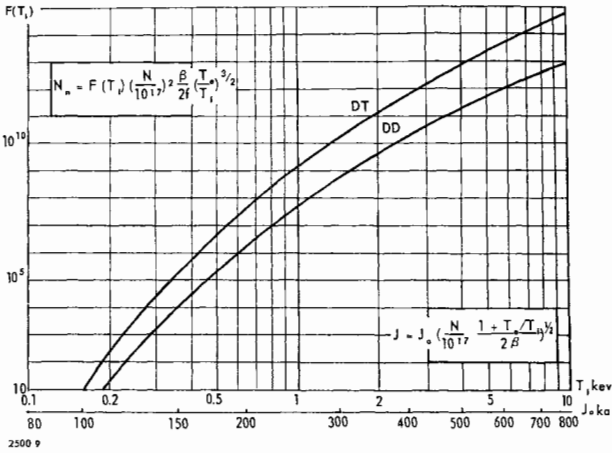


Figure 9. The total number of reactions per impulse per unit length of the plasma pinch

dimensions is the same for the same temperature. This result is analogous to the conclusion that the number of reactions is independent of the rate of current growth in processes without a longitudinal field.

Figure 9 gives the total number of reactions per impulse for $T < 10$ keV, in accordance with (3.23), the time of the process being determined by (3.7): $t/a^2 = (\beta/f)(s_1/2)T^{3/2}$. The ion and electron temperatures are taken to be the same. The values $\langle \sigma_n v \rangle$ are from the work of Klimov and Zubarev (1952).

The power generated in the plasma by the reactions is

$$Q_n = (N^2/\pi a^2) \langle \sigma_n v \rangle \varepsilon, \quad (3.24)$$

where ε is the energy generated in the plasma by each reaction. The neutrons emerging in the reactions freely leave the plasma and are ignored here (although they may be utilized, of course, in one way or another). The fast charged particles circulate in the strong magnetic field and impart their energy to the plasma.

For each neutron formed in the DD reactions a total energy of 7.25 MeV is generated; out of this there remains in the plasma the energy of the charged particles, 4.825 MeV; for DT reactions the corresponding numbers are 17.6 and 3.52 MeV.

Figure 10 shows curves of the power generated per unit length of the pinch for deuterium plasma and also for a plasma consisting of an equal number of deuterium and tritium ions. The numerical data were taken from the work of Klimov and Zubarev. The tritium formed in the reactions may also "burn up", participating in reactions with deuterons. Figure 10 gives two curves for the power generation in deuterium plasma, on the assumption (a) that there is enough time for tritium to burn up, and (b) that this is not the case.

In addition to generation of nuclear energy (Q_{DD}, Q_{DT}) Fig. 10 shows also the generation of heat due to the electric field ($\frac{3}{2}Q_J$, see [3.7']) and radiation losses—

bremsstrahlung Q_T and magnetic Q_H —per unit length of pinch. Q_J and Q_T depend upon N in different ways from Q_H and Q_{DD}, Q_{DT} . The curves for them are given for $N = 10^{16}, 10^{17}$ and 10^{18} . Figure 10 shows graphically the relationship between the different types of energy generation and energy losses.

The points of intersection of the $Q_{nuclear}$ curve with the $\frac{3}{2}Q_J$ curve determine the temperature at which the nuclear energy generated in the pinch becomes equal to $3NT$. For the DT system this temperature is about 6–7 keV at $N = 10^{18}$.

At high temperatures an energetically advantageous thermonuclear reactor using a mixture of deuterium and tritium is quite feasible. For the DD system the energy conditions are far more strict. In this case a narrow operating range at a temperature of about 40–50 keV might be possible. Magnetic radiation, as was pointed out by Budker, may be considerably reduced by the use of metal reflectors. In this case the operating range for the DD system widens.

Similitude Laws

For plasma pinches with different parameters, simple relations of similitude may be established. First, physical similitude requires, of course, geometrical similarity of the systems and of the time dependence of the current and the external field. If these conditions are observed, the systems, as may be seen from the fore-going, will be physically similar

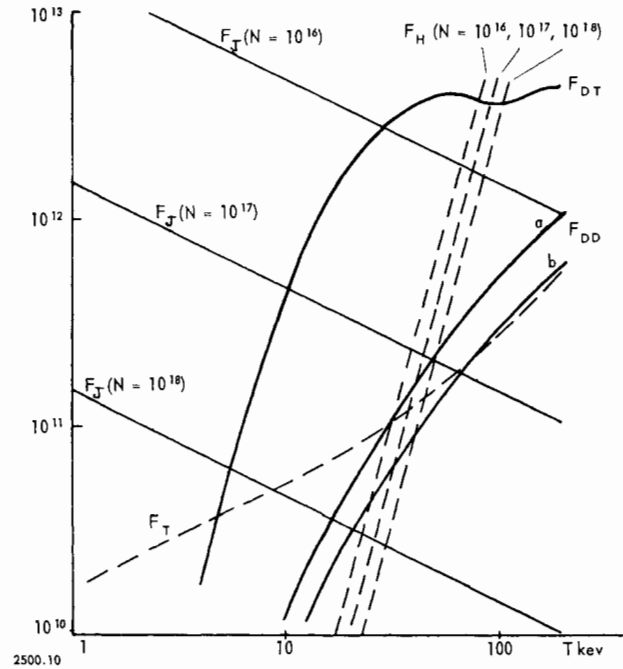


Figure 10. Generation of energy per unit length of pinch

$$Q_J = \frac{N_1 b}{a^2 \beta} F_J, \quad Q_T = \frac{N_1^2}{a^2} F_T$$

$$Q_H = \frac{N_1^{7/4}}{a^2 \beta^{5/4}} F_H; \quad Q_{DD,DT} = \frac{N_1^2}{a^2} F_{DD,DT}$$

provided they have in common the following dimensionless parameters—criteria of similitude—viz.:

1. *Number of particles per classical proton radius—*

$$\Pi = e^2 N / M c^2. \quad (3.25)$$

The ratio T_i/T_e depends upon Π , which is also involved in the relation between the current and the temperature.

The ratio of the ion cyclotron frequency ω_c to the frequency of the magnetohydrodynamic oscillations of the pinch $v_A/a = H/(4\pi p)^{1/2}a$, upon which the character of the oscillations possible in the plasma substantially depends, is also expressed in terms of Π : $\omega_c a / v_A = \Pi^{1/2}$. For instance, the concept that the magnetic lines of force are “frozen” in for oscillations with a wavelength $\sim a$ is applicable only if $4\pi e^2 \Pi / M c^2 \gg 1/a^2$, i.e., $\Pi \gg 1$.¹² Hence, if wave processes play a substantial role in the pinch their similitude also depends on Π . The value of Π determines the ratio of the Larmor ion radius to the pinch radius ($r_{ci}/a \sim \Pi^{-1/2}$) as well as the ratio of the time of particle collisions to the skin-time s (3.5), upon which depends the character of the microscopic motion of the particles. The dimensionless criterion Π was first introduced by Braginsky.¹

2. *Relation between skin-time and process time—*

$$S = s/t = \pi a^2 \sigma / c^2 t. \quad (3.26)$$

The depth of the skin-effect and consequently the transverse distribution of all the quantities in the pinch, the degree to which the magnetic field is “frozen in”, and the relative role of Joule heating and heating due to compression depend upon this dimensionless parameter.

It may be convenient also to use, instead of the parameter S , the dimensional quantity $\bar{S}$ which directly involves the current and radius of the discharge chamber:

$$\bar{S} = b^2 J^3 / t. \quad (3.27)$$

3. *Characteristic ratio of the longitudinal field to the current field—*

$$Z = cbH_0/2J = H_0/H_J(b). \quad (3.28)$$

This value defines the stability conditions. The role of the longitudinal field in equilibrium and its influence upon the process of constriction or expansion of the pinch, Joule heat generation, heat conductivity, etc. depend upon the value of the parameter Z .

The other dimensionless quantities, such as $\beta = 2 e^2 N (T_e + T_i) / J^2$ and T_e/T_i , are functions of these three criteria of similitude, Π , S and Z .

The equality of the criteria Π , S or Z in various systems does not, of course, guarantee exact similitude of the processes at the initial stage when the ionization is

still incomplete.|| Also, equality is not maintained of the ratio of cyclotron frequency to collision frequency, on which the character of the movement of particles in the magnetic field depends, but this is not essential if $\omega_c \tau \gg 1$. For deuterons, $\omega_c \tau \sim 10^{-3} a T_e^2 v (\beta \Pi)^{-1/2}$. If $\beta \Pi \approx 1$ and $a \approx 10$ cm, then $\omega_c \tau \gg 1$ for $T \gg 10$ ev. Similitude in Π , S and Z thus takes place when the temperature reaches several tens of ev. In this case, the duration of the initial stage with a low temperature must be small compared with the whole process. It is also assumed that the characteristic inertia time (Artsimovich, 1953)¹³ is also much shorter than the process period so that the inertia does not matter.

At high temperatures, when bremsstrahlung, magnetic radiation, and generation of nuclear energy contribute to the energy balance, similitude is preserved if the temperatures in the respective systems are the same. In such systems with equal temperatures, as follows from the equality of criterion S , the ratios t/a^2 should be the same.

Among the similitude criteria mentioned above, only one, namely S , directly depends on the electrical conductivity connected with Coulomb collisions. If turbulence in the plasma is essential, the direct dependence of S on the plasma parameters (temperature, etc.) may change. However, there is reason to hope that at equal temperatures (currents) the systems will continue to be similar for the same values of Π , Z , t/a^2 .

STABILIZATION BY A WEAK MAGNETIC FIELD AND METAL WALLS

The microscopic stability ensuring a sufficiently long existence of the pinch as a whole is absolutely necessary for its heat insulation. Two stabilization variants have been mentioned above. One of them is stabilization by a weak magnetic field and well-conducting walls. In this case, to stabilize the pinch, conditions must be created under which the longitudinal magnetic field outside the pinch would be sufficiently small, while inside it would be of about the same order as the current field $H_J = 2J/ca$.

Such a condition can be achieved, for instance, if a sufficiently highly-conducting plasma pinch is compressed in a previously established longitudinal magnetic field H_0 . The magnetic field in the pinch is trapped by the plasma and increases with its constriction in proportion to $1/a^2$, while the longitudinal field outside the pinch diminishes, for the magnetic flux remaining outside distributes itself over the large area between the pinch and the conducting tube.

Such conditions can be created, for instance, if, simultaneously with the current rise in the pinch,

|| It should be noted that the quantity E/P , which is important at the initial stage and is proportional to $(dJ/dt)/(N/a^2)$ is the same in similar systems if the maximum currents and, consequently, the ratios t/a^2 , are the same. At the ionization stage, when the temperature of the electrons and the electric conductivity are more or less constant, systems with similar t/a^2 also preserve similitude of the processes connected with criterion S (the skin effect, etc.).

the external longitudinal field is removed by reducing the current in the solenoid producing the field (the metal tube must have a longitudinal insulated gap in this case). We shall consider below the first regime only, i.e., that with trapping and compressing the field.

We shall stress the basic features of this regime, assuming at first, for the sake of simplicity, that the magnetic field is completely dragged along by the plasma when compressed.

It is clear from the condition of equilibrium that the constriction can begin only after the pressure of the current field exceeds that of the longitudinal field and the plasma. Prior to detaching from the walls, the pinch is intensively cooled by heat conduction and the plasma pressure is relatively small. Considering the main mechanism of energy-losses to be the "magnetized" ion heat conductivity and assuming $H_i = H_J$, we obtain, taking Q_J to be equal to the heat flow to the walls, an estimate for $\beta = (m/M)^{1/2}$. Neglecting the plasma pressure, we get for the moment of detaching:

$$2J_0/cb = H_0. \quad (4.1)$$

As H_i required to stabilize the longitudinal field must be close to H_J , it is possible as the first approximation to the detached pinch to omit from the condition of equilibrium the kinetic gas pressure and take, approximately,

$$H_J = H_i. \quad (4.2)$$

The longitudinal field increases with the constriction of the pinch according to the relation

$$H_i \approx H_0 b^2/a^2 \approx H_J$$

and the pinch radius is therefore inversely proportional to the current

$$a/b = J_0/J. \quad (4.3)$$

Hence in the stability region, $a_0/a < 5$ (see Fig. 6), the current at the initial stage of the constriction must be equal to at least 1/5 of its maximum value. The initial longitudinal field must be adjusted from the condition that

$$H_0 \approx (a/b)2J_{\max}/cb. \quad (4.4)$$

It may be seen that the pinch begins to contract at a rather high current, and there exists danger of contaminating the gas by the interaction between the hot plasma and the chamber walls.

Let us now examine whether the conditions for the maintenance of the longitudinal field in the pinch are compatible with the conditions for Joule heating. Since the electrical conductivity of the plasma is not infinitely high, the longitudinal magnetic field trapped by the pinch gradually "flows" out of it, which spoils the stability conditions. When a considerable part of the magnetic flux has passed outside and the fields have appreciably equalized, the stability will vanish. Equalization of the fields takes place in a period of time of the order of s or even faster with a strong skin-effect (Artsimovich, 1953). This quantity, s , sets the limit for the time for which the

pinch can remain stable and heat-insulated. Heating must therefore be completed in less time

$$t \ll s. \quad (4.5)$$

A strong skin-effect will take place in this case. The condition for heating (3.7) gives, in the absence of heat losses,

$$s = 2t/\beta \quad (4.6)$$

where, according to relation (3.9') $f = f'^2/2\beta$. Joule heat is generated both by the current flowing along the pinch and by the azimuthal currents which create an internal longitudinal field. Since $H_i^2 - H_e^2 = H_J^2(1 - \beta)$, the density of the azimuthal currents, for small β , is almost equal to the density of the axial current. The additional heating due to the azimuthal currents can be approximately calculated by taking $f' = 2 - \beta$; then $s/t = 2f/\beta = (2 - \beta)^2/\beta^2$. The value of β for $H_e < H_i$ is always less than 1. To fulfil the stability conditions it is necessary for β not to exceed a value of the order of $\frac{1}{2} - \frac{1}{3}$ (see Fig. 6). Thus, for $f \sim 2.5 - 4$, the ratio $2f/\beta$ is comparatively high (10–20), and heating time is much less than s so that it is possible to combine the condition for holding the trapped longitudinal field within the pinch with the conditions for heating by Joule heat.

For Joule heat to be imparted to the ions, the number of particles per unit of length in the pinch must be sufficiently large, as has been pointed out above. The optimum value of dimensionless parameter, $\Pi = e^2 N/Mc^2$, is given by $\Pi_{\text{opt}} = 0.5s/t \approx f/\beta$. Assuming, for instance, $\beta = 0.4$ we get $f' = 1.6$, $f = 3.2$, $\Pi_{\text{opt}} = 8$, which corresponds to $N \simeq 10^{17}$ particles/cm for deuterium.

The variation of temperature with time, in a pinch detached from the walls, is defined, with a strong skin-effect, by formula (3.10). With $f = 1.6$ it appears as

$$T_0 = \left(\frac{10^{14}}{N} \frac{J^2 t^2}{a} \right)^{1/2}. \quad (4.7)$$

The choice of heating conditions for an installation of a given size at a given maximum current consists mainly in the choice of an optimum time for the process. If this time is chosen correctly, the maximum temperature permitted by the conditions of equilibrium and stability can be attained. If the chosen time is more than the optimum, the stability will be disturbed before the current reaches its maximum and the temperature will be lower. If the time is less than the optimum, stability will be maintained but the plasma will not be heated to the maximum temperature (β will be small).

These estimates are very approximate. Even within the framework of the accepted ideal scheme (a completely ionized plasma pinch with a distinct boundary) an exact definition of the coefficients entering the formula requires complex computations. Experimental data being available, the installation for obtaining high temperatures can be designed with the aid of the theory of similitude.

Let us assume, for the purpose of estimation, that $a/b = \frac{1}{3}$, $\beta = 0.4$, and accordingly $f = 3.2$; $\Pi = 8$, $N = 10^{17}$, $T_i/T_e = 0.6$. Ignoring the "outflow" of the longitudinal field from the pinch, we get for criterion Z , assuming complete capture and applying the condition of equilibrium: $Z = (a/b)(1 - \beta)^{\frac{1}{2}}$, so that $Z = 0.26$. The outflow factor increases this figure. Taking $\Pi = 8$; $S = 2/f\beta = 16$; $Z = 0.3$, we get in accordance with Eqs. (3.28), (1.2) and (3.7):

$$H_0 = 6 \times 10^{-2} J/b, \quad (4.8)$$

$$T_i = 0.6 T_e = 4.7 \times 10^{-6} J^2, \quad (4.9)$$

$$t = 10^{-13} b^2 J^3. \quad (4.10)$$

The numerical coefficients of these formulae cannot claim to be of particular accuracy as they are sensitive to the choice of permissible values of β . More precise values of optimum Π , S , Z and respective coefficients in Eqs. (4.8), (4.9) and (4.10) must be determined experimentally.

The choice of the optimum dimensions of the system involves a number of practical considerations. An increase of the radius demands an increase in heating time proportional to a^2 , in accordance with Eq. (4.10). Ratio t/a^2 having a fixed value and maximum current J being fixed, quantities such as the total number of reactions, Joule heating and generation of nuclear energy, losses on bremsstrahlung and magnetic radiation, as well as the energy of the particles in the plasma, and the energy of the magnetic field per unit of length of the pinch do not depend on the radius. If, therefore, we take into account the above energy relations alone, the dimensions of the installation may prove to be largely arbitrary.

A number of considerations may be submitted, however, in regards to the choice of the radius. Thus, the radius cannot be too small from the standpoint of the strength of the design, as small dimensions lead to an increase of the magnetic field pressure. For instance, the pressure on the metal tube, $H^2/8\pi = J^2/2\pi c^2 b^2$, at $J = 10^3$ ka and $b = 10$ cm amounts to 15.5 atmos. Another factor requiring an increase in dimensions is the heat load of the discharge chamber walls which creates the danger of contamination of the deuterium by foreign admixtures. The energy, generated in the discharge chamber per unit of length does not depend upon the radius, and the amount of energy per unit of surface in the chamber is proportional to $1/a$. Besides, at $t/a^2 = \text{const.}$, this energy, if the dimensions increase, takes more time to heat the walls and is distributed in a deeper layer, its depth being proportional to $\sqrt{t} \sim a$.

On the other hand, Π being fixed, the initial gas pressure, with an increase of the chamber radius, must decrease ($P_0 = 0.45/b^2$ mm Hg at $N = 10^{17}$), which places stricter demands on the vacuum system.

The need for prolonging the process time may also set a limit for the dimensions of the installations, as very long current impulses are difficult to realize practically.

STABILIZATION BY A STRONG MAGNETIC FIELD

The other method of stabilizing the plasma pinch, noted earlier, is based upon the use of a strong external longitudinal magnetic field. According to condition (2.2), a rather short pinch can be stabilized by an external longitudinal field H_e greatly exceeding the current field, H_J .

Consider briefly the behaviour of a plasma pinch in a strong longitudinal field. A strong longitudinal field frozen into the plasma prevents the constriction and expansion of the pinch, as appears from (3.16). The change of the pinch radius in time may then substantially depend upon the initial period of the discharge development and the variation of the external field.

Let the external longitudinal field be constant, for instance. Then, if a pinch is formed as a result of the initial development, occupying the entire cross section of the discharge chamber, the radius will remain more or less constant, for the current field will be unable to compress strongly the great longitudinal field "frozen into" the plasma.

It follows from (3.16) that in a condition of constriction the pinch is $\dot{J}/J > \bar{f}s$ where $\bar{f}$ is a coefficient of the order of 1. If this condition is fulfilled, the pinch radius diminishes with time; but the rate of constriction will be very low: $-a \sim (a/s)(H_J^2/H^2)$. The border of the pinch will, throughout the process, be very close to the chamber wall. For the present it is not yet clear whether such a "precision" plasma pinch can be created with sufficiently reliable heat insulation.

Let us therefore consider a "rougher" variant of a plasma pinch with a radius, say, half as large as that of the discharge chamber. For compression of the pinch the process of increasing the external field may itself be used. With a large longitudinal field $H = H_1 \approx H_e \gg H_J$ and ideal conductivity of the plasma, the product Ha^2 does not change with time (see [3.16]). If, for instance, ionization and good conductivity are achieved by the time $H = H_{\max}/4$, then with the field reaching a maximum the radius is halved. By increasing the pinch current and the external field simultaneously but with some properly chosen phase shift, the required constriction of the pinch can be realized.

The higher the plasma temperature, the bigger must be the current and the greater, according to (2.2), the stabilizing longitudinal field. The maximum value of this field intensity is limited, however, by the strength of the solenoid coil. With stabilization by a strong field, the conditions of strength imposed on the design condition impose far more rigid limitations than in case of stabilization by a weak field. In practice it seems difficult apparently to reach a field greater than 5×10^4 gauss, which corresponds to a pressure of $H^2/8\pi = 100$ atmos. Here the condition of stability (2.2) sets a limit for the value of the current field:

$$H_J \leq Ha/R = H/h. \quad (5.1)$$

It is thus rational to have a minimum ratio, R/a . In practice, however, the curvature of the torus is restricted by the need to find place for the winding to create the field and current. Since, in addition, we agree that the radius of the pinch must be equal to half of the radius of the chamber, in computing the numerical coefficients below we shall for clarity take $h = 5$.

In practice, the value of the longitudinal field H is limited by the mechanical strength requirement and time t , during which the longitudinal field and the current can be maintained, by the capacity of the capacitor bank. The energy required to create a longitudinal field is much greater than the energy of the current field (at $H/H_J = 5$, $b/a = 2$, it is about 50 times). As a result, the determining value is H rather than J . The specific number of particles is determined by the condition of ion heating.

Considering values H , t and N as given, the maximum possible temperature can be found by combining the heating condition (3.8) with the stability condition (5.1). This produces:

$$T = 1.88(h/5)^{-4/5} (10^{15} f H^2 t / N)^{2/5} \text{ kev} \quad (5.2)$$

where H is in kilogauss.

Here the current must be chosen on the basis of the condition $H_J = H/h$ or at, $h = 5$,

$$J = \frac{1}{5} c H a / h \text{ ka}, \quad (5.3)$$

where H is in kilogauss and a in cm.

Taking $f = 1$ and $h = 5$, if we let		
$H = 10$	32	50 kgauss,
$t = 10^{-3}$	10^{-2}	1 sec,
and $N = 5$	5	10×10^{16} ,
then $T = 0.15$	1	6.7 kev.

The time for heating the plasma to the given temperature, according to (5.2), does not depend on the pinch radius, since in (5.2) it is assumed that with a change in the radius the current changes correspondingly in accordance with (5.3). The energy of the magnetic field grows with the growth of the radius, because of the increase in the volume occupied by the field. It is therefore rational to decrease the dimensions of the system. The coefficient of utilization of the current, β , characterizing the ratio of the energy of the plasma to the energy of the field, then increases:

$$\begin{aligned} \beta &= 4c^2 NT / J^2 \\ &= 6.4 \times 10^{-13} (NT / H^2 a^2) (h/5)^2. \end{aligned} \quad (5.4)$$

But the value of β , from the conditions of toroidal equilibrium, cannot be very large ($\beta \lesssim R/b$) as with an increase of "diamagnetism" the toroidal stretching force grows. Hence the radius cannot be very small; there are certain optimum dimensions of the installation defined by the relationships

$$a = 1.1 \times 10^{-3} (f/\beta)^{1/2} (Hh/5)^{1/2} (tN^{2/3})^{1/2}. \quad (5.5)$$

For the foregoing examples, at $\beta = 1$, if

$a = 7$	5.6	13 cm,
$R = 35$	28	65 cm,
then $J = 70$	180	650 ka.

In a system with stabilization by a strong field, a supplementary mechanism for heating the plasma is highly desirable. In the case of supplementary heating, formula (5.2) is inapplicable as, during the same time, the plasma can be heated to a higher temperature. In this case, the dimensions of the system, according to the confinement condition (5.4), must be larger than follows from formula (5.5).

Let us compare the systems in which stabilization by a strong field and by a weak field are employed. Stabilization by a strong field is energetically less advantageous than stabilization by a weak field. Let us consider the ratio of the energy of the plasma to the energy of the magnetic field in these systems:

$$K = 3NT / W_H = \frac{3}{2} \pi p a^2 / W_H. \quad (5.6)$$

Disregarding the magnetic energy in the space outside the conducting tube we get

$$K = \frac{\frac{3}{2} \beta}{(H_e/H_J)^2 (b^2/a^2) + 2 \ln(b/a) + \frac{3}{2} - \beta}. \quad (5.7)$$

Internal self-inductance (per centimeter) is here taken to be $\frac{1}{2}$. For a system with a weak field, at $\beta = \frac{1}{3}$, $b/a = 3$ and $H_e = 0$ we get $K = 0.15$. For an installation with a strong field at $H_e/H_J = 5$, we get $K = 0.015$, i.e., 10 times less. A system with a strong current thus becomes energetically more advantageous with an increase of β and a decrease of b/a . Thus, if it is possible to achieve $b/a = 1$ ("precision" pinch), $\beta = 2.5$ and $H_e/H_J = 2.5$, we get $K = 0.7$. These evaluations are in large measure arbitrary, but they show that, in principle, the energy difference between these two types of installations may not be very great.

In a system with a strong field, the external longitudinal field can be used together with the current to oppose the plasma pressure; while, with stabilization by a weak field, the pressure of the longitudinal field itself must be compensated. Therefore, with equal temperatures of the plasma the current may be several times weaker in the system with a strong field than in a system with a weak field. In this sense the confinement of the plasma here is much simpler. The time for heating the plasma increases, of course. However, with the aid of supplementary heating this defect can be eliminated.

The substantial difference between the two types of systems lies also in the fact that with stabilization by a weak field the very mechanism for maintaining stability limits the life of the pinch, because of the outflowing of the trapped field; no such limitation exists in case of stabilization by a strong field. In the latter case the life of the pinch is limited only by the time the current and the field are maintained. A plasma pinch stabilized by a strong field approximates to a stationary system.

It is still too early to draw final conclusions as to the advantages of one or the other of these systems. Experimental investigations must be carried out in both directions.

ACKNOWLEDGEMENT

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Theory of Runaway Electrons

By H. Dreicer *

This paper treats the problem of electrons moving through an infinite gas of positive ions under the influence of a static uniform electric field of arbitrary strength. In evaluating the electrical conductivity of such a gas the conventional treatment involves a perturbation solution of the time-independent Boltzmann equation, and results in the well-known (temperature)^{3/2} law.¹ Two assumptions are basic to these treatments: (1) that a steady state electron velocity distribution is attained several mean-free collision times after the electric field is applied, and (2) that the terminal electron drift velocity is small compared to the average random electron speed. Both assumptions are avoided in this paper. In the next section the problem is formulated starting with the Boltzmann equation and a review of approximate analytic solutions appropriate to the weak and strong electric field cases is presented. We then describe a time-dependent numerical solution to the Boltzmann equation and compare these results with the approximate solutions.

All of these treatments lead to the conclusion that this problem does not admit a time-independent solution. Because of the strong energy dependence of the Rutherford scattering law, the electron drift velocity is not bounded by a terminal value, rather it grows monotonically with time. This is the so-called runaway effect predicted by Giovanelli.²

Collective effects, or plasma oscillations, are ignored in this work, although these undoubtedly play an important role in the conduction of electricity through a plasma.

REVIEW OF APPROXIMATE ANALYTIC TREATMENTS

Following standard procedure, the statistical behavior of the electrons is described by a velocity distribution function F which satisfies the Boltzmann equation

$$\partial F / \partial t + (eE/m) \partial F / \partial c_z = \partial F / \partial t_{\text{coll}}. \quad (1)$$

The electric field is applied along the negative z axis of a stationary cartesian coordinate system. The

distribution function F therefore possesses cylindrical symmetry and we may consider it to be a function of the radial velocity variable c and c_z . We use the Fokker-Planck equation to describe the effect of Coulomb encounters on F . However, in the interest of clarity we defer the details of this collision term to the next section.

Strong Field Regime

In the limit of strong fields we may neglect the interaction between particles compared to their interaction with the applied field. The solution to Eq. (1) which satisfies the initial condition

$$F(c, c_z, 0) = (m/2\pi kT_0)^{3/2} \exp - [(m/2kT_0)(c^2 + c_z^2)] \quad (2)$$

is then simply the displaced Maxwellian distribution

$$F(c, c_z, t) = (m/2\pi kT_0)^{3/2} \exp - \{(m/2kT_0)(c^2 + [c_z - v(t)]^2)\}, \quad (3)$$

where v , the electron drift velocity, is determined by the equation of motion

$$dv/dt = eE/m. \quad (4)$$

If we assume that this solution has approximately the correct form for all values of applied field, then we may use it to satisfy the Boltzmann equation on

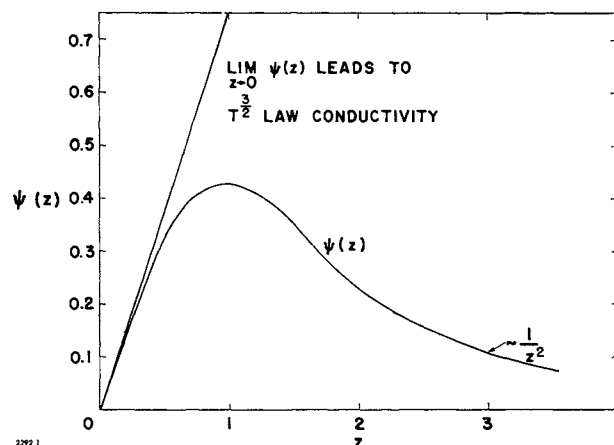


Figure 1. The velocity dependence, $\Psi(Z)$, of the dynamical friction force as a function of Z

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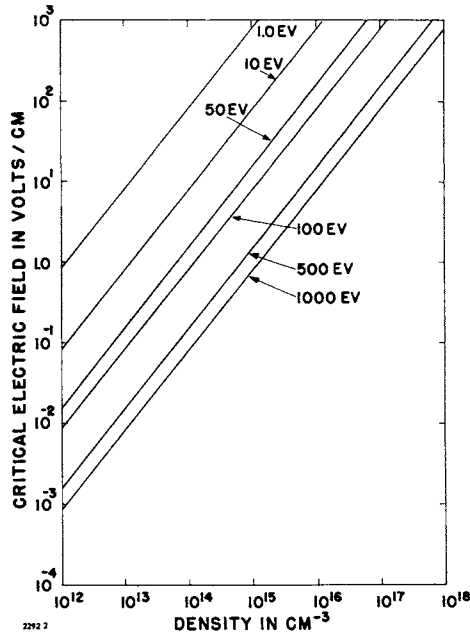


Figure 2. The critical electric field as a function of electron density with average electron energy in volts as parameter

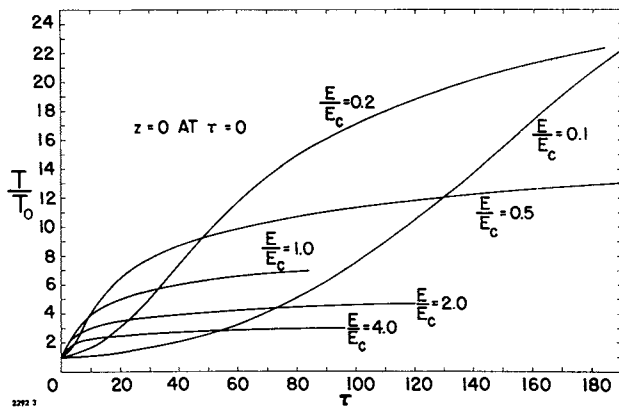


Figure 3. Normalized temperature, T/T_0 , as a function of τ with E/E_c as a parameter

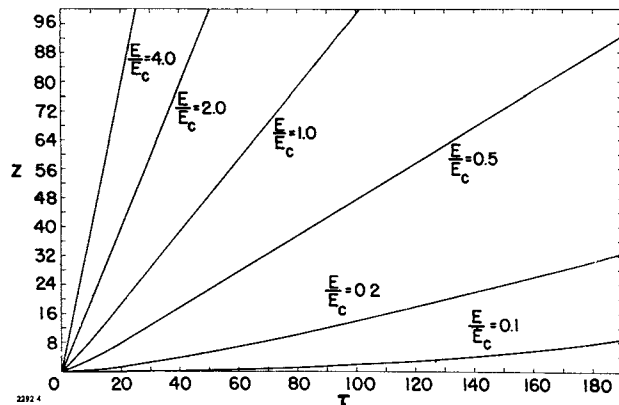


Figure 4. Normalized drift velocity, Z , as a function of τ with E/E_c as parameter

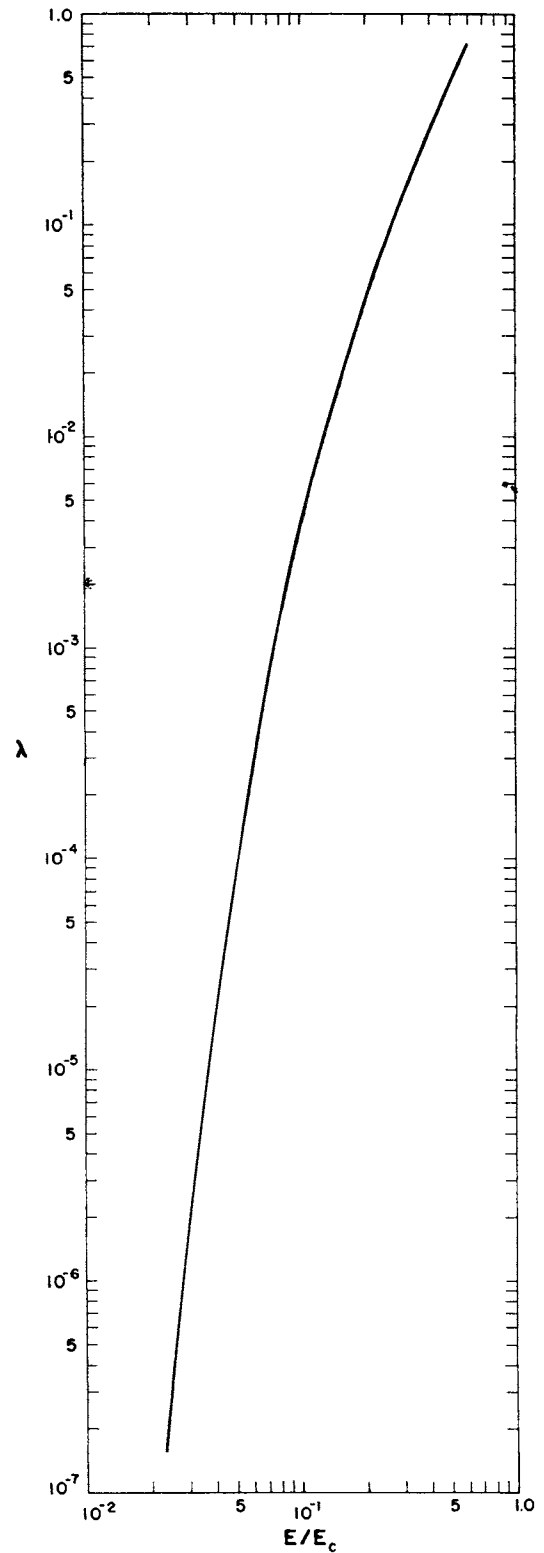


Figure 5. Runaway rate λ as a function of E/E_c

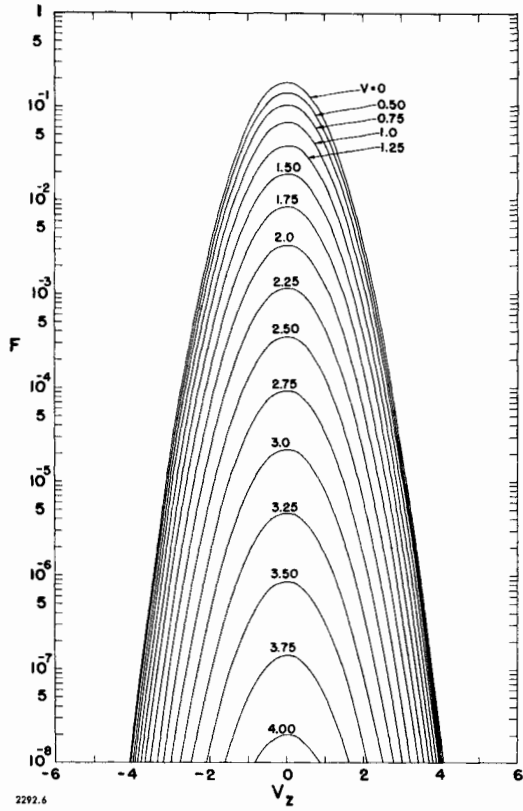


Figure 6. The Maxwellian velocity distribution shown at the initial instant of time with V as a parameter

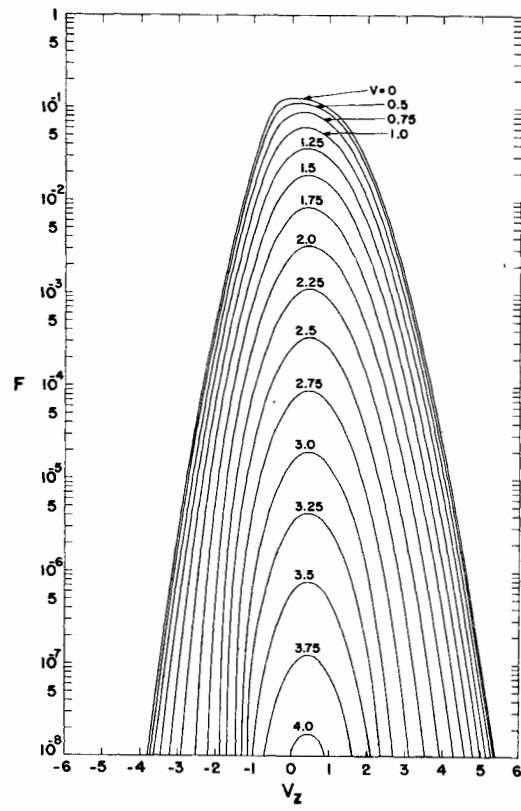


Figure 7. The velocity distribution shown at $\tau = 1$ with V as a parameter

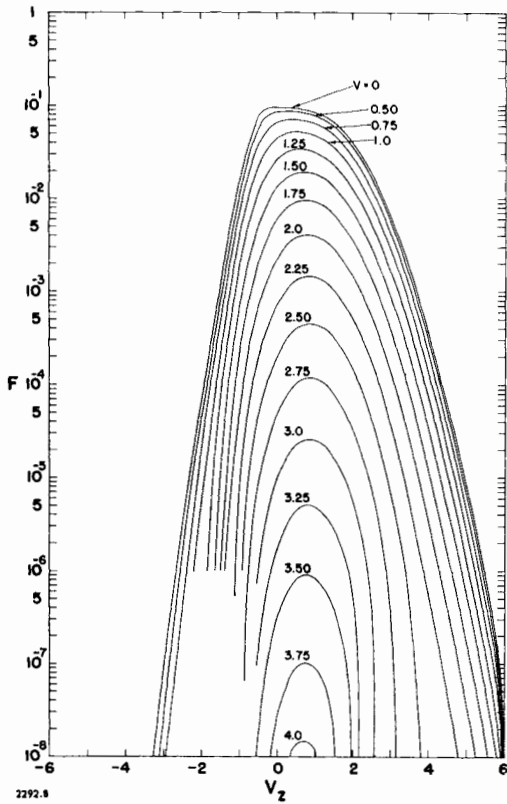


Figure 8. The velocity distribution shown at $\tau = 2$ with V as a parameter

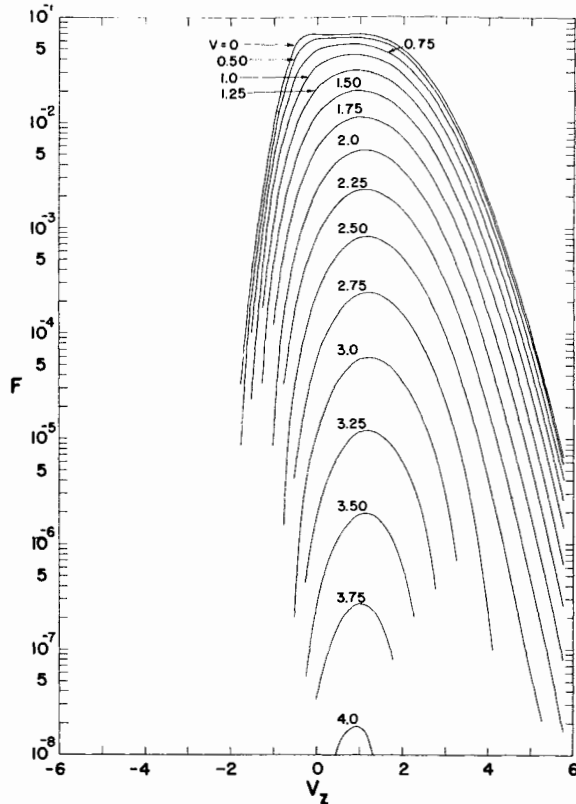


Figure 9. The velocity distribution shown at $\tau = 3$ with V as a parameter

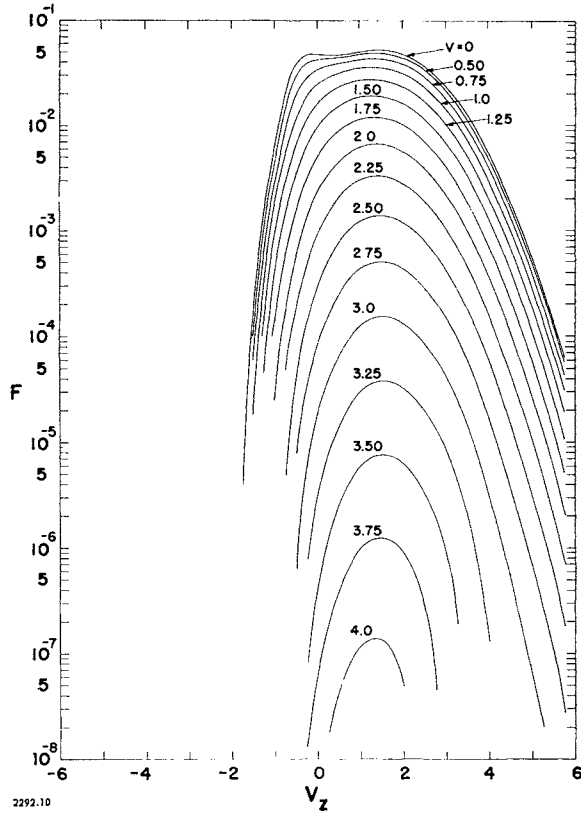


Figure 10. The velocity distribution shown at $\tau = 4$ with V as a parameter

the average. The drift velocity in Eq. (3) is then the solution of an equation of motion which includes the effects of collisions and has the form [†]

$$dv/dt + eE_c \Psi(Z)/m = eE/m, \quad (5)$$

where

$$eE_c/m = 4\pi n (Z_1^2 e^4/m^2) (m/2kT_0) \ln \Lambda$$

n = electron particle density,

$Z_1 e$ = ionic charge,

$Z = (m/2kT_0)^{1/2} v$

$$\Psi(Z) = E_2(Z) - Z \frac{dE_2}{dZ} / Z^2$$

$$E_2(Z) = \frac{2}{\sqrt{\pi}} \int_0^Z e^{-t^2} dt. \quad (6)$$

Z is a dimensionless measure of the electron velocity v . The additional term in Eq. (5) arises from the dynamical friction force exerted by the ions (assumed to be stationary) upon the electrons. As illustrated in Fig. 1, this friction force follows a Stokes law for small Z , whereas for $Z \gg 1$ the strong velocity dependence of the Rutherford scattering law leads to a Z^{-2} decay. The friction force is a maximum at $Z = 1$. Examination of Eq. (5) shows that it admits no static solution when E exceeds $E_c \Psi(1)$. The parameter E_c therefore plays the role of a critical field in the theory

[†] Details of the strong and weak field approximation will be submitted for publication elsewhere.

Table 1

τ	T/T_0	Z
0	1.00	0
1	1.04	0.36
2	1.22	0.59
3	1.48	0.82
3.75	1.71	1.00
4	1.79	1.07
5	2.14	1.35
6	2.51	1.67
7	2.89	2.01
8	3.27	2.38
9	3.64	2.77
10	4.00	3.18

and furnishes a useful criterion for separating the weak and strong field cases. Figure 2 gives values of E_c for a range of temperatures and densities.

Since the positive ions are assumed to be stationary Eq. (5) also yields the rate of Joule heating

$$d(\frac{3}{2}kT)/dt = eE_c (2kT_0/m)^{1/2} Z \Psi(Z). \quad (7)$$

The appropriate dimensionless variables for this problem are

$$E/E_c, T/T_0, Z, \text{ and } \tau = (eE_c/m) (m/2kT_0)^{1/2} t, \quad (8)$$

with E_c and Z defined in terms of the initial electron temperature T_0 , and τ measured in terms of the mean-free time for collisions between electrons moving with the thermal speed $(2kT_0/m)^{1/2}$. Equations (5)

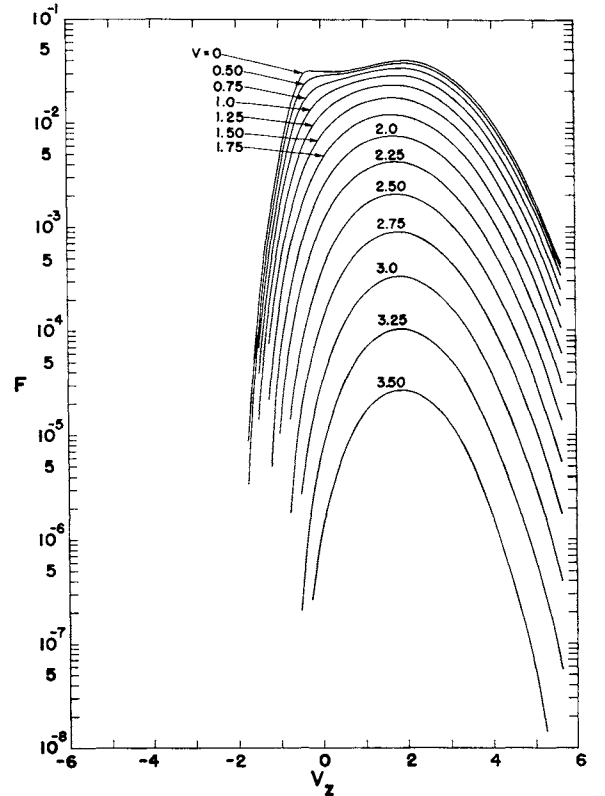


Figure 11. The velocity distribution shown at $\tau = 5$ with V as a parameter

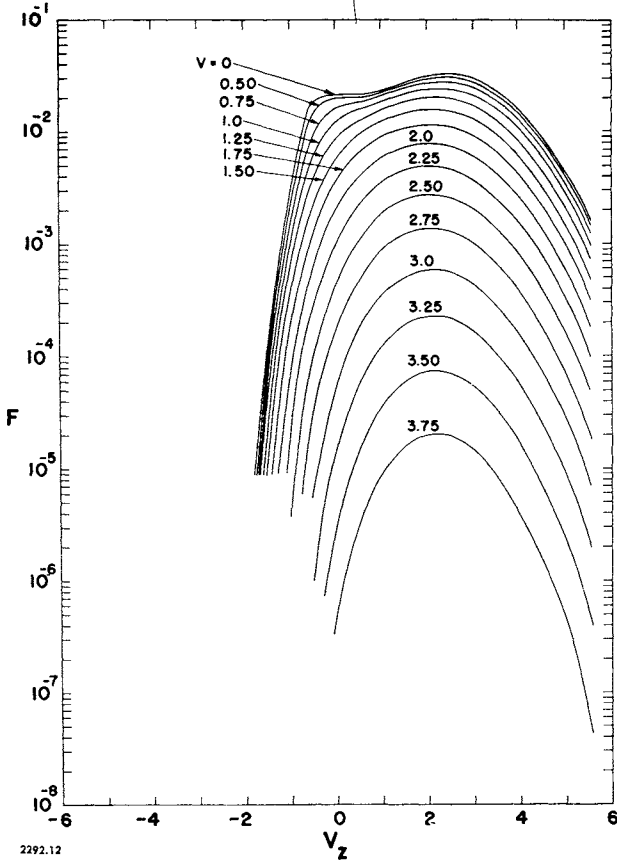


Figure 12. The velocity distribution shown at $\tau = 6$ with V as a parameter

and (7) have been solved simultaneously, by numerical means, for a variety of applied fields. From the results shown in Figs. 3 and 4 we may conclude that runaway occurs even when $E < E_c \Psi(1)$. This is simply a statement of the fact that Joule heating transforms any weak field into a strong field as time proceeds. The variation of T/T_0 and Z as a function of time is tabulated in Table 1 for $E/E_c = \frac{1}{2}$.

Weak Field Regime

In the weak field limit, $E \ll E_c$, Eq. (5) leads to the well-known $T^{3/2}$ conductivity law. This result is not strictly valid, quite apart from the Joule heating effects just discussed. By making use of the displaced Maxwellian distribution we ignore the fact that certain fast electrons in the high-energy "tail" of the distribution make collisions so infrequently that for these almost any applied field may be considered to be strong. When this effect is examined, we find that velocity space can roughly be divided into a runaway region where the applied field plays the role of a strong field and into a non-runaway region where the same field is weak. The time scale for appreciable depletion of the original distribution is therefore determined by the diffusion of electrons into the runaway region. We have employed the time-dependent Boltzmann equation, supplemented by Fokker-Planck collision terms, to calculate the

probability $Q(\tau)$ that all electrons have crossed into the runaway region in the time τ . To a good approximation $Q(\tau)$ is given by

$$Q(\tau) = 1 - e^{-\lambda\tau}. \quad (9)$$

The variation of λ with E/E_c is shown in Fig. 5. Again τ and E_c are defined in terms of the initial temperature of the electrons.

We conclude, therefore, that runaway in the weak field limit proceeds under the combined action of Joule heating and diffusion into the high-energy tail of the distribution.

NUMERICAL SOLUTION

A more detailed analysis of this problem was carried out with the aid of an IBM 704 digital computer. A straightforward difference equation version of the Boltzmann equation was solved subject to the initial condition given in Eq. (2). In terms of dimensionless variables, this equation has the form

$$\partial F(V, V_z, \tau)/\partial \tau + (E/E_c) \partial F/\partial V_z = (\partial F/\partial \tau)_{\text{coll}}, \quad (10)$$

where the velocity components are defined by

$$V = (m/2kT_0)^{1/2} c$$

$$V_z = (m/2kT_0)^{1/2} c_z.$$

In cylindrical coordinates the Fokker-Planck collision term describing electron encounters with electrons and stationary ions has the form

$$\begin{aligned} \left(\frac{\partial F}{\partial \tau} \right)_{\text{coll}} = & 4\pi(F)^2 - \frac{V}{(V^2 + V_z^2)^{3/2}} \frac{\partial F}{\partial V} \\ & - \frac{V_z}{(V^2 + V_z^2)^{3/2}} \frac{\partial F}{\partial V_z} \\ & + \frac{1}{2} \left(\frac{\partial^2 G}{\partial V^2} + \frac{V_z^2}{(V^2 + V_z^2)^{3/2}} \right) \frac{\partial^2 F}{\partial V^2} \\ & + \frac{1}{2} \left(\frac{\partial^2 G}{\partial V_z^2} + \frac{V^2}{(V^2 + V_z^2)^{3/2}} \right) \frac{\partial^2 F}{\partial V_z^2} \\ & + \frac{1}{2V^2} \left(\frac{\partial G}{\partial V} + \frac{V}{(V^2 + V_z^2)^{3/2}} \right) \frac{\partial F}{\partial V} \\ & + \left(\frac{\partial^2 G}{\partial V \partial V_z} - \frac{V V_z}{(V^2 + V_z^2)^{3/2}} \right) \frac{\partial^2 F}{\partial V \partial V_z}. \quad (11) \end{aligned}$$

The function $G(V, V_z)$ describes Coulomb encounters between electrons and is defined by

$$\begin{aligned} G(V, V_z) = & \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} F(V', V'_z, \tau) \\ & \times (V'^2 + V^2 - 2VV' \cos \phi \\ & + (V_z - V'_z)^2)^{3/2} V' dV' dV'_z d\phi. \quad (12) \end{aligned}$$

Its second derivatives (i.e., $\partial^2 G/\partial V^2$, $\partial^2 G/\partial V_z^2$, $\partial^2 G/\partial V \partial V_z$) may be related to the rate at which fluctuations about the dynamical friction force bring about a diffusion of particles in velocity space. Rosenbluth, MacDonald and Judd,³ have shown that G may be related to F through an auxiliary $H(V, V_z)$ function as follows

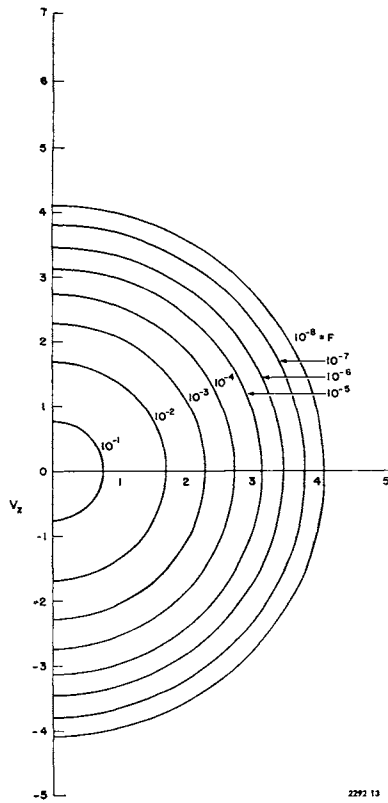


Figure 13. Curves of constant F in velocity space shown at $\tau = 0$. These curves represent the Maxwellian distribution

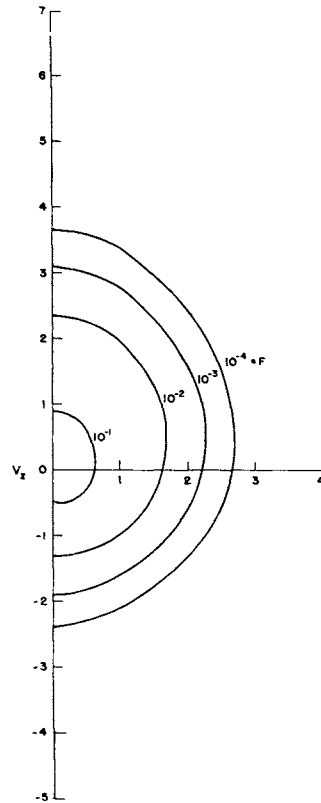


Figure 14. Curves of constant F in velocity space shown at $\tau = 1$

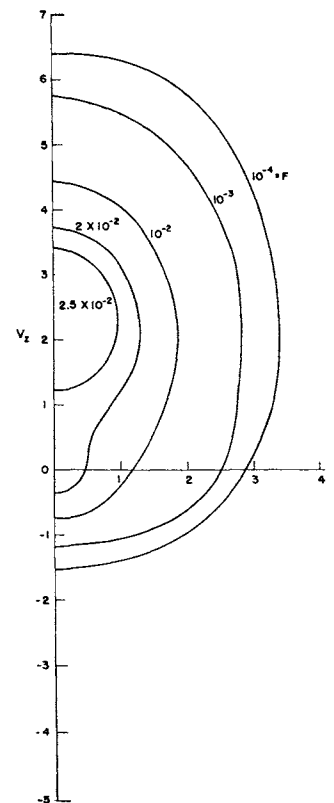


Figure 15. Curves of constant F in velocity space shown at $\tau = 2$

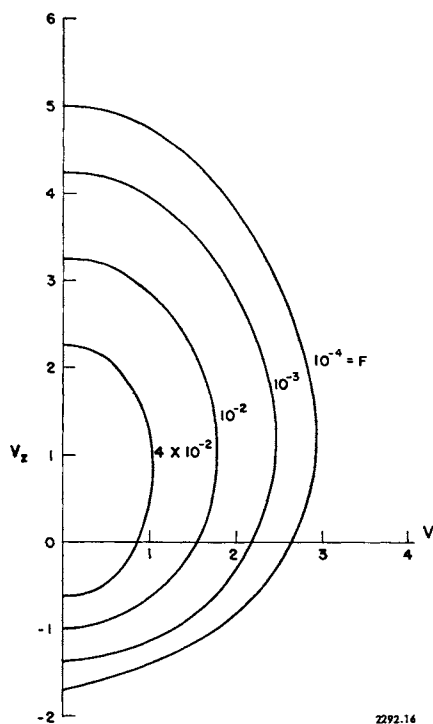


Figure 16. Curves of constant F in velocity space shown at $\tau = 3$

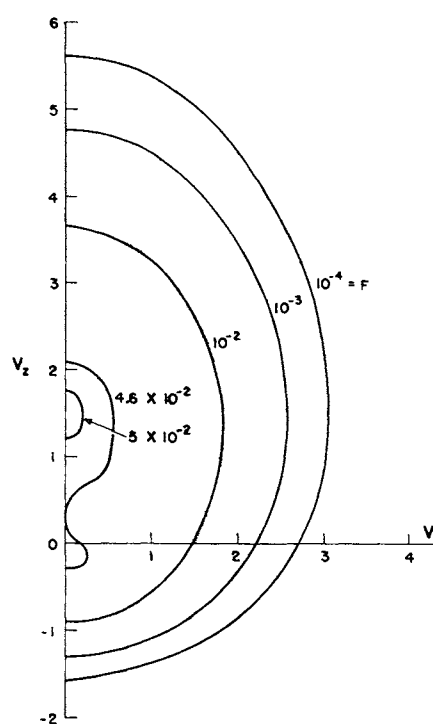


Figure 17. Curves of constant F in velocity space shown at $\tau = 4$

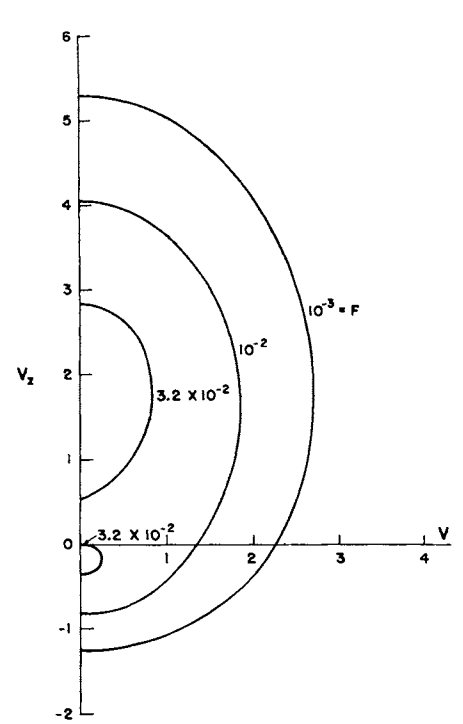
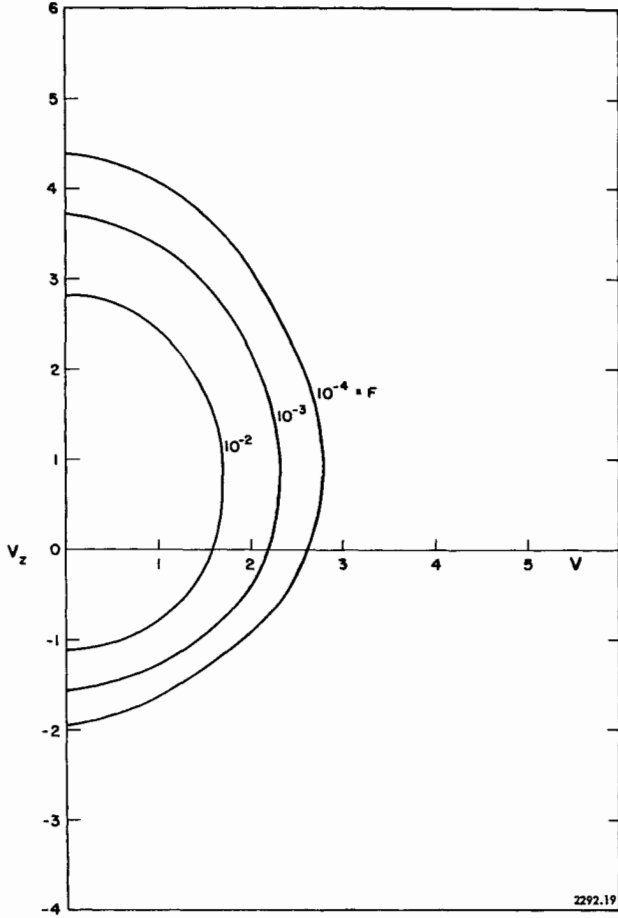


Figure 18. Curves of constant F in velocity space shown at $\tau = 5$

Figure 19. Curves of constant F in velocity space shown at $\tau = 6$

$$\nabla_v^2 G = H(V, V_z) \quad (13a)$$

$$\nabla_v^2 H = -8\pi F(V, V_z, \tau). \quad (13b)$$

The definition of $H(V, V_z)$ is

$$H(V, V_z) = 2 \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{F(V', V_z', \tau) V' dV' dV_z' d\phi}{(V'^2 + V^2 - 2VV' \cos \phi + (V_z - V_z')^2)^{\frac{3}{2}}} \quad (14)$$

and its gradient in velocity space gives the dynamical friction force.

In our numerical scheme G is obtained from Eqs. (13a) and (13b) by a relaxation method. The neces-

sary boundary values of H and G are obtained from the asymptotic limits of Eqs. (12) and (14) in the form

$$G(V, V_z) \sim [V^2 + (V_z - \langle V_z \rangle)^2]^{\frac{1}{2}} \begin{cases} V \rightarrow \infty \text{ or} \\ V_z \rightarrow \pm \infty \end{cases} \quad (15)$$

$$H(V, V_z) \sim 2[V^2 + (V_z - \langle V_z \rangle)^2]^{-\frac{1}{2}} \begin{cases} V \rightarrow \infty \text{ or} \\ V_z \rightarrow \pm \infty \end{cases} \quad (16)$$

where

$$\langle V_z \rangle = \frac{\int_{-\infty}^{\infty} \int_0^{\infty} F V_z V dV dV_z}{\int_{-\infty}^{\infty} \int_0^{\infty} F V dV dV_z}.$$

The entire calculation is carried out over the rectangular region in velocity space bounded by the lines

$$V = 0, \quad V = 6, \quad V_z = \pm 6$$

and the mesh points in this region are spaced apart by the intervals

$$\Delta V = \frac{1}{4}; \quad \Delta V_z = \frac{1}{4}.$$

A single-cycle time-step of size $\Delta\tau = 10^{-3}$ was used and G was recalculated after every 10 cycles.

The following physical quantities were recalculated periodically

$$N = \int_{-\infty}^{\infty} \int_0^{\infty} F 2\pi V dV dV_z \quad (17)$$

$$\begin{aligned} \frac{1}{2} \langle V^2 \rangle &= m \langle c^2 \rangle / 4kT_0 \\ &= N^{-1} \int_{-\infty}^{\infty} \int_0^{\infty} F \pi V^3 dV dV_z \end{aligned} \quad (18)$$

$$\frac{1}{2} \langle V_z^2 \rangle = m \langle c_z^2 \rangle / 4kT_0, \quad (19)$$

$$\langle V_z \rangle = m \langle c_z \rangle / 2kT_0. \quad (20)$$

$$\int_0^{\tau} (E/E_c) \langle V_z \rangle d\tau = (eE/2kT_0) \int_0^{\tau} \langle c_z \rangle dt. \quad (21)$$

The normalization N provides a check on the conservation of particles, and the energy integrals in Eqs. (18), (19) and (21) can be related to check the conservation of energy in the following way

$$\begin{aligned} \frac{1}{2} [\langle V_z^2 \rangle + \langle V^2 \rangle]_{\tau=\tau_1} - \frac{1}{2} [\langle V_z^2 \rangle + \langle V^2 \rangle]_{\tau=0} \\ = \int_0^{\tau_1} (E/E_c) \langle V_z \rangle d\tau. \end{aligned} \quad (22)$$

Random average energies are defined by

$$\begin{aligned} \frac{1}{2} m \langle W_z^2 \rangle &= \frac{1}{2} m [\langle c_z^2 \rangle - \langle c_z \rangle^2] \\ &= [\langle V_z^2 \rangle - \langle V_z \rangle^2] kT_0 \end{aligned} \quad (23)$$

Table 2

τ	$\langle V_z \rangle$	N	$\frac{m}{2} \langle W_z^2 \rangle$	$\frac{m}{2} \langle c^2 \rangle$	$\frac{\frac{1}{2} [\langle V_z^2 \rangle + \langle V^2 \rangle]_{\tau}^T - \int_0^T E/E_c \langle V_z \rangle d\tau}{\frac{1}{2} [\langle V_z^2 \rangle + \langle V^2 \rangle]_0^T}$
0	0	1.00	0.50 kT_0	1.00 kT_0	0
1	0.43	1.096	0.61	1.02	-0.088
2	0.77	1.196	0.76	1.13	-0.150
3	1.07	1.296	0.97	1.32	-0.100
4	1.39	1.296	1.22	1.56	-0.071
5	1.72	1.296	1.49	1.84	-0.052
6	2.06	1.296	1.76	2.13	-0.034

$$\frac{1}{2}m\langle v^2 \rangle = \langle V^2 \rangle kT_0. \quad (24)$$

In Table 2 we have recorded (17), (20), (23), (24) and the degree to which (22) is satisfied over a duration covering 6 mean-free collision times. The applied field in this calculation had the value $E/E_c = \frac{1}{2}$. Figures 6 through 19 illustrate the evolution of $F(V, V_z, \tau)$ during the same interval of time.

If we choose as our runaway criterion the equality of the drift velocity and the initial thermal speed; i.e., $\langle V_z \rangle = 1$, then comparison of Tables 1, 2 and Fig. 5 yields reasonably good agreement among the various runaway times, τ_f .

	<i>Numerical treatment</i>	<i>Strong field approximation</i>	<i>Weak field approximation</i>
τ_f	3.0	3.75	2.0

The normalization tabulated in Table 2 indicates that the number of particles in the rectangular mesh at first increases with time and then reaches a steady value. This is associated with the error in the assumed boundary values of F . The algebraic sign of the error in the energy balance indicates that fictitious particles are probably entering the mesh

across the negative V_z boundary. With the onset of the runaway effect, particles also begin to leave the mesh across the positive V_z boundary, and a balance between these currents produces the steady value for the normalization calculated after $\tau = 3$. In spite of these inaccuracies, we believe that the absence of a time-independent solution has also been illustrated in this numerical calculation.

ACKNOWLEDGEMENTS

It is a pleasure for the author to express his gratitude to Mr. A. Feldstein, who carried through the numerical solution of Eqs. (5) and (7). The author is also greatly indebted to Mr. E. Kinney, who coded Eqs. (10) and (11) for the computer.

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Individual Particle Motion and the Effect of Scattering in an Axially Symmetric Magnetic Field

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T. G. Northrop† and J. E. Roberts†

The possibility of confining charged particles with magnetic "mirrors" has long been recognized. A mirror field has axial symmetry and a magnitude that increases along the axis away from a central region in which the particles are to be contained. Heretofore, the likelihood of confinement has been based on the approximate invariance of the magnetic moment as described by Alfvén.¹ If the magnetic moment of a particle with given energy is too small the particle escapes axially through the mirror. The moment can become small because it is not a rigorous constant of the motion or because of Coulomb scattering of the particle. Both these effects have been studied; the first by analytic and numerical methods and the second by numerical solution of the Fokker-Planck equation.

PARTICLE CONTAINMENT IN THE ABSENCE OF COLLISIONS

Under what conditions would a particle that makes no collisions remain indefinitely in a mirror field? Before trying to answer this question we give the vector potential used in our numerical orbit calculations,

$$A_\theta(r, z) = (LB_0/2\pi)[\varrho/2 - \alpha I_1(\varrho) \cos \zeta] \\ \equiv B_0\psi(\varrho, \zeta)/\varrho, \quad A_r = A_z = 0,$$

where α is a constant, L = distance between mirrors, $\varrho = 2\pi r/L$, $\zeta = 2\pi z/L$, and B_0 is the magnetic field at $\varrho = 0$, $\zeta = \pi/2$. This is a static, axial-symmetric, curl-free field with mirrors at $\zeta = -\pi$ and π . Further, for the two known rigorous constants of the motion we use the dimensionless parameters $V = 2\pi v/\omega_0 L$ and $P_\theta = 4\pi^2 \dot{\varphi}_\theta/m\omega_0 L^2$, where v is the speed, $\dot{\varphi}_\theta$ the canonical angular momentum in conventional units and $\omega_0 = eB_0/mc$.

One notes that the motion in the (ϱ, ζ) plane is determined by a Hamiltonian from which θ has been eliminated:

$$H = \frac{1}{2}(P_\rho^2 + P_\zeta^2) + \frac{1}{2}[(\psi - P_\theta)/\varrho]^2 = \frac{1}{2}V^2 \\ = \text{constant},$$

where $P_\rho = (1/\omega_0)d\varrho/dt$ and $P_\zeta = (1/\omega_0)d\zeta/dt$. Certain classes of particles remain indefinitely in the mirror field given by A_θ above. All orbits for which P_θ is sufficiently negative (these encircle the axis) are rigorously contained. In that case the effective potential for the (ϱ, ζ) motion, $\frac{1}{2}[(\psi - P_\theta)/\varrho]^2$, is such that the equipotentials less than or equal to the total particle energy are closed curves, with the potential less on the inside. For particles not encircling the axis, the potential surface is an open-ended trough with a flat bottom; therefore, for most such particles we cannot make as strong a statement about containment as we can for those encircling the axis. However, for a denumerably infinite set of orbits not encircling the axis, one can demonstrate that there is rigorous confinement by invoking the symmetry properties of the field and the equations of motion.

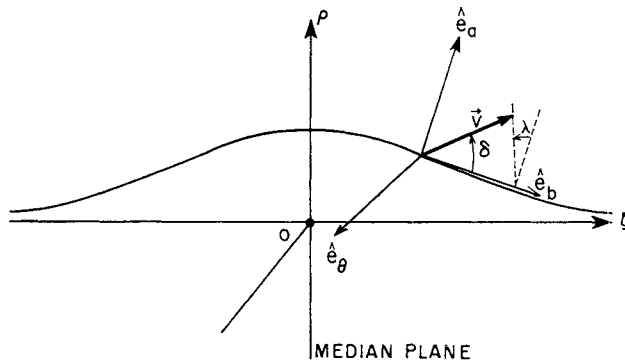
Since P_θ and V are constants of the motion and since the field has axial symmetry, one needs only three independent variables to describe the motion. For the present purpose, we use the set (ζ, P_ζ, P_ρ) to describe a point P in phase space. Remembering that ψ and hence H is symmetric in ζ , one can see that the canonical equations of motion in the (ϱ, ζ) space are invariant with respect to the transformations T and R

$$T: \zeta \rightarrow \zeta, \varrho \rightarrow \varrho, P_\zeta \rightarrow -P_\zeta, P_\rho \rightarrow -P_\rho; \quad t \rightarrow -t \\ R: \zeta \rightarrow -\zeta, \varrho \rightarrow \varrho, P_\zeta \rightarrow -P_\zeta, P_\rho \rightarrow P_\rho; \quad t \rightarrow t.$$

These transformations, when applied to all the phase points P of a given trajectory Θ , generate, respectively, two other real trajectories, $T\Theta$, the time-reversed trajectory, and $R\Theta$, the one reflected in the median ($\zeta = 0$) plane. One can show that there is a countably infinite number of trajectories for which $\Theta = T\Theta = R\Theta$, and that these trajectories have the property

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$$\begin{aligned} \text{FOR } \zeta = 0: \quad \dot{\zeta} &= P_{\zeta} = V \cos \delta \\ \dot{\rho} &= P_{\rho} = V \cos \lambda \sin \delta \\ \rho \dot{\theta} &= \psi - P_{\theta} = V \sin \lambda \sin \delta \\ &\quad \rho \end{aligned}$$

Figure 1. Definition of angles

of being periodic, and consequently of being permanently contained. Any such periodic trajectory may be recognized by the fact that it contains two special points: $P_0 = (\zeta, 0, 0)$ and $P_1 = (0, P_{\zeta}, 0)$.

Unfortunately, the trajectories that satisfy none of the above criteria for rigorous containment are of great practical importance. Belief in their containment has usually been based on the alleged approximate constancy of the magnetic moment, or "adiabatic invariant", $J = |\mathbf{v} \times \mathbf{B}|^2 / B^3$. Our first numerical computations examined the behavior of J for single transits between reflections. We found that for orbit sizes of interest, its variations are disturbingly large. Characteristically J oscillates about a fixed value with the particle rotation frequency until the particle crosses the median plane, near which J suffers a large transient change. Subsequently, J oscillates about a new mean value; ΔJ , the residual change in J between successive reflections, depends on the various parameters that define the orbit. Some of these dependences are indicated by Figs. 1-3. Perhaps the dependence on velocity (Fig. 3) is most interesting; one can well approximate $\Delta J/J$ by a function of the form $a \exp - (b/V)$.[†]

We have carried out analyses which predict qualitatively the observed dependences of $\Delta J/J$. To lowest order in the radius of gyration, a , we obtain by a variation-of-parameters approach (see Fig. 1 for definition of the unit vectors)

$$dJ/dt \cong -2av_{\parallel}^2 (\mathbf{e}_b \cdot \nabla \mathbf{e}_a) \cdot \mathbf{e}_b \cos \phi,$$

[†] The wavy curve in Fig. 3 was obtained when a higher harmonic was added to the vector potential.

Even though J does not seem to be a very good invariant, perhaps another function is. The approach adopted by Helwig,² Kruskal,³ and Gardner and Berkowitz⁴ is to develop successive corrections to the magnetic moment in terms of the variation of the magnetic field over the finite diameter of the particle's helical path. The function $a \exp - b/V$ is asymptotically zero to all orders of V , consistent with Kruskal's results.

where $\phi(t) \equiv \int_0^t \omega dt + \phi(0)$, $\phi(0) \equiv$ phase of particle about its circle of gyration at $t = 0$ ($z = 0$), $v_{\parallel}$ is the component of velocity parallel to $\mathbf{B}$ and $\omega \equiv eB/mc$. If the motion as given by constant J is substituted into the right-hand side of the equation we can integrate and obtain

$$\begin{aligned} \frac{\Delta J(z_0)}{J} &\cong -\frac{4}{\sin \delta} \int_0^{z_0} dz \cos \phi(t(z)) \frac{\mathbf{e}_a \cdot \nabla \mathbf{B}}{B} \\ &\quad \times \left[\left(1 - \frac{B}{B_0} \sin^2 \delta \right) \frac{B_0 B}{B_z^2} \right]^{\frac{1}{2}}. \end{aligned}$$

This expression yields the observed sinusoidal dependence on the phase $\phi(0)$, the characteristic transient near $\zeta = 0$ and the magnitude of $\Delta J/J$ per reflection within a factor of two.

In addition to the above information, however, one would like to know the cumulative effect of many traversals of the machine. For example, one might expect that in multiple passages J would change as in a random walk, or in a regular fashion, perhaps merely oscillating so that the particle would be effectively bound. To investigate this question, orbits were computed for which the particle made many reflections.

As mentioned earlier, three variables suffice to describe the orbits. Let us now use (ζ, δ, χ) , where (see Fig. 1) δ and χ are the spherical angles of the velocity vector:

$$\delta = \cos^{-1} P_{\zeta}/V, \quad \chi \equiv 3\pi/2 - \lambda = \tan^{-1} \rho P_{\rho}/(P_{\theta} - \psi).$$

We have followed a variety of orbits with digital computers for many traversals of the median plane, in one case about forty. Most of these were for $\alpha = 0.2$, $V = 0.4$, $P_{\theta} = 0.1$, which correspond to orbits with diameters about 1/10 of the distance between mirrors. At each traversal of the median

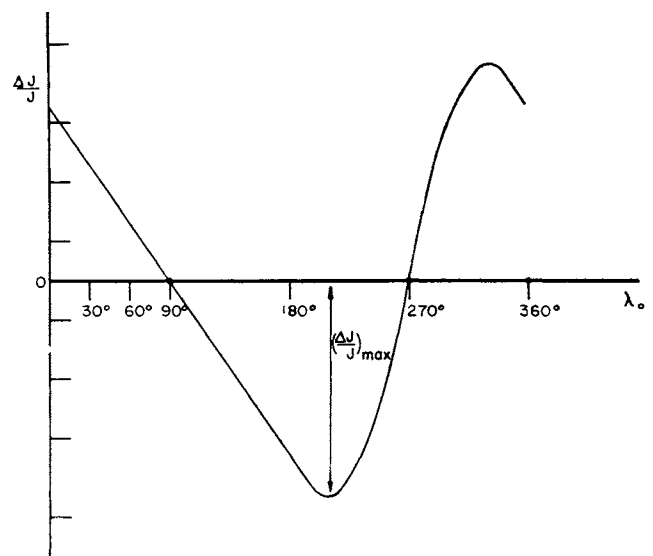


Figure 2. An example of the change in the adiabatic invariant as a function of the phase with which the particle passes through the median plane

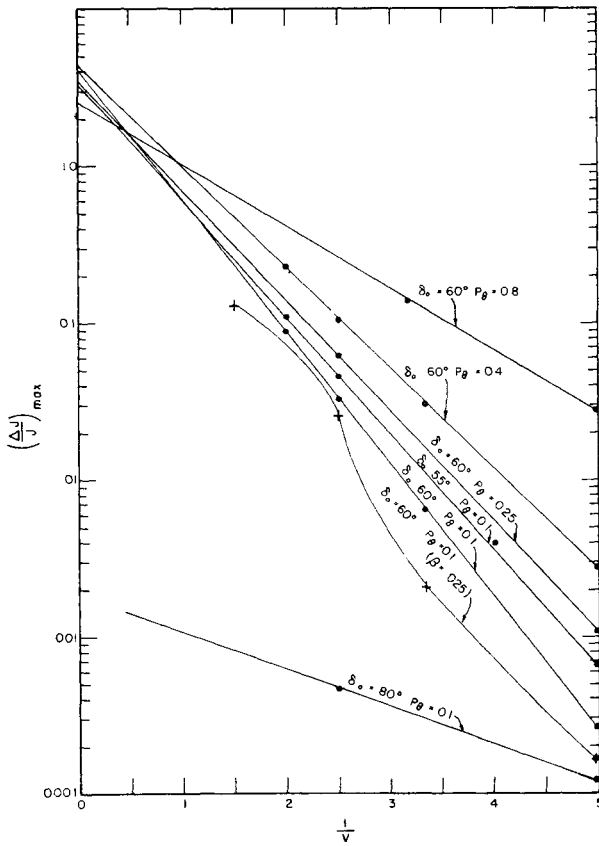


Figure 3. Change in the adiabatic invariant between reflections as a function of the particle velocity

plane we plotted δ or $\pi - \delta$, whichever was smaller, vs. χ . This plot, for the choice of parameters above, is shown in Fig. 4, with δ plotted radially and χ azimuthally, both in degrees. (Note that the origin of Fig. 4 is at $\delta = 50^\circ$.)

The orbits fall into two main classes, which may tentatively be called stable and unstable. The "stable" orbits intersect the median plane in points that may easily be joined by smooth curves. Each curve is designated by a capital letter and the points for successive traversals of each orbit are numbered. There are two types of stable orbits, viz.: B, C, M ; and L, J, N, P .

The "unstable" orbits are not shown in Fig. 4; they all lie in the cross-hatched area. Successive points on these orbits skip about in a rather erratic way and in two cases were followed long enough to see the particles escape through the ends. All the "unstable" orbits were inside (i.e., have smaller δ than) L , the innermost circumpolar curve.

Inside L there appear to be stable islands such as B, C . These are the regions surrounding the intersection with the median plane of the simplest kind of periodic orbits; namely, those which *always* traverse the median plane at $\chi = \pi$, and two values of δ : δ_0 and $\pi - \delta_0$. We call these intersections *fixed points*. For example, the one at the center of the B, C system belongs to an orbit that makes exactly four turns between transits through the median plane.

A similar system represented by orbit M surrounds the fixed point belonging to the orbit that makes three turns between transits. Two orbits belonging to fixed points and one periodic orbit intersecting the median plane in three distinct points are shown in Fig. 5, where we regard (χ, δ) and $(\chi, \pi - \delta)$ as one point.

One is tempted to infer that these smooth curves represent the intersection with the median plane of two-dimensional surfaces in the (ζ, δ, χ) or (ζ, P_ζ, P_ρ) space, which are invariant in the sense that all particles on one of them at one time remain on it for ever. It may be shown that if such surfaces really exist they have the same symmetric properties as do the periodic orbits discussed above. All our data seem to be at least consistent with the existence of such surfaces.

If curve L of Fig. 4 really is the intersection of an invariant surface with the $\zeta = 0$ plane, then it is rigorously true that all orbits outside it (i.e., with larger δ) are permanently contained. Hence, we may tentatively identify curve L as the effective loss cone for $V = 0.4, P_\theta = 0.1$. It has an average δ of about 65° , which may be compared to the value predicted if one assumes the constancy of μ , which is 55° .

We have studied to some extent the variations with V and P_θ . As might be expected from the strong dependence of $\Delta J/J$ on V discussed previously, the loss cone defined by the curves corresponding to curve L of Fig. 4 also varies. This is shown in Table 1 in which $\delta_{\min}$ is the average value of δ for the innermost "stable" curve and $\delta_{\max}$ is the value of ζ at reflection for the corresponding orbit. We also examined one rigorously bound orbit ($P_\theta < 0$). It gave a curve similar to type P in Fig. 4.

To what extent is it legitimate to infer permanent stability from these results? In a rigorous sense we

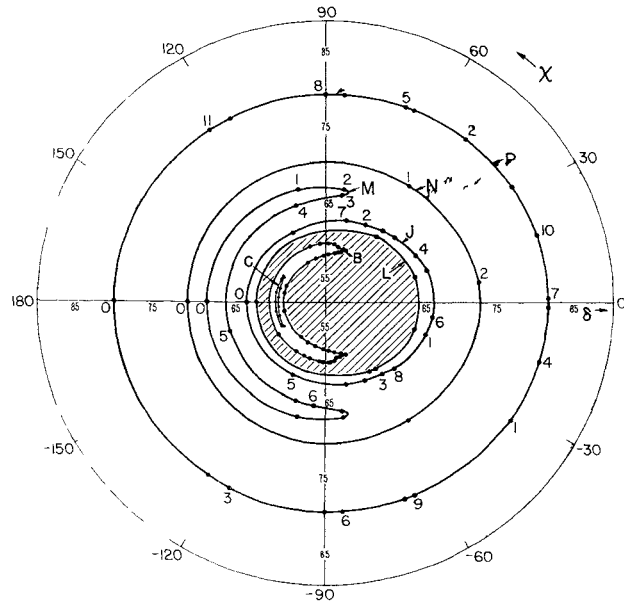


Figure 4. Values of δ and χ with which various orbits intersect the median plane. Successive transits are numbered except on curves C, B , and L . Unnumbered dots on the other curves are the time reverses of the numbered points

Table 1. Effective Loss Cone Defined by $\delta_{\min}$ as a Function of V and P_θ

V	P_θ	$\delta_{\min}$	$\delta_{\text{adiabatic}}$	$\zeta_{\max}$
0.2	+ 0.1	$\sim 55^\circ$	55°	$\sim \pi$
0.4	+ 0.1	~ 65	55	$\sim \pi/2$
0.6	+ 0.1	~ 80	55	$\sim \pi/4$
0.4	+ 0.4	~ 65	55	$\sim \pi/2$

can conclude nothing because we have made plots of points (δ, χ) for only a finite number of successive transits of the median plane. We have attempted, however, to give some practical measure of our uncertainty by comparison with multiple Coulomb scattering, as follows: we ran two orbits, starting at $\zeta = 0$ from the same value of χ and from values of δ differing by 10^{-3} degree. By plotting δ vs. χ for the two cases on a greatly expanded scale of δ , we found that the δ values of one orbit remained consistently greater than those of the other for corresponding χ values for about twenty traversals of the median plane, after which the points became interspersed. Hence, we concluded that the apparent departure from one of the invariant surfaces is of the order of 10^{-3} degree for twenty transits. In the case studied, this corresponded to a total distance of about twenty-five times the distance between mirrors. While we do not know what part, if any, of this effect is real, and what part results from errors in the machine calculation, the difference is practically unimportant so long as multiple Coulomb scattering dominates. Scattering leads to a mean square deflection in a distance S ⁵

$$\langle \theta^2 \rangle_{\text{sec}} \sim 6 \times 10^{-19} nS/W^2,$$

where n is the number of particles per cm^3 , W is the energy in keV and S is the distance traveled (in this case $S \sim 25 L$). We conclude that any possible intrinsic instabilities of particles are of no concern, at least for this particular V , so long as $(\langle \theta^2 \rangle_{\text{sec}})^{1/2}$ is greater than $10^{-3} \times \pi/150$, or so long as we have $nL/W^2 \gtrsim 2 \times 10^7$.

We studied two slightly less idealized fields which were asymmetric about the median plane. One type gave smooth curves (like P , Fig. 4) but with a splitting of the odd- and even-numbered traversals of the median plane. The other type did not give smooth curves. The points scattered in a manner suggestive of unstable motion. Thus the effect of magnetic-field imperfections on the particle containment requires further study.

EFFECT OF COLLISIONS ON CONTAINMENT

Through successive Coulomb collisions with other ions the δ of a given ion eventually becomes so small that the particle is in the effective loss cone and escapes through a mirror. Even if curves of type L of Fig. 4 are the intersections of invariant surfaces with the $\zeta = 0$ plane, this scattering loss constitutes an intrinsic limitation on mirror geometry. To study this loss, we have solved the Fokker-Planck equation numerically with several simplifying assumptions.

We have considered only ion-ion collisions in a spatially uniform, azimuthally symmetric, neutral plasma in a uniform magnetic field. The fact that electrons may scatter out through the mirrors at a different rate from that of the ions and thereby produce an electric field is neglected. The mirror effect is represented by a critical $\delta \equiv \delta_c$, below which particles are assumed to be lost immediately. This δ_c is assumed to be independent of V , with the justification that the results are insensitive to δ_c .

With these assumptions, the equation to be solved is ⁶

$$\begin{aligned} \partial f / \partial t + (F^j / m) \partial f / \partial v^j &= \langle \partial f / \partial t \rangle_{\text{coll}} \\ &= \partial (f \langle \Delta v^j \rangle) / \partial v^j + \frac{1}{2} \partial^2 (f \langle \Delta v^j \Delta v^k \rangle) / \partial v^j \partial v^k \end{aligned}$$

where

$$\begin{aligned} F^j &= (e/c) [\mathbf{v} \times \mathbf{B}]_j, \\ \langle \Delta v^j \rangle &= \int d^3 v' f(v') \int d\Omega \sigma(u, \Omega) u \Delta v^j, \\ \langle \Delta v^j \Delta v^k \rangle &= \int d^3 v' f(v') \int d\Omega \sigma(u, \Omega) u \Delta v^j \Delta v^k, \end{aligned}$$

$u = |\mathbf{v}' - \mathbf{v}|$ is the relative velocity of the colliding particles, σ = differential scattering cross section, $d\Omega$ = solid angle element and Δv^j = change in velocity of the particle due to the collision. Here

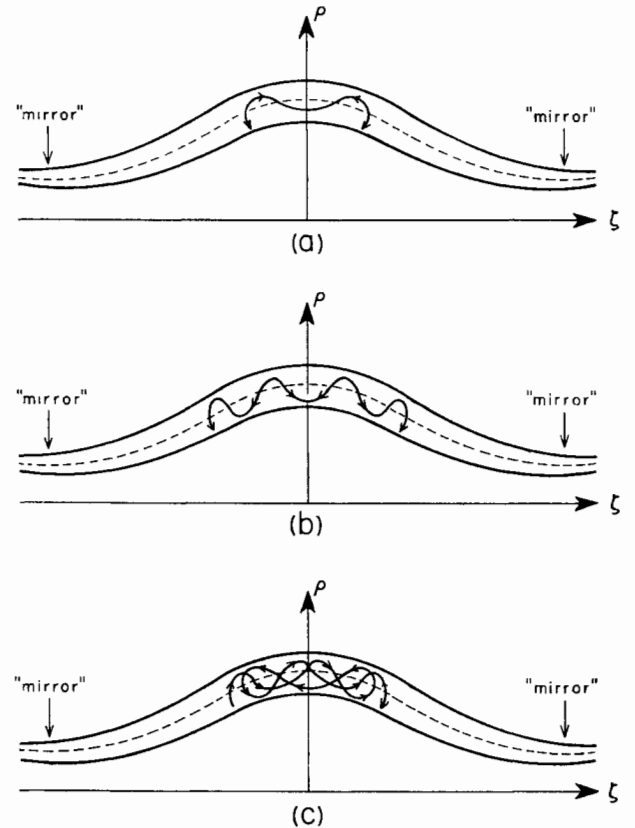


Figure 5. Examples of periodic orbits: (a) intersects median plane in one "fixed point", two turns between transits, (b) same as (a) but four turns between transits, (c) intersects median plane in two points

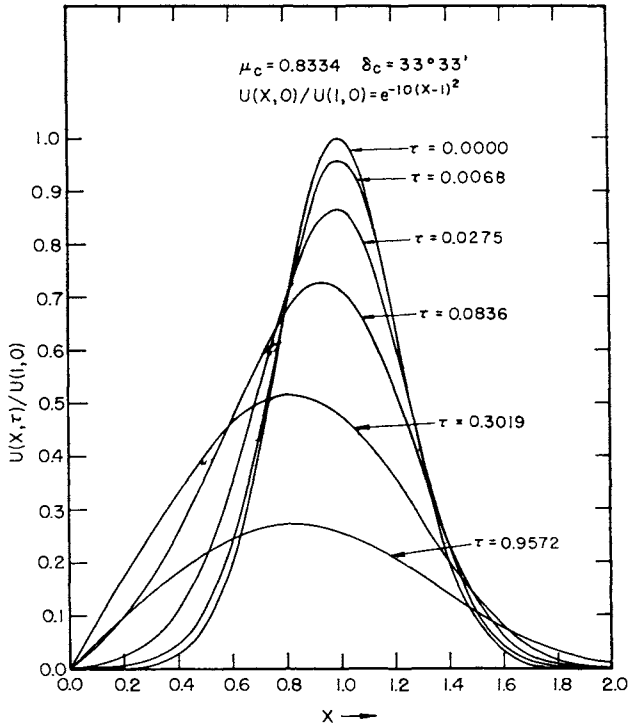


Figure 6. Development in time of an initial distribution function for $\delta_c = 43^\circ$ by use of the "normal-mode" approximations to the Fokker-Planck equation. This corresponds adiabatically to a mirror ratio $R = 3$, where $R \equiv B_{\max}/B_{\min}$.

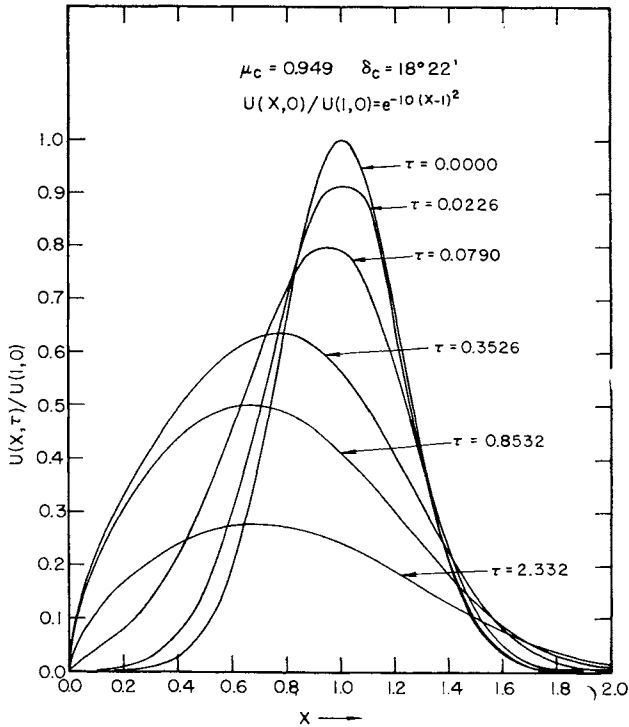


Figure 7. Development in time of an initial distribution function for $\delta_c = 18^\circ$ ($R = 10$) by use of the "normal-mode" approximations to the Fokker-Planck equation

$(F^j/m)\partial f/\partial v^j = (-eB/mc)\partial f/\partial \phi = 0$, where ϕ is the azimuthal angle of the velocity vector about the magnetic field $\mathbf{B}$.

The Fokker-Planck equation is to be solved with the boundary condition $f = 0$ for $0 \leq \delta \leq \delta_c$ and $\pi - \delta_c \leq \delta \leq \pi$. The independent variables that will be used are the dimensionless quantities.

$$\mu = \cos \delta, \quad X = v/v_0,$$

$$\tau = (n_0 t / v_0^3) (4\pi e^4 / m^2) \ln(b_1/b_0);$$

v_0 is a characteristic velocity whose meaning will be apparent in each problem; b_1/b_0 = ratio of maximum to minimum impact parameter; n_0 = initial number of ions/cm³; $(4\pi e^4 / m^2) \ln(b_1/b_0) \cong 1.2 \times 10^{12}$ cm⁶/sec⁴.

To simplify the Fokker-Planck equation an approximate separation of variables can be made if one

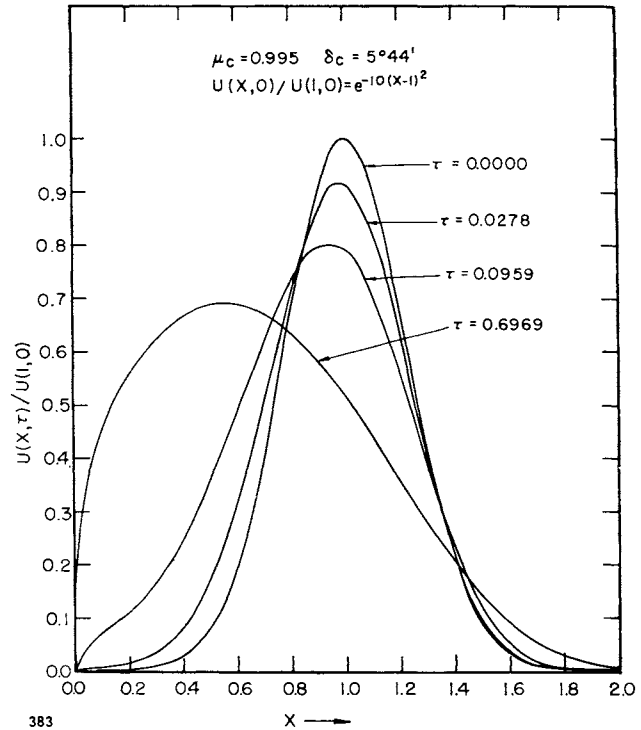


Figure 8. Development in time of an initial distribution function for $\delta_c = 60^\circ$ ($R = 100$) by use of the "normal-mode" approximations to the Fokker-Planck equation

assumes that the average relative velocity is independent of μ .§ Then, if we write $f(\mu, v, t) = (n_0/v_0^3) U(X, t)M(\mu)$, the Fokker-Planck equation becomes

$$(1 - \mu^2) \frac{d^2 M}{d\mu^2} - 2\mu \frac{dM}{d\mu} + \Lambda M = 0$$

$$\frac{\partial U}{\partial \tau} - 4\pi U^2 - \frac{1}{2} \frac{\partial^2 G}{\partial X^2} \frac{\partial^2 U}{\partial X^2} - \frac{1}{X^2} \frac{\partial G}{\partial X} \frac{\partial U}{\partial X} + \frac{\Lambda}{2X^3} \frac{\partial G}{\partial X} U = 0,$$

§ This method (unpublished) was originated by the authors of Reference 6.

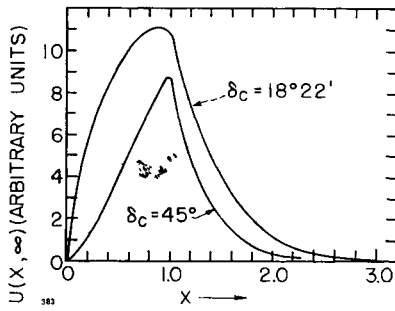


Figure 9. Steady-state distribution functions with a source of particles for two values of δ_c

where the ratio of the average relative velocity to v_0 is

$$G(X, \tau) = (1/n_0 v_0) \int f(\mu', v', t) |\mathbf{v}' - \mathbf{v}| d^3 v',$$

a quantity assumed independent of μ . The $2U^2$ term should be $2U^2 M(\mu)$. The approximation $M(\mu) \cong 1$ was made to achieve separation of the μ variable. The effects of these approximations can be seen by comparison with more exact calculations described below. The Legendre equation for M is to be solved subject to the boundary conditions $M(\pm \mu_c) = 0$ and $M(\mu) = M(-\mu)$. Figures 6, 7, and 8 show numerical solutions $U(X, \tau)$ corresponding to the eigenvalue Λ ; Λ is related to μ_c approximately by $\Lambda = -1/\log_{10}(1 - \mu_c^2)$ for the most slowly decaying (normal) mode. It can be seen that the loss rate is very insensitive to δ_c .

The above approximate method of solving the Fokker-Planck equation was also used with a steady source term which represents particles injected at an energy W_0 and normal mode in μ . The resulting distributions in X at steady state are shown in Fig. 9. For $X > 1$ the curves are well fitted by the Maxwellian form $\exp(-\alpha X^2)$. Since we have $\alpha < 1$ for both values of δ_c , the mean energy of the actual distribution is greater than W_0 , the injection energy. Numerical integration gives the number of particles/cm³: $n = 2.23 \times 10^7 D (W_0^{3/2} J)^{1/2}$ cm⁻³, where W_0 = injected energy in kilovolts, J = number of particles injected/cm³ sec, $D = 3.14 \times 10^5$ for $\mu_c = 0.949$, and 1.67×10^5 for $\mu_c = 0.833$. From these numbers, a quantity η of importance in the power economy of a mirror machine may be derived. If η = thermonuclear reaction energy/kinetic energy of the injected particles, we have for pure deuterium

$$\eta = (\frac{1}{2} n^2 \langle \sigma_r v \rangle W_r) / W_0 J = \frac{1}{2} (2.23)^2 \times 10^{14} D^2 \langle \sigma_r v \rangle W_0^{1/2} W_r,$$

where W_r = the energy in kilovolts released per fusion and $\langle \sigma_r v \rangle$ is the mean fusion cross section times velocity. For a (D-T) mixture the above expression for η should be divided by two. Table 2 gives η at two values of δ_c for pure deuterium and a deuterium-tritium mixture. It can be seen that scattering loss presents a severe problem for the

Table 2. Values of η , the Ratio of Thermonuclear Reaction Energy to Kinetic Energy of Injected Particles, under Various conditions

W_0 (kilovolts)	η for deuterium (10 Mev/reaction)		η for 50% D + 50% T (15 Mev/reaction)	
	$\delta_c = 45^\circ$	$\delta_c = 18^\circ 22'$	$\delta_c = 45^\circ$	$\delta_c = 18^\circ 22'$
50	0.026	0.140	0.58	2.2
100	0.078	0.32	0.76	2.8
200	0.206	0.80	0.92	3.4
400	0.48	1.84	1.10	4.2

economical operation of devices based on containment by mirrors.

A more elaborate numerical solution of the Fokker-Planck equation has been obtained without the simplifying assumptions that made the equation separable. Figures 10 and 11 show the development in time of the same initial distribution as in Fig. 6,

$$f(\mu, X, 0) \sim \exp - 10(X - 1)^2 \\ \times [1 - \exp - 3.6(\mu_c - |\mu|)].$$

This initial μ dependence is an approximate fit to the slowest normal mode described above. Comparison with Fig. 6 shows that the normal mode approximations lead to too high a loss rate by about 25%. From Figs. 10 and 11 it can be seen that the angular distribution remains approximately constant as particles are lost, while the velocity distribution tends slowly towards a Maxwellian one at high velocities. Low-energy particles are quickly scat-

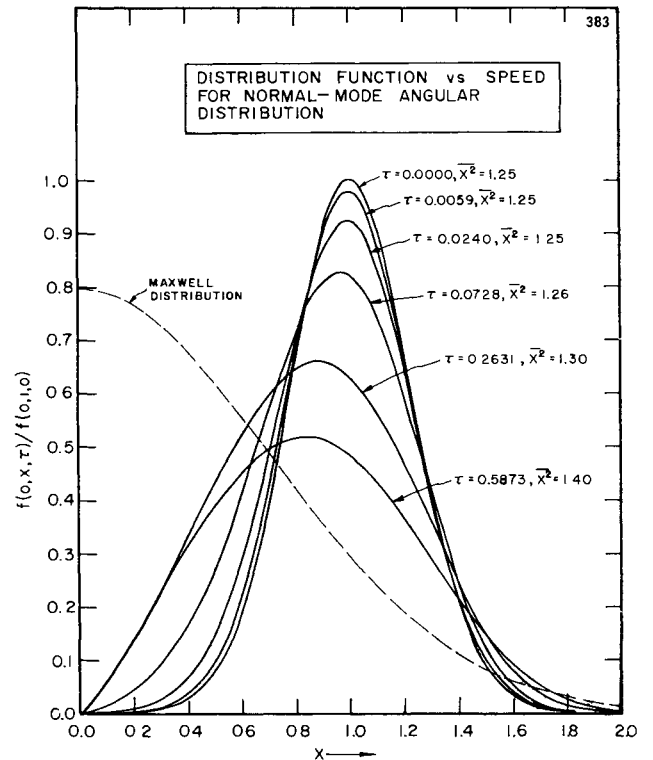


Figure 10. Development in time of an initial distribution function by direct numerical integration of the Fokker-Planck equation, as a function of speed

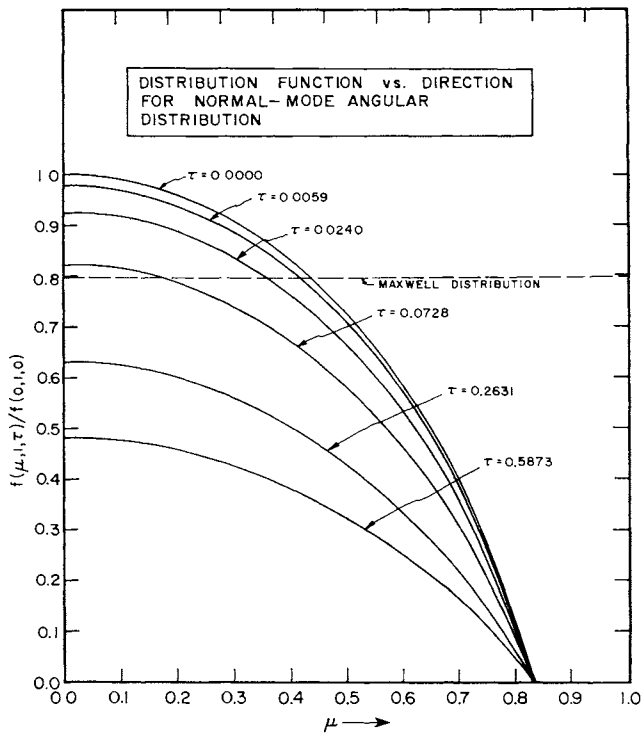


Figure 11. Development in time of an initial distribution function by direct numerical integration of the Fokker-Planck equation, as a function of direction

tered out because of the u^{-4} dependence of the Coulomb cross section. This results in a distribution that is deficient in low velocities compared with a Maxwellian. Note the increase in $\langle X^2 \rangle$ with τ . The self-collision time defined by Spitzer⁷ as

$$t_c = m^{\frac{1}{2}} (3kT)^{3/2} / (8 \times 0.714 \pi n e^4 \ln b_1/b_0),$$

where T is the temperature, corresponds to $\tau_c = 0.7$.

The fraction N of the original number of particles left after time τ in the normal-mode case is given in Table 3. In the third column is given a number κ defined by $(1/N(\tau_2)) - (1/N(\tau_1)) = \kappa(\tau_2 - \tau_1)$. This

Table 3. Fraction N of Particles Left after Time τ for Normal-mode Case

τ	N	κ
0.0000	1.000	—
0.0014	0.997	2.12
0.0057	0.988	2.09
0.0173	0.961	2.41
0.0625	0.864	2.63
0.1395	0.734	2.64

is the integrated form of a binary-collision loss mechanism over a time interval τ_1 to τ_2 . From the fact that κ becomes constant one can see that the process is of the form $(dN/d\tau) = (-\kappa N^2)$ at large τ . With $\kappa = 2.64$ the definition of τ gives the time in seconds for one-half the original particles to be lost: $t_{1/2} = 4.2 \times 10^{10} W_0^{3/2} n_0^{-1}$ seconds.

Calculations are in progress with initial distributions which are sharply peaked about $\mu = 0$. Thermonuclear reaction rates $\langle \sigma v \rangle$ are also being calculated as a function of τ for these peaked distributions and for those of Fig. 10.

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Effect of Distant Collisions on Magnetic Autostriction in an Arc

By Max Hoyaux and Paul Gans *

The purpose of this paper is to draw up a theory of the pinching of an arc formed in hydrogen or its isotopes due to magnetic autostriction, which might help in interpreting some experiments tending to a controlled release of the energy developed in fusion.

Magnetic autostriction has been investigated by various authors,¹⁻⁴ who assumed the plasma to be in radial equilibrium under the action of its kinetic pressure, the outside electrical field and the forces arising out of its own magnetic field. The radial distribution of the electronic and ionic densities, for the case of a concentration which vanishes at wall contact, tends to become a Dirac function when a parameter A , which is approximately proportional to the current, is made to increase. On the other hand, if the radial distribution curves are made to be in mutual osculation, their limit is a well-defined curve. We feel that conditions favorable for thermonuclear reactions practically correspond to $A = \text{infinity}$.

Some years ago, the writers improved on the theory reviewed above in order to allow for "distant collisions", the influence of which had been revealed by Gvosdover. Post mentions that these "distant collisions" or "electron scatter" through the Coulomb microfield of the positive ions may well come to play a part in the controlled thermonuclear reactions. To this day, we have found no theory which allowed for them.

In some instances, the Gvosdover effect modifies the radial distributions, always in the sense of a reduction in the autostriction. In some extreme cases, a current increase even gives rise to a slight swelling of the arc instead of a striction.

These conclusions, which were initially drawn up for the mercury vapor arc, can be used for an arc formed in hydrogen or its isotopes by reason of the close similarity between the two gases. For both, the electron scattering cross section for a neutral atom varies inversely to electron speed, and it is further known that this same cross section, for ionized atoms, follows the same law for all gases. The effect of the differences between the ionizing potentials and the

absence, in the case of hydrogen, of the metastable level which is characteristic of mercury, can readily be allowed for in the computation.

The practical conclusion is that in order to achieve final conditions of maximal pinch, initial conditions are required in which the Gvosdover effect is not marked; for instance, a low degree of ionization combined with a high electron temperature.

The direct application of this theory to current experiments, which are characterized by a very high dI/dt , requires that the dynamic phenomena be taken into account. The computations now under way will probably be finished by the date the conference is convened.

GENERALITIES ON MAGNETIC AUTOSTRICITION

Starting Equations

For clarity of exposition, we shall quote extensively from Ref. 4, adapting those quotations to the new problem we shall consider. It is known (see for instance Ref. 5, first part, III, I) that an electron or an ion subjected to a force $\mathbf{f}$ acquires a transport velocity

$$\mathbf{v} = (b/e)\mathbf{f} \quad (1)$$

where the algebraic values of the mobility b and the elementary charge e are used. This formula has been demonstrated for the cases of electrical forces and diffusion forces; there is no reason whatever why it should not hold for the case of magnetic forces or for any combination of these three types of forces. In the present case, only the radial components are of interest. Let us designate by v_z^+ and v_z^- the axial components of the ionic and electronic transport velocities. We have then

$$v_z^+ = b^+ E_z \quad v_z^- = b^- E_z, \quad (2)$$

E_z being the axial component of the electric field. In the absence of magnetic striction, we know that the radial velocities can be written as follows:

$$\begin{aligned} v_r^- &= b^- \left(E_r - \frac{kT^-}{e^-} \frac{dN}{Ndr} \right), \\ v_r^+ &= b^+ \left(E_r - \frac{kT^+}{e^+} \frac{dN}{Ndr} \right), \end{aligned} \quad (3)$$

Original language: French.

* Division Nucléaire des Ateliers de Constructions Electriques de Charleroi.

(k = Boltzmann constant; T^- = electron temperature; T^+ = ion temperature; N = electron concentration; E_r = radial component).

The existence of a magnetic field (reduced to its tangential component) causes the development of supplementary forces of the form eBv_z , which upon replacing the v_z by their value, gives:

$$\begin{aligned} v_r^- &= b^-(E_r - \frac{kT^-}{e^-} \frac{dN}{Ndr} - b^-BE_z), \\ v_r^+ &= b^+(E_r - \frac{kT^+}{e^+} \frac{dN}{Ndr} - b^+BE_z), \end{aligned} \quad (4)$$

where B = magnetic induction.

The ambipolar diffusion phenomenon (*loc. cit.*) leads to an adjustment of E_r to such a value that $v_r^- = v_r^+$; and by subtracting member from member of the equations (4), we have:

$$\begin{aligned} (b^- - b^+)E_r &= \left(b^- \frac{kT^-}{e^-} - b^+ \frac{kT^+}{e^+} \right) \frac{dN}{Ndr} \\ &\quad + [b^- - b^+]BE_z \end{aligned} \quad (5)$$

from which

$$\begin{aligned} E_r &= \left(\frac{b^-}{b^- - b^+} \frac{kT^-}{e^-} - \frac{b^+}{b^- - b^+} \frac{kT^+}{e^+} \right) \frac{dN}{Ndr} \\ &\quad + (b^- + b^+)BE_z. \end{aligned} \quad (6)$$

By substitution,

$$\begin{aligned} v_r &= v_r^+ = v_r^- = \\ &b^- \left[\frac{b^+}{b^- - b^+} \left(\frac{kT^-}{e^-} - \frac{kT^+}{e^+} \right) \frac{dN}{Ndr} + b^+BE_z \right]. \end{aligned} \quad (7)$$

Since $b^+ \ll b^-$ and $T^+ \ll T^-$ we can write

$$v_r = -b^+ \frac{kT}{e} \frac{dN}{Ndr} + b^-b^+BE_z. \quad (8)$$

where, in order to simplify the notation, we have written $T = T^-$ and $e = -e^- = e^+$.

Let us consider (Fig. 1) a layer of unit height cut out of the positive column having radius R and an infinitely thin layer comprised between cylinders of radii r and $r + dr$. Let us generalize the reasoning to be found in the literature (see for instance Ref. 6): in unit time and unit height, the number of carriers which diffuse to the outside through the cylinder having radius $r + dr$ must be equal to that which diffuses from the interior through the cylinder having radius r plus the number of carriers created by ionizing collisions in the intermediate annular space.

From this

$$\begin{aligned} 2\pi(r + dr) \left[-b^+ \frac{kT}{e} \left(\frac{dN}{dr} \right)_{r+dr} + b^-b^+N_{r+dr}B_{r+dr}E_z \right] \\ - 2\pi r \left(-b^+ \frac{kT}{e} \frac{dN}{dr} + b^-b^+N_rB_rE_z \right) \\ = \zeta_i N 2\pi r dr \end{aligned} \quad (9)$$

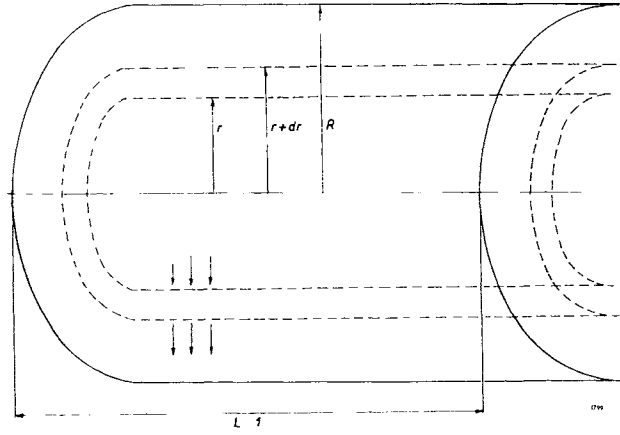


Figure 1. Representation of a layer cut out of the column

(with ζ_i = number of pairs of carriers created per electron per second).

Or again:

$$\begin{aligned} -b^+ \frac{kT}{e} \frac{dN}{dr} + b^-b^+NBE_z \\ + r \left[-b^+ \frac{kT}{e} \frac{d^2N}{dr^2} + b^-b^+ \frac{d(BN)}{dr} E_z \right] = \zeta_i N r. \end{aligned} \quad (10)$$

Recall that $b^+kT/e = D_a$, where D_a is the "ambipolar diffusion coefficient". Then we have

$$\begin{aligned} -D_a \frac{d^2N}{dr^2} - D_a \frac{dN}{dr} - \zeta_i N r + b^-b^+E_zNB \\ + r b^-b^+E_z \frac{d(BN)}{dr} = 0, \end{aligned} \quad (11)$$

whence

$$\frac{d^2N}{dr^2} + \frac{1}{r} \frac{dN}{dr} + \frac{\zeta_i}{D_a} N = \frac{b^-b^+E_z}{D_a} \frac{1}{r} \frac{d(NBr)}{dr}. \quad (12)$$

It will be recalled that the equation with the second member zero is that of the theory of Schottky. The second member is clearly the expression of magnetic striction.

In order to verify the homogeneity of the above equation, we note that mobility (which in the Giorgi system is expressed in m/sec (v/m), namely, in m²/v sec) has the dimensions of a reciprocal of a magnetic induction.

Computation of B

B is a function of r and of N . We have

$$B_r = \mu I_r / 2\pi r, \quad (13)$$

I_r being the current through a circumference of radius r and μ the magnetic permeability. We find readily that

$$I_r = \int_0^r N e |v_z| 2\pi r dr = 2\pi e b^- E_z \int_0^r N r dr \quad (14)$$

from which

$$NBr = -\mu e b^- E_z N \int_0^r N r dr. \quad (15)$$

By substitution

$$\frac{d^2N}{dr^2} + \frac{1}{r} \frac{dN}{dr} + \frac{\zeta_1}{D_a} N = -\frac{b^+ b^- E_z \mu e b^- E_z}{D_a r} \frac{d}{dr} \left(N \int_0^r N r dr \right). \quad (16)$$

Let us write

$$x = r(\zeta_1/D_a)^{\frac{1}{2}} \quad y = N/N_0 \quad (17)$$

where N_0 is the electron concentration in the axis of the discharge. We introduce

$$A = N_0 b^+ (b^-)^2 E_z^2 \mu e / \zeta_1. \quad (18)$$

We can readily show that A is a dimensionless number. Equation (16) becomes

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + y = -\frac{A}{x} \frac{d}{dx} \left(y \int_0^x y x dx \right). \quad (19)$$

This equation can be written

$$\begin{aligned} \frac{1}{x} \frac{d}{dx} \left(x \frac{dy}{dx} \right) + \frac{A}{x} \frac{d}{dx} \left(y \int_0^x y x dx \right) + y &= 0, \\ \frac{1}{x} \frac{d}{dx} \left(x \frac{dy}{dx} \right) + A y \int_0^x y x dx + y &= 0. \end{aligned} \quad (20)$$

Integrating member by member

$$x \frac{dy}{dx} + A y \int_0^x y x dx + \int_0^x y x dx = 0,$$

from which we have the final form

$$x \frac{dy}{dx} + (1 + Ay) \int_0^x y x dx = 0. \quad (21)$$

Let us write

$$u = x(1 + A)^{\frac{1}{2}} = r(\zeta_1/D_a)^{\frac{1}{2}} (1 + A)^{\frac{1}{2}}. \quad (22)$$

Then

$$\frac{dy}{du} = -\frac{1}{u} \frac{1 + Ay}{1 + A} \int_0^u y u du. \quad (23)$$

This equation is in a particularly interesting form for numerical solution by an iterative method.

Two values are of particular interest.

- (1) For $A = 0$, the magnetic striction correction obviously is zero.

Furthermore, Eq. (19) reduces to the well-known Bessel equation.

- (2) For $A = \infty$, the equation takes a very simple form.

$$\frac{dy}{du} = -\frac{y}{u} \int_0^u y u du. \quad (24)$$

Figure 2 shows the results obtained by the iterative solution method for the indicated values of A .

The curve obtained for $A = 0$ is compared to the Bessel function J_0 . The results obtained can be

extended toward larger values of A as follows: examination of the curves obtained for $A = 5, 10$ and ∞ suggest that in this range the value of y corresponding to a given value of u is approximately a linear function of $1/A$. Therefore we interpolated linearly between the values $A = 10$ and $A = \infty$ in terms of $1/A$, in order to obtain curves which correspond to $A = 20, 50, 100$ and 1000 .

Radial Distribution of Plasma Density

In Fig. 3, we changed abscissas so as to bring to the same point all the u intercepts of the curves. Therefore these curves represent N/N_0 as a function of $r/R = u/U$, for various values of A , U being the value of u for which y becomes zero.

Computation of Discharge Current

Consider again the value of A as given by Eq. (18). In this expression, the current density near the axis is given by

$$i_0 = N_0 e b^- E_z. \quad (25)$$

Thus by substituting (25) in (18) we can write

$$A = i_0 b^+ b^- E \mu / \zeta_1. \quad (26)$$

For simplicity we shall henceforth substitute E for E_z . From (26) we have:

$$i_0 = A \zeta_1 / b^+ b^- E \mu. \quad (27)$$

The current density at any point is given by

$$i = N e b^- E. \quad (28)$$

From Eqs. (28), (25), and (17), we have

$$i/i_0 = y \quad (29)$$

and

$$\begin{aligned} I &= \frac{2\pi R^2 i_0}{U^2} \int_0^U y u du \\ &= \pi R^2 \frac{\zeta_1}{b^+ b^- \mu E} \frac{2A}{U^2} \int_0^U y u du. \end{aligned} \quad (30)$$

Since U and $\int_0^U y u du$ are functions of A by (23), the over-all expression $(2A/U^2) \int_0^U y u du$ is a function of A and we can also write

$$(2A/U^2) \int_0^U y u du = (I/\pi R^2) b^+ b^- E \mu \sigma / \zeta_1, \quad (31)$$

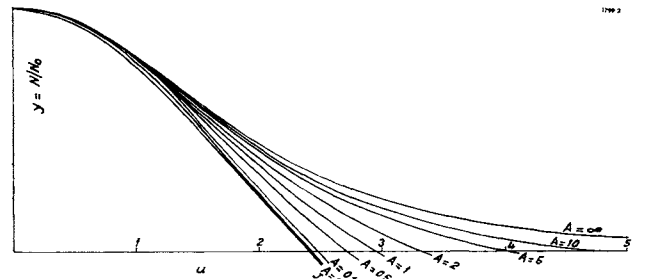


Figure 2

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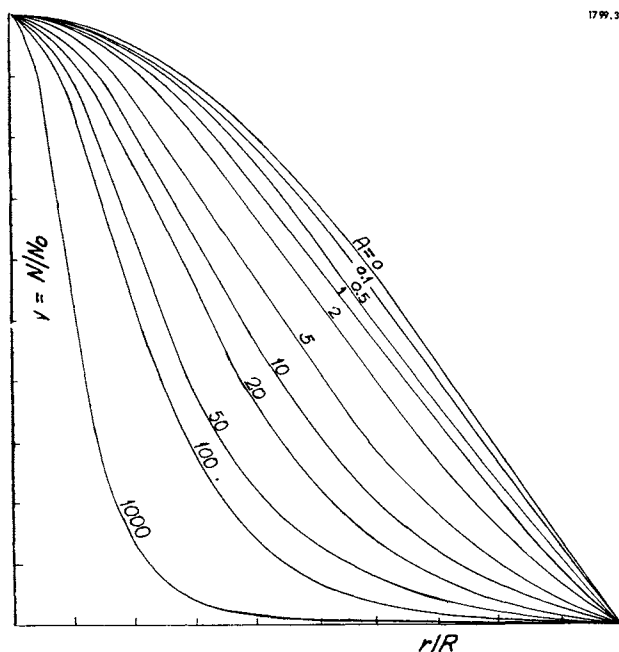


Figure 3

$\sigma = \mu/\mu_0$ being the magnetic susceptibility of the plasma. From Eq. (31), it can be seen that the parameter A is roughly proportional to the current. Since the above theory applies only to the first phases of the contraction, we shall assume in the following that the conditions which favor the appearance of thermonuclear reactions correspond to $A = \infty$. It will readily be seen that as A tends towards infinity, the radial distribution of the ionization density tends toward a Dirac function on Fig. 3 and toward a well-defined curve on Fig. 2.

GVOSDOVER EFFECT

Some years ago,⁴ the authors improved the theory mentioned above in order to allow for the "distant collisions", the influence of which have been shown by Gvosdover.⁷ In the first part of this paper, we considered the mean-free path of electrons to be determined solely by their collisions with neutral molecules. In 1937, Gvosdover⁷ showed that in a strongly ionized plasma we must allow for diffusion of the electrons in the Coulomb field of the positive ions. The theoretical computations of Gvosdover show that everything happens as though the electron mean-free paths were shortened according to

$$\frac{1}{\lambda_c} = \frac{1}{\lambda} + \frac{1}{S}. \quad (32)$$

λ_c is the "corrected" mean-free path (to be inserted into the new values of the mobility and diffusion coefficients); λ is the mean-free path which would be found if the only collisions were those with neutral molecules; and S is a length given by

$$S = \frac{k^2 T^2}{\gamma^4} \quad (33)$$

where

$$\gamma = (\pi/2) \ln \left(\frac{3}{2} kT e^{-2} N^{-\frac{1}{2}} \right). \quad (34)$$

Formulas (33) and (34) were given by Gvosdover in cgs esu. It will readily be seen that in mksa units we have

$$S = \frac{(4\pi\epsilon_0 kT)^2}{\gamma e^4 N} \quad (35)$$

and

$$\gamma = (\pi/2) \ln \left(6\pi\epsilon_0 kT / \epsilon^2 N^{\frac{1}{2}} \right), \quad (36)$$

ϵ_0 being the permittivity of empty space.

Schmidt¹⁰ computed electron mean-free paths in large mercury-vapor rectifiers (for currents of about 1000 amperes) by the mobility formula; he found that the values thus obtained are close to those given by the Gvosdover formula and sometimes quite different from those obtained when this effect is neglected. It is to be noted, however, that Schmidt seems to have assumed, without any experimental proof, that the radial distribution of N is a Bessel function J_0 , which often is inaccurate as we shall see below. Maecker^{11, 12} has studied experimentally the Gvosdover effect in several types of arcs, in particular the low pressure water vapor arc and the ordinary carbon arc. He arrived at the conclusion that the experimental values accord well with the Gvosdover equation, the equivalent cross section of the positive ions being reduced throughout by a coefficient 0.55, independently of the nature of the gas. In the following, we shall apply the correction factor of Maecker. Equation (35) consequently takes the form

$$S = \frac{(4\pi\epsilon_0 kT)^2}{0.55 \gamma e^4 N}, \quad (37)$$

the value of γ still being given by (36).

Let us note in conclusion of this introductory review that the equivalent cross section which one is thus led to attribute to the positive ions is large in comparison with that of the neutral molecules, so much so that S may be of the same order as λ or even smaller when the concentration of positive ions remains perceptibly less than that of the neutral molecules.

IONIZING POWER, EXCITING POWER, ENERGY EXCHANGE COEFFICIENT

It can readily be shown that the Gvosdover effect does not affect the validity of the reasoning of Ref. 5, first part, II D,E,F,G. However, regarding the last paragraph cited, we should note that, in addition to the h collisions per unit of time which the electron experiences with neutral molecules, there are also h^+ "collisions" with positive ions, for which we may introduce an energy exchange coefficient κ^+ , in analogy with the κ of the cited reference.

But κ^+ results almost exclusively from diffusion, which may, from the energetic point of view, be included in the elastic collisions, and $h^+\kappa^+$ is, in any case, much smaller than $h\kappa$. Therefore Eqs. (29) and (30), Ref. 5, remain valid on the condition that the mean used be taken exclusively for collisions with

neutral atoms, the number of which is given by the same expression as if the Gvosdover effect did not exist. We remark, moreover, that this effect does not modify the velocity distribution; an electron scattered from the Coulomb field of a positive ion always departs with a velocity practically equal to that with which it arrived. In what follows, we shall continue using Eqs. (24), (26) and (30), Ref. 5, which we write here again:

$$\xi_i/p = \xi_i^{(1)} = f(T), \quad (38)$$

(the index (1) will be used to denote magnitudes which apply to unit pressure)

$$\zeta_e = f(T), \quad (39)$$

and

$$\kappa = f(T), \quad (40)$$

with p = reduced pressure, ζ_e = ratio of number of exciting impacts to number of ionizing impacts, κ = ratio of mean energy-loss per collision to the mean energy prior to the collision.

These conclusions, established for the mercury-vapor arc, are applicable to an arc in hydrogen or its isotopes, due to the close similarity between the two gases. Indeed, in both cases the electronic diffusion cross section for the neutral atom varies as the reciprocal of the electron velocity. It is known, further, that the corresponding cross section for the ionized atoms varies according to the same law in all gases.

An essential difference lies in the existence, for mercury, of a metastable level at 5.4 eV, slightly above half of the ionization potential, which may give rise to ionization phenomena, either in cascade or through impacts of the second kind. We shall refer the reader to the appendix of Ref. 4 for a discussion of this phenomenon, which appears to play but a very limited role in the investigations of interest here.

Mobility

On the other hand, Eqs. (32) and (38) no longer can be retained when we allow for the Gvosdover effect. We shall assume — and as a matter of fact this is Schmidt's fundamental hypothesis — that the new formula for mobility is simply deduced from the old one by substituting λ_e in place instead of λ . We have, thus:

$$b_e = \frac{K_1}{2} \frac{e}{m} \frac{\lambda_e}{w} = \frac{K_1}{2} \frac{e}{m} \frac{\lambda}{w} \frac{1}{1 + \frac{\lambda}{S}} = b \left(1 + \frac{\lambda}{S} \right)^{-1}, \quad (41)$$

where m = the mass of the particle, w = the random component of the electronic velocity; K_1 = correction factor in the formula giving the mobility (Ref. 5) in which it suffices to replace S by the value given by Eq. (37). However, we can inquire whether it is permissible to apply the same correction to the two types of carriers. We do not think so; certainly an interference of the nature being investigated involves simultaneously a positive ion and an electron, furthermore, the ions indubitably interfere with one another in a similar fashion (the force being simply of the opposite sign).

However the demonstration of Eqs. (35) and (38), Ref. 5, assumes that the scattered particle is thrown back isotropically following interaction. In the Gvosdover effect, such isotropy may very likely be postulated for an electron which is scattered by an ion, but an ion which is scattered by an electron undergoes only a negligible disturbance. An ion which is scattered by another ion, as well as an electron which is scattered by another electron, need not affect the mobility formula, since the sum of the projections of the momenta on the axis of the discharge is not changed, and the particles involved have the same sign.

We feel, therefore, that we can conclude that Eq. (41) applies only to electrons, the mobility of the positive ions having, all other things being equal, the same value as though there were no Gvosdover effect.

Ambipolar Diffusion

The reasoning in Ref. 5, first part, III. G, does not appear to be weakened by the intervention of the Gvosdover effect; Eq. (40), Ref. 5, holds both for electrons and positive ions.

For the latter, the value of the diffusion coefficient remains unchanged, all other things being equal, whereas the diffusion coefficient of the electrons is reduced, as is their mobility coefficient, in the same ratio as the free path. From Ref. 5, second part, I. B, it will be seen readily that the value of the ambipolar diffusion coefficient still is given by Eq. (40) and therefore is not changed by the Gvosdover effect, all other things being equal.

Basic Equations

Energy Balance

In accordance with the preceding discussion, Eq. (63), Ref. 5, second part, II. A, remains valid despite the Gvosdover effect. But as we propose, hereafter, to allow simultaneously for this effect and for magnetic autostriction, it should be rejected anyway, for the reasons given in the first part. Equation (64), Ref. 5, first part, I. B, is not rendered invalid by autostriction; it is logical to assume that the Gvosdover effect here again comes into play in the same manner as above, namely, that it suffices to replace the mobility coefficient by its new value. In particular, for the reasons given above, in the denominator of Eq. (64), Ref. 5, neither h_1 nor κ need be modified.

We shall rewrite the above formula in the form

$$E/p = (3kh_1\kappa T/2eb_e^{(1)})^{\frac{1}{2}}. \quad (42)$$

According to Eq. (41)

$$b_e^{(1)} = b^{(1)}[1 + \lambda/S]^{-1}. \quad (43)$$

By substituting in (42),

$$\begin{aligned} E/p &= \{3kh_1\kappa T[1 + (\lambda/S)]/2eb^{(1)}\}^{\frac{1}{2}} \\ &= (E/p)_0[1 + \lambda/S]^{\frac{1}{2}}, \end{aligned} \quad (44)$$

$(E/p)_0$ designating the function of T which appears in the second member of Eq. (64), Ref. 5.

Transport Velocity, Current Density

In the old theory, $b\rho$ and E/ρ being functions of T , the transport velocity $v = bE$ was also a function of T . This is not valid here. We have

$$\begin{aligned} v_c &= b_c E = b_c \rho E / \rho, \\ &= b \rho [1 + \lambda/S]^{-1} \cdot (E/\rho)_0 [1 + \lambda/S]^{\frac{1}{2}}, \\ &= v [1 + \lambda/S]^{-\frac{1}{2}}, \end{aligned} \quad (45)$$

v_c being the transport velocity corrected for the Gvosdover effect and v the velocity which would correspond to the same electronic temperature if this effect did not exist.

Since the current density is given except for sign by the formula

$$i = Nev_c, \quad (46)$$

we shall write

$$i = Nev [1 + \lambda/S]^{-\frac{1}{2}} \quad (47)$$

and

$$N/i = (N/i)_0 [1 + \lambda/S]^{\frac{1}{2}}. \quad (48)$$

Radial Distribution of the Density of Carriers

The simultaneous introduction of the Gvosdover effect and the magnetic autostriction, investigated in the first part of this paper, does not appear to raise any special problem, it being understood that in the preceding conclusions we shall replace the free paths, the diffusion coefficients, the transport velocities, etc., by their corrected values throughout.

The Gvosdover effect, as it shortens the free paths, reduces more and more the influence of the magnetic field in combination with the random velocity, and enables us *a fortiori* to neglect this effect. Equation (8) remains valid provided we replace b^- by b_c^- :

$$v_r = b_c^- b^+ E_z - D_a \frac{d(\ln N)}{dr}. \quad (49)$$

Equation (11) is, in principle, valid with the same reservation. However, we note that b_c^- (which enters into the expression of Nv_r) now is a function of r and cannot be moved across a derivative sign. A brief computation shows that the correction factor remains under the sign and that the equation becomes

$$\begin{aligned} \frac{d^2 N}{dr^2} + \frac{1}{r} \frac{dN}{dr} + \frac{\zeta_i N}{D_a} \\ = \frac{b_c^- b^+ E_z}{D_a} \frac{1}{r} \frac{d}{dr} N B r [1 + \lambda/S]^{-1}. \end{aligned} \quad (50)$$

Variation of the Electron Temperature

Another essential difference appears when we endeavor to compute magnetic induction as a function of carrier density, to obtain an equation in terms of N alone. In the first part of this paper, we had the advantage that since the transport velocity was everywhere the same, the radial distribution of current density was identical with that of carrier density. In a medium which has translational symmetry, Eq. (64), Ref. 5, leads to the conclusion that the electron temperature is the same at all points of the medium.

This equation being now replaced by Eq. (44) of this paper, the logical conclusion would be that the electron temperature must be different, in view of the sign of variation of E/ρ as a function of T . Equation (44), applied for a constant field in a horizontal layer of the medium, leads to the conclusion that the electron temperature is lower on the axis of the discharge, where the carrier concentration is greater and therefore the Gvosdover effect correction $1/S$ is large, than it is near the wall, where this concentration tends to zero and with it this correction. Such a conclusion neglects an effect which is well known, although we have not yet expressed it in an equation: it is the thermal conductivity of the plasma, which tends to equalize, over short distances, the variations in the electron temperature. We will not err greatly in assuming that the electron temperature has everywhere the value computed using Eq. (44) on the axis of the discharge. By assuming T constant in a horizontal layer, we simplify computations greatly.

Computation

Despite the assumption that T is independent of the distance from the axis of the discharge, the transport velocity is not the same at all points; we should note, however, that in virtue of the consideration mentioned in the preceding paragraph, the radial variation of v is not determined by Eq. (45) but rather by Eq. (43) combined with an axial field which is independent from the distance of the axis.

Since v_z is considered as given by

$$v_z^- = b_z^- E_z [1 + \lambda/S]^{-1}, \quad (51)$$

the value of I_r to be introduced into Eq. (14) is

$$I = -2\pi e b^- E_z \int_0^r \frac{N r dr}{1 + \lambda/S}. \quad (52)$$

Here b^- represents the "uncorrected" value of the mobility for the existing electron pressure and temperature, and E_z the effective axial field which prevails in the discharge independently of any other consideration. Equation (15) is to be replaced by

$$N B r = -\mu e b^- E_z N \int_0^r \frac{N r dr}{1 + \lambda/S}. \quad (53)$$

Let us consider the value of function λ/S , λ is the uncorrected value of the electron mean free paths and obeys Eq. (22), Ref. 5:

$$\lambda \rho = \lambda^{(1)} = f(T), \quad (54)$$

S being given by Eqs. (37) and (36). In these equations, N appears both as a factor to the power 1 and in the logarithm to the power $1/3$. This second effect is somewhat complicating and we shall use an approximation, adequate for our computations, which eliminates this difficulty. Thanks to the fact that the second appearance of N is in a logarithm and at a power close to zero, we shall make no great error if there and only there we replace N by a constant mean value. It is to be noted that the value of the logarithm in (37) is generally approximately 7. If our

approximation is wrong by a factor of one or two, there will be an error of 0.3 in $\ln N$, of 0.1 in $\ln N^{\frac{1}{2}}$ and of 0.23 in $\ln N^{\frac{1}{3}}$. The error in γ in this case will be 0.23 divided by 7, i.e., approximately 3%. The advantage of this approximation is that γ becomes a function of T alone and that λ/S can be written in the form

$$\lambda/S = N/pC, \quad (55)$$

where C is a function of T given by

$$1/C = 0.55\lambda^{(1)}\gamma(e^2/4\pi\epsilon_0 kT)^2. \quad (56)$$

Equation (53) becomes

$$NBr = -\mu eb^- E_z N \int_0^r \frac{Nrdr}{1 + N/pC}. \quad (57)$$

We must substitute this value of NBr in Eq. (50), the b_c^- used in this equation being replaced by its value as given by Eq. (41) in which λ/S is given by (55). Thus we have

$$\begin{aligned} \frac{d^2 N}{dr^2} + \frac{1}{r} \frac{dN}{dr} + \frac{\zeta_i N}{D_a} = -\frac{\mu e(b^-)^2 b^+ E_z^2}{D_a} \frac{1}{r} \frac{d}{dr} \\ \times \left(\frac{N}{1 + N/pC} \int_0^r \frac{Nrdr}{1 + N/pC} \right). \end{aligned} \quad (58)$$

Setting

$$x = r(\zeta_i/D_a)^{\frac{1}{2}}, \quad (59)$$

$$y = N/N_0, \quad (60)$$

$$A = N_0 b^+ (b^-)^2 E_z^2 \mu e / \zeta_i, \quad (61)$$

$$y_s = pC/N_0, \quad (62)$$

we have

$$\begin{aligned} \frac{1}{x} \frac{d}{dx} \left(x \frac{dy}{dx} \right) + y + \frac{A}{x} \frac{d}{dx} \left(\frac{y}{1 + y/y_s} \right) \\ \times \int_0^x \frac{xydx}{1 + y/y_s} = 0. \end{aligned} \quad (63)$$

This equation can be written

$$\frac{d}{dx} \left(x \frac{dy}{dx} \right) + A \frac{y}{1 + y/y_s} \int_0^x \frac{xydx}{1 + y/y_s} + xy = 0. \quad (64)$$

or again

$$\frac{dy}{dx} = -\frac{1}{x} \left(\int_0^x xydx + \frac{Ay}{1 + y/y_s} \int_0^x \frac{xydx}{1 + y/y_s} \right). \quad (65)$$

Let us write

$$u = x(1 + A)^{\frac{1}{2}}, \quad (66)$$

$$\begin{aligned} \frac{dy}{du} = -\frac{1}{u} \left(\frac{1}{1 + A} \int_0^u yudu \right. \\ \left. + \frac{A}{1 + A} \frac{y}{1 + y/y_s} \int_0^u \frac{yudu}{1 + y/y_s} \right). \end{aligned} \quad (67)$$

This equation has been integrated at our request by I.B.M. using the method presented for Eq. (23) but with two parameters A and y_s . Equations (61) and (62) show that A and $1/y_s$ are proportional to N_0 ; they

are therefore increasing functions of the current. However, A , which also involves the ordered electronic velocity, is more directly related to the current than is $1/y_s$, which involves the pressure and thus actually is related to the percentage of ionized atoms. In reality, we can consider A and $1/y_s$ as being relatively independent. Typical radial distributions as obtained are found in Figs. 4-7. On the other hand, the current of the arc is given by Eq. (53) in which the upper limit of integration is R instead of r . This equation is written allowing for Eqs. (55) and (59)-(62).

$$I = -2\pi eb_{(1)}^- \left(\frac{E_z}{p} \right) \int_0^R \frac{Ny \left(\frac{Ru}{U} \right) d \left(\frac{Ru}{U} \right)}{1 + y/y_s}. \quad (68)$$

In view of (44) and of the observation made above regarding the relationship between (E_z/p) and T we have

$$\begin{aligned} I = -2\pi e R^2 N b_{(1)}^- (E/p)_0 \\ \times (1 + y_s^{-1})^{\frac{1}{2}} \frac{1}{U^2} \int_0^U \frac{yudu}{1 + \frac{y}{y_s}}. \end{aligned} \quad (69)$$

If the Gvosdover effect becomes so significant that the mean free path of the electrons is determined completely by the density of positive ions, S becomes small compared to λ and y_s small compared to y . Equation (67) then assumes the simplified form

$$\begin{aligned} dy/dx = -x^{-1} \left(\int_0^x yxdx + A y_s \int_0^x y_s xdx \right) \\ = -x^{-1} \left(\int_0^x yxdx + \frac{1}{2} A y_s^2 x^2 \right). \end{aligned} \quad (70)$$

Let us write

$$B = \frac{1}{2} A y_s^2. \quad (71)$$

(Note that B does not represent magnetic induction.) Then

$$\frac{dy}{dx} = -x^{-1} \int_0^x yxdx - Bx. \quad (72)$$

Let us write:

$$v = x(1 + B)^{\frac{1}{2}}. \quad (73)$$

A simple computation gives

$$\frac{dy}{dv} = -\frac{1}{1 + B} \left(\frac{1}{v} \int_0^v yv dv + Bv \right). \quad (74)$$

Even though y_s is small, we cannot neglect the term Bv of Eq. (74). Indeed, we have seen that in a first approximation the factor A , which characterises magnetic autostriction, tends to increase under precisely the circumstances in which y_s tends to decrease, and it is difficult to decide *a priori* about the value of $B = \frac{1}{2} A y_s^2$. At the most, we may state that B would be large if a very high degree of ionization were accompanied by a low current. The physical interpretation of this approximation is easy: When the Gvosdover effect is important, it tends to distribute the current uniformly throughout a horizontal layer, the increase in density toward the center being

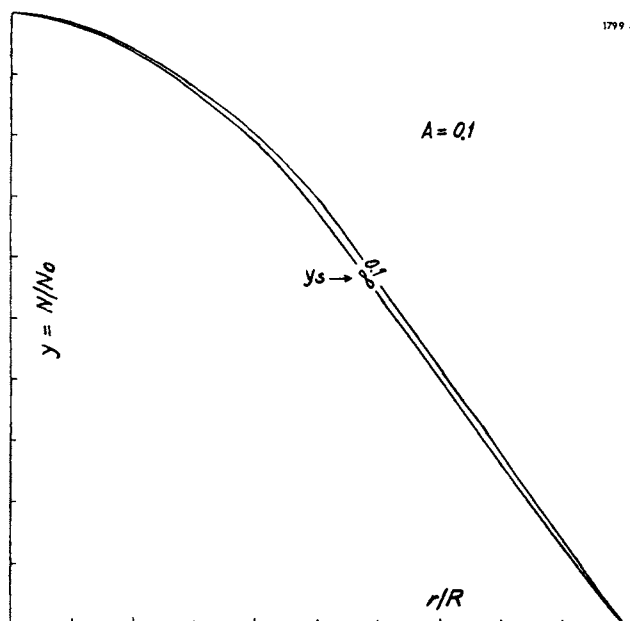


Figure 4

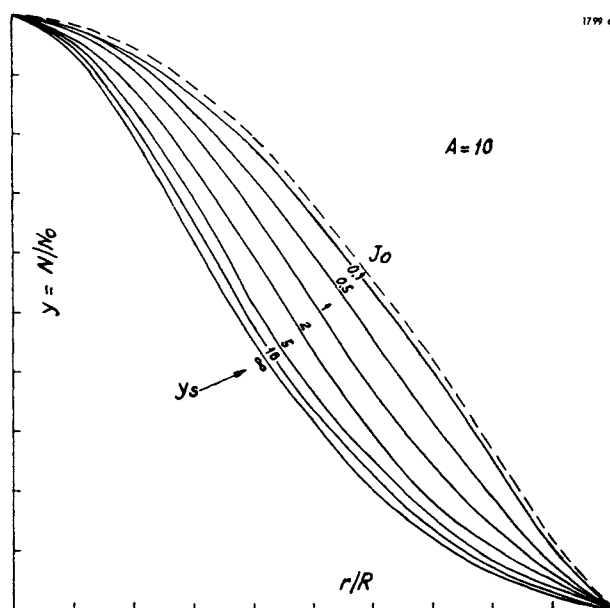


Figure 6

exactly compensated for by a decrease in mobility. Eq. (74) has been integrated by an iterative method identical to that used above; the curves obtained are given in Figs. 8 and 9. For large values of B , we carried out a linear interpolation in terms of $1/B$ between the curves obtained for $B = 10$ and $B = \infty$. The latter curve plainly is the graph of the equation $y = 1 - \frac{1}{2}v^2$.

Conclusions

Examination of Figs. 4, 5, 6, 7 and 9 enables us to draw the following conclusions:

(a) The theory presented in the first part appears to be a limiting case of the general theory when y_s

tends towards infinity. This is in accord with our hypotheses, since y_s tending towards infinity corresponds to S tending to infinity, i.e., to a Gvosdover effect which is negligible according to Eq. (32).

(b) When y_s is small, i.e., when the Gvosdover effect is substantial for a given value, the radial distribution is always less pinched of A than on Fig. 3.

(c) In particular, the extreme Gvosdover effect (Fig. 9) leads to non-pinched distributions which are even "swollen" as compared to the Bessel function, regardless of the current. Such a situation plainly is extremely unfavorable to magnetic confinement of the form envisaged for use in controlled thermo-

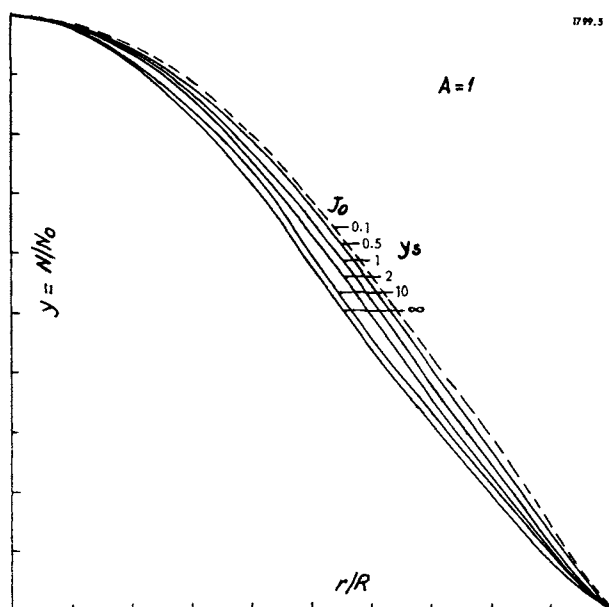


Figure 5

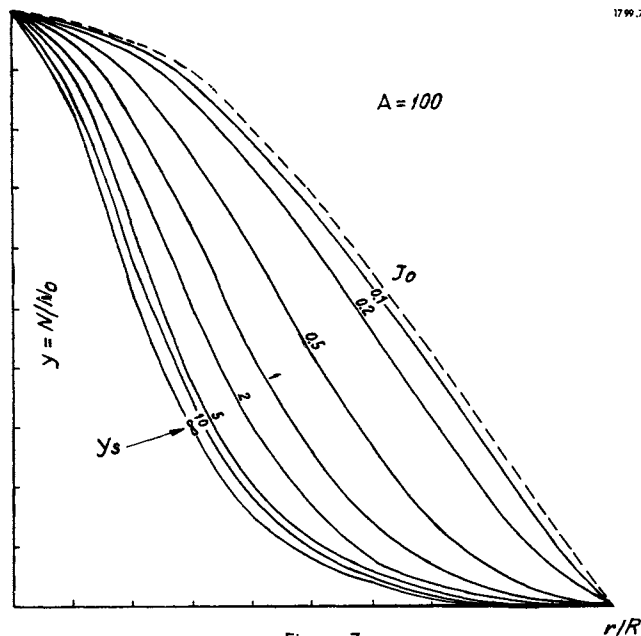


Figure 7

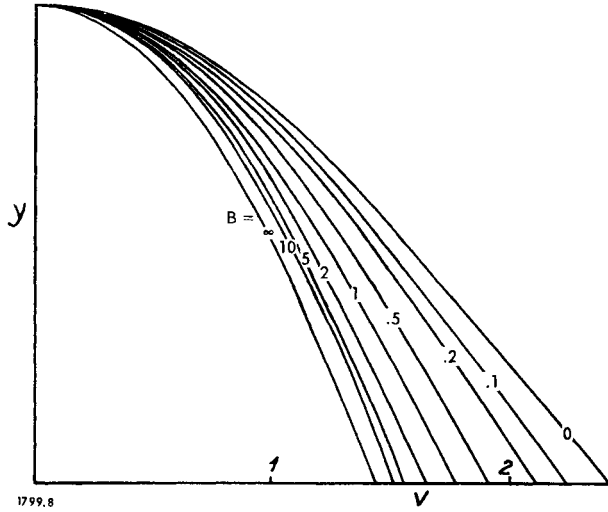


Figure 8. (The curve for $B = \infty$ was integrated by quadrature)

nuclear phenomena. Thus, the arc will contract all the faster in the first instants of its existence, and therefore will have all the more chance of reaching a configuration favorably confined by its own magnetic field, as the initial value of S , given by (37), will be high. Thus, an initial condition in which the degree of ionization is small and the electronic temperature high must be sought.

The degree of ionization cannot increase until satisfactory magnetic confinement is achieved.

INTRODUCTION OF DYNAMIC PHENOMENA

In most of the experiments aimed at achieving a stable magnetic confinement, the current varies very rapidly with time.¹³⁻¹⁵ Therefore one might fear that the above conclusions could be rendered invalid. Our computations show that this is not the case. To introduce the evolution of the phenomena with time, we must consider, in the balance expressed by Fig. 1 and Eq. (9), the possibility of a variation of N as a function of time. Examination reveals that it suffices to replace $\zeta_i N$ by $\zeta_i N - \partial N / \partial t$, and total derivatives with respect to r by partial derivatives with respect to the same variable in Eq. (9) and in all which follow. In addition, for simplicity of notation, we shall designate by N^* the concentration of the plasma in the case where I is a function of time.

A simple computation leads to replacing (58) by

$$\frac{\partial^2 N^*}{\partial r^2} + \frac{1}{r} \frac{\partial N^*}{\partial r} + \frac{\zeta_i N^*}{D_a} - \frac{1}{D_a} \frac{\partial N^*}{\partial t} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left[\frac{N^*}{1 + N^*/pC} \int_0^r \frac{N^* r dr}{1 + N^*/pC} \right] \quad (76)$$

with

$$\alpha = -\mu e b^+ (b^-)^2 E_z^2 / D_a. \quad (77)$$

This equation, which is not linear, does not lend itself to solution by separation of variables nor by

variation of parameters. Therefore, we shall substitute for it a linear equation considering the time variation as a perturbation with respect to Eq. (58). This equation having been solved, we shall calculate the solution of (76) from the solution of (58) by setting

$$N^* = N + n \quad (78)$$

and substituting in (76). N will be considered a function of r alone and the perturbation n as a function of r and t . We shall assume that n is small with respect to N . The development of (76) allowing for (78) will include terms of various degrees in n/N ; the terms of zero degree cancel, by reason of (58); the terms of degree two and higher will be neglected.

Finally, we consider that the Gvosdover effect is of the first order, namely, that $1 + N^*/pC$ may be replaced by $1 + N/pC$. A lengthy but uncomplicated computation shows that n satisfies the following linear equation

$$\begin{aligned} \frac{\partial^2 n}{\partial r^2} + \frac{1}{r} \frac{\partial n}{\partial r} + \frac{\zeta_i n}{D_a} - \frac{1}{D_a} \frac{\partial n}{\partial t} \\ = \frac{\alpha}{r} \frac{\partial}{\partial r} \left[\frac{N}{1 + N/pC} \int_0^r \frac{n r dr}{1 + N/pC} \right. \\ \left. + \frac{n}{1 + N/pC} \int_0^r \frac{N r dr}{1 + N/pC} \right] \end{aligned} \quad (79)$$

which is readily solved by separation of variables.

Let us write

$$n(r, t) = \varrho(r) \theta(t). \quad (80)$$

D_a , α , p and C are assumed to be functions of t but not of r . This hypothesis obviously applies only during the first instants of the contraction. By

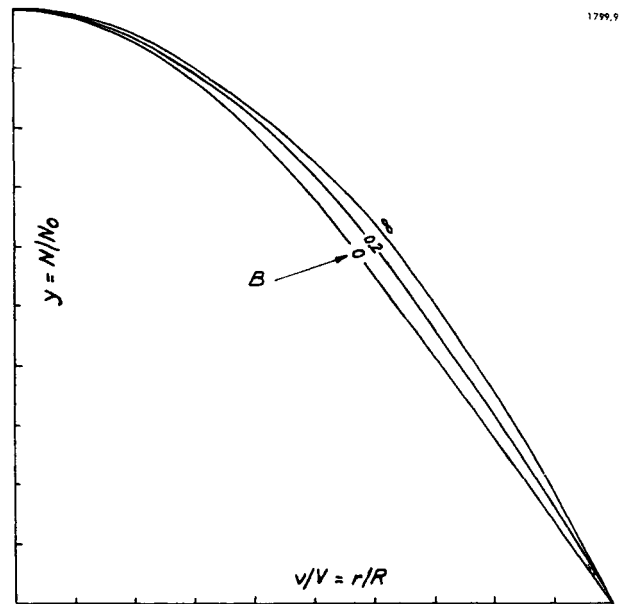


Figure 9

substituting and dividing each term by $\varrho \cdot \theta$ we obtain:

$$\begin{aligned} \frac{1}{\varrho} \frac{d^2 \varrho}{dr^2} + \frac{1}{r\varrho} \frac{d\varrho}{dr} + \frac{\zeta_i}{D_a} - \frac{1}{D_a \theta} \frac{d\theta}{dt} \\ = \frac{\alpha}{r\varrho} \frac{d}{dr} \left[\frac{N}{1 + N/pC} \int_0^r \frac{\varrho r dr}{1 + N/pC} \right. \\ \left. + \frac{\varrho}{1 + N/pC} \int_0^r \frac{Nr dr}{1 + N/pC} \right]. \end{aligned} \quad (81)$$

By introducing a parameter β which in general will depend on time, this equation can be separated into the following two equations:

$$\frac{1}{\theta} \frac{d\theta}{dt} = \beta, \quad (82)$$

$$\begin{aligned} \frac{d^2 \varrho}{dr^2} + \frac{1}{r} \frac{d\varrho}{dr} + \left(\frac{\zeta_i - \beta}{D_a} \right) \varrho \\ = \frac{\alpha}{r} \frac{d}{dr} \left[\frac{N}{1 + N/pC} \int_0^r \frac{\varrho r dr}{1 + N/pC} \right. \\ \left. + \frac{\varrho}{1 + N/pC} \int_0^r \frac{Nr dr}{1 + N/pC} \right]. \end{aligned} \quad (83)$$

Setting

$$z = \varrho/\varrho_0, \quad (84)$$

$$v = r[(\zeta_i - \beta)/D_a]^{\frac{1}{2}}, \quad (85)$$

$$y = N/N_0, \quad (86)$$

$$y_s = N_0/pC, \quad (87)$$

we have

$$\begin{aligned} \frac{d^2 z}{dv^2} + \frac{1}{v} \frac{dz}{dv} + z = \frac{A'}{v} \frac{d}{dv} \left[\frac{y}{1 + y/y_s} \int_0^v \frac{zv dv}{1 + y/y_s} \right. \\ \left. + \frac{z}{1 + y/y_s} \int_0^v \frac{yv dv}{1 + y/y_s} \right] \end{aligned} \quad (88)$$

with

$$A' = N_0 b^+ (b^-)^2 E z^2 \mu e / (\zeta_i - \beta). \quad (89)$$

Setting:

$$w = v(1 + A')^{\frac{1}{2}}, \quad (90)$$

we obtain

$$\begin{aligned} \frac{dz}{dw} = -\frac{1}{w} \left\{ \frac{1}{1 + A'} \int_0^w zw dw + \frac{A'}{1 + A'} \right. \\ \times \left[\frac{y}{1 + y/y_s} \int_0^w \frac{zw dw}{1 + y/y_s} \right. \\ \left. + \frac{z}{1 + y/y_s} \int_0^w \frac{w dw}{1 + y/y_s} \right] \Bigg\}. \end{aligned} \quad (91)$$

This equation lends itself to iterative integration by the method used for Eqs. (23), (67) and (74). Such an integration has not been carried out as yet for the general case. We have investigated only certain special cases.

Special Case

Let us assume that A' is very great; Eq. (91) becomes

$$\begin{aligned} \frac{dz}{dw} = -\frac{1}{w} \left[\frac{y}{1 + (y/y_s)} \int_0^w \frac{zw dw}{1 + (y/y_s)} \right. \\ \left. + \frac{z}{1 + (y/y_s)} \int_0^w \frac{y w dw}{1 + (y/y_s)} \right]. \end{aligned} \quad (92)$$

If at the same time y_s tends to infinity, i.e., if the Gvosdover effect tends towards zero, the equation is further simplified to

$$\frac{dz}{dw} = -\frac{1}{w} \left[y \int_0^w zw dw + z \int_0^w y w dw \right]. \quad (93)$$

Furthermore, comparison of Eqs. (59), (66) and (61) with

$$w = v(1 + A')^{\frac{1}{2}} \quad (94)$$

shows readily that

$$w = u. \quad (95)$$

Iterative integration of Eq. (93) starting from the value of y given in Fig. 2 for $A = \infty$ leads to the result shown in Fig. 10. It will be seen that the perturbation z is not markedly different from the unperturbed function y . Figure 11 combines y and z linearly according to the formula

$$y^* = (y + \varrho^* z)/(1 + \varrho^*) \quad (96)$$

for positive and negative values of ϱ^* .

It is obvious that y^* is simply N^*/N_0^* and that the corresponding N^* is an approximate solution of the equation (76).

We now have to establish a relationship between ϱ^* and the logarithmic derivative of the current $dI/I dt$; this relationship may be established for the general case and not only for the specific case we have just treated. To this end, let us note that

$$I = -2\pi e R^2 N_0 b_{(1)}^-(E/p)_0 (1 + y_s^{-1})^{\frac{1}{2}} \times$$

$$\left[\frac{1}{U^2} \int_0^U \frac{y u du}{1 + y/y_s} + \frac{\varrho_0 \theta}{W^2} \int_0^W \frac{z w dw}{1 + y/y_s} \right]. \quad (97)$$

W is the value of w at the wall (from Eq. (114), Ref. 4). Under our hypotheses, the second integral term is negligible relative to the first. Also, we can write

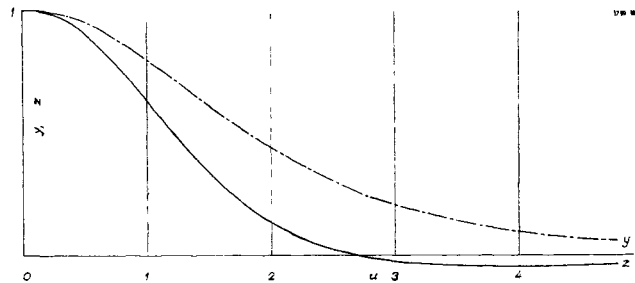


Figure 10

$$\frac{dI}{dt} = -2\pi e R^2 N_0 b_{(1)} (E/p)_0 (1 + y s^{-1})^{\frac{1}{2}} \times \frac{1}{W^2} \int_0^W \frac{z w dw}{1 + y/y_s} \varrho_0 \frac{d\theta}{dt} \quad (98)$$

since we assumed $N = N_0 y$ is independent of time while $n = N_0 \varrho_0 d\theta/dt$ accounts for all the variation with time. Dividing each member of (98) by the same member of (97), we have:

$$\frac{1}{I} \frac{dI}{dt} \simeq \frac{\frac{1}{W^2} \int_0^W \frac{z w dw}{1 + y/y_s} \varrho \frac{d\theta}{dt}}{\frac{1}{U^2} \int_0^U \frac{y u du}{1 + y/y_s}} \quad (99)$$

However, we are not interested in the value of $d\theta/dt$ which expresses the evolution of the distribution in time, but rather in the perturbation introduced, for a given value of I , by the fact that the current is variable with time. By analogy with many well-known phenomena of gaseous discharges, it is natural to consider this type of perturbation as being strictly equivalent to the cumulative effect of dI/dt changing sign for a time equal to the relaxation time of the ionization phenomena itself, which is none other than $1/\zeta_i$. We thus replace $d\theta/dt$ by $\theta^* \zeta_i$ in the second member. The value of ϱ^* to be selected in Fig. 11 is thus given by

$$\varrho^* = \varrho_0 \theta^* = - \frac{\frac{1}{U^2} \int_0^U \frac{y u du}{1 + y/y_s} \frac{dI}{dt}}{\frac{1}{W^2} \int_0^W \frac{z w dw}{1 + y/y_s} \frac{dI}{dt}} \quad (100)$$

Each of the integrals is defined and the value could be found if Eq. (1) had been integrated. Let us denote their ratio by k^* . We then have

$$\varrho^* = - \frac{k}{\zeta_i} \frac{1}{I} \frac{dI}{dt}; \quad (101)$$

k^* is a constant which can be computed in each specific instance. Thus, in the case of Eq. (93), we readily find that the value of k is close to 4.

Extreme Gvosdover Effect

Equation (79) is not valid in the case of an extreme Gvosdover effect since we assumed that $N^*/pC \ll 1$. When N^*/pC is sufficiently large for unity to become negligible, we must return to Eq. (76) which can readily be put in the form

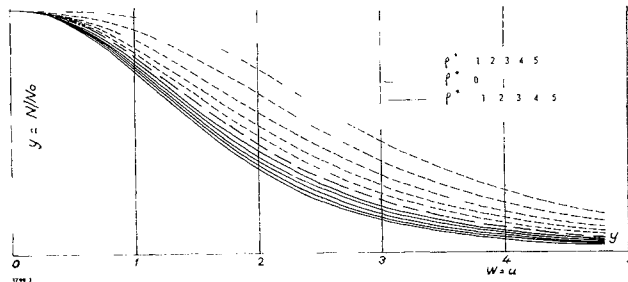


Figure 11

$$\frac{\partial^2 N^*}{\partial r^2} + \frac{1}{r} \frac{\partial N^*}{\partial r} + \frac{\zeta_i}{D_a} N^* - \frac{1}{D_a} \frac{\partial N}{\partial t} = \beta \quad (102)$$

with

$$\beta = \alpha p^2 C^2. \quad (103)$$

As a general rule, β will be a function of time but in the very first instants is not dependent on r . Equation (102) is not linear and is not soluble by separation of variables, nor by variation of parameters. On the other hand, it can easily be solved by writing

$$N^* = N + n, \quad (104)$$

N being a solution of the non-perturbed equation

$$\frac{d^2 N}{dr^2} + \frac{1}{r} \frac{dN}{dr} + \frac{\zeta_i}{D_a} N = \beta. \quad (105)$$

This equation is equivalent to Eq. (74). By substitution and subtraction, member for member, we easily find that n obeys the linear equation

$$\frac{\partial^2 n}{\partial r^2} + \frac{1}{r} \frac{\partial n}{\partial r} + \frac{\zeta_i}{D_a} n - \frac{1}{D_a} \frac{\partial n}{\partial t} = 0. \quad (106)$$

This is clearly a conventional equation, the solution of which is known. The part which is a function of r is a Bessel function J_0 having its first zero on the wall. As in the above special case, we note that the perturbation, to first order, is strikingly similar to the unperturbed function, as shown by the comparison of the various curves on Fig. 9, in which the Bessel function corresponds to $B = 0$. Therefore, here again the radial distribution of the concentration will not be much altered by the introduction of a non-zero time derivative of the current. Figure 12 carries out, in the above special case $B = \infty$ the same linear combination of the fundamental and the first-order perturbation, as Fig. 11, and Eq. (96). The relation between the ϱ^* and the logarithmic derivative of the current given by Eq. (100) or (101) holds good provided we assume $y_s \ll y$. Examination of the curves of Fig. 9 show that the constant k^* is very close to unity.

Generalization and Conclusions

Since both extreme cases give us quite comparable results, we feel we can interpolate our conclusions and state that, generally, the introduction of a temporal variation, even if it is relatively substantial on the scale of the relaxation time, does not palpably modify the radial distribution of the carriers. In the intermediate cases, the constant k^* of Eq. (101) would obviously assume a value which is intermediate between 4, corresponding to the vanishing of the Gvosdover effect, and the value 1, corresponding to the extreme Gvosdover effect. An examination of Figs. 11 and 12 seems to indicate that the conclusions drawn earlier will not be challenged in any case.

It should be noted, however, that the theory given above remains an approximation to the extent that it rules out radial variation of E_z during dynamic phenomena.

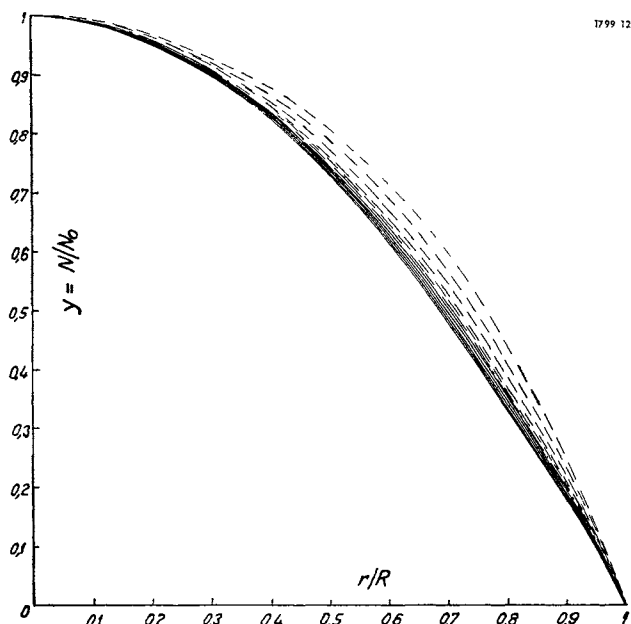


Figure 12. ϱ^* values identical to those of Fig. 11

Such a variation may indeed take place, if we superimpose on the electrostatic or almost electrostatic main field a field of the form $\mathbf{E} = -\partial \mathbf{A}/\partial t$, $\mathbf{A}$ being the vector potential due to the variations in current density in the various layers. These fields are functions of these variations only, and their relative importance will be great according as the main field is small compared to this perturbing field, or as the conductivity of the plasma and the logarithmic derivative of the current are high. Their injection into the above computation is very com-

plicated, except for a special case in which either the conductivity of the plasma, or the logarithmic derivative of the current is considered infinite, in which case the Lenz' law requires that $\partial \mathbf{B}/\partial t = 0$ in the interior of the plasma, i.e., a purely superficial current, such as that described by Post.⁸ Such a description may apply in the plasma of a thermonuclear reactor, and also appears to be achieved at the very beginning of the contraction,¹³ but does not strike us as adequate for describing its essential phase, during which experiment shows that the current density is re-established according to a distribution which is essentially in keeping with our theory.¹³ It would doubtless be most interesting to investigate the transitions between the phase of conductivity throughout the mass, and those of purely pellicular conductivity at the very beginning (when the logarithmic derivative of the current is enormous) or at the end of the contraction (when the plasma, completely ionized and at a very high temperature, has acquired practically infinite conductivity). It would not seem impossible to generalize the above theory to the case of a longitudinal field which is a function of r provided we retain some simplifying hypotheses, such as ζ , D_a , T and p independent of r . However, even these hypotheses then seem dubious to us.

Nevertheless, by reasoning in a purely physical manner, we get the impression that this investigation could hardly lead to a modification of our main conclusion, that the Gvosdover effect, which is marked in the first moments, should oppose the pinch of the discharge. Indeed, we cannot see how the fact that the current is occasionally partially or totally pellicular would favor a contraction which, in its absence, has no tendency at all to appear.

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The Static Pinch

By Conrad L. Longmire *

In a pinch, the outward diffusion of plasma due to collisions can be balanced by the inward drift resulting from $\mathbf{E} \times \mathbf{B}$, where $\mathbf{E}$ is the applied electric field and $\mathbf{B}$ the magnetic field. From the equation expressing the balance of these two effects, together with the pressure balance equation, one obtains the "perpendicular" conductivity, which is about one-half of the classical "parallel" conductivity. This result has been applied to the problem of a static pinch under the assumptions:

- (1) there is an applied longitudinal (B_z) magnetic field;
- (2) the plasma is isothermal;
- (3) the solution depends only on the radial coordinate.

From the Maxwell $\nabla \times \mathbf{B}$ equation, and from the parallel and perpendicular conductivity laws, one can deduce the equations governing the radial distribution of $\mathbf{B}$:

$$b_z(db_z/dr) + (b_\theta/r)d(rb_\theta)/dr = b_\theta, \quad (1)$$

$$(b_z/r)d(rb_\theta)/dr - b_\theta(db_z/dr) = \frac{1}{2}b_z,$$

where $b_z = B_z/B_z(0)$, $b_\theta = B_\theta/B_z(0)$, and $B_z(0)$ is the value of B_z at the axis of symmetry. The unit of length in Eq. (1) is

$$l = B_z(0)/2\pi\sigma E \text{ (Gaussian units),} \quad (2)$$

where E is the applied electric field and σ is the parallel conductivity.¹ The equation for the pressure balance relates the particle density n to the magnetic field:

$$d(n/n_0)/dr = b_\theta/\beta; \quad \beta \equiv 8\pi n_0 kT/B_z^2(0), \quad (3)$$

where n_0 is the particle density at the center of the pinch.

Equations (1) have a universal solution inside the plasma, and Eq. (3) has a solution that depends very simply on the parameter β . Typical results are shown in Fig. 1. The universal solution of Eq. (1) is terminated at the plasma boundary, $r = r_1$. For $r > r_1$, we have $b_z = \text{constant}$ and $b_\theta \sim 1/r$. The position r_1 is determined by the parameter β .

Equations (1) do not admit negative b_z . It seems likely that this result is a consequence of the assumption of isothermality. The general form of b_z and b_θ is quite similar to some experimentally determined distributions, except that a negative b_z has been observed in certain experiments.

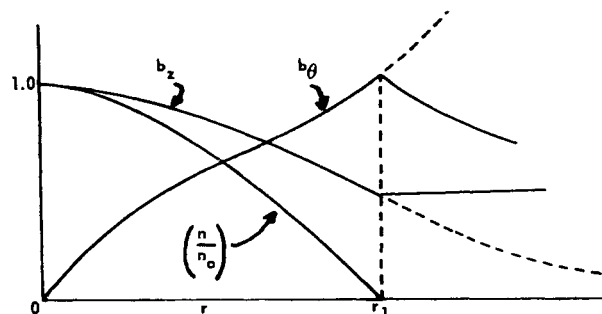


Figure 1

In spite of diffusion, it is interesting to note that the plasma can have a sharp edge which need not be in contact with the walls of the system. However, the plasma distribution is unstable according to the Suydam² criterion for interchange instability. This result is independent of β . Furthermore, an attempt to find a solution with a convective (unstable) core but with a stable envelope has been unsuccessful.

A more detailed account of this work will be published as a Los Alamos report.

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Stability and Heating in the Pinch Effect

By M. N. Rosenbluth *

One of the most promising types of thermonuclear device is the stabilized pinch.^{1, 2} This consists of a pinched cylindrical plasma in which a longitudinal magnetic field is trapped. This field provides rigidity against the various types of instability to which the pinch is subject. An external conductor which confines the magnetic field of the pinch provides additional stability, so that, with a proper choice of parameters to define the equilibrium, the configuration may be made linearly stable against all perturbations.

In the first part of this paper we shall discuss some surface instabilities which may arise in the stabilized pinch. In the second part we shall discuss the disassembly and heating of the plasma which results from collisions.

SURFACE-LAYER INSTABILITIES

Previous calculations of stability^{1, 2} have dealt mainly with an equilibrium with a very sharp surface layer such as might be expected with a very highly conducting plasma.³ That is, for $0 < r < r_0$, we have

$$B_\theta = 0; \quad B_z = \alpha_p B_0; \quad p = (1 + \alpha_p^2 - \alpha_p^2)(B_0^2/8\pi) \\ \text{and for } \beta r_0 > r > r_0 + \delta, \text{ we have} \\ B_\theta = B_0 r_0/r; \quad B_z = \alpha_v B_0; \quad p = 0.$$

(There is a glossary of symbols at the end of the paper.)

The radius of the external conductor is βr_0 . In the thin region between r_0 and $r_0 + \delta$, large surface currents must flow. However, it would appear at first sight that in the limit of small δ this complicated region need not be considered explicitly, since, in the dynamic equations for the perturbation, one may use the conditions of continuity of the normal components of the magnetic field and stress tensor to relate the interior solutions to the exterior ones. We shall now try to give a more complete discussion of the above-defined problem.

It has been shown previously⁴ that if the magnitude of B does not change along a field line, as is the case here, and if all distances are large compared with a

Larmor radius and we confine ourselves to the marginal-stability case $\omega = 0$, then the standard magneto-hydrodynamic equations are valid. That is, the governing equations for the perturbation are

$$\nabla \delta p = [(\nabla \times \mathbf{B}) \times \delta \mathbf{B} + (\nabla \times \delta \mathbf{B}) \times \mathbf{B}]/4\pi \quad (1)$$

$$\nabla \cdot \delta \mathbf{B} = 0, \quad (2)$$

where $\delta \mathbf{B}$ and δp are the perturbed quantities and $\mathbf{B}$ is the equilibrium magnetic field.

The boundary conditions are regularity at the origin and $\delta B_r = 0$ at $r = \beta r_0$. For the simple cylindrical geometry of the problem, the perturbations may be expanded in normal modes, i.e.,

$$\delta p, \delta \mathbf{B} = (p_1, \mathbf{B}_1) e^{i(kZ + m\theta)}. \quad (3)$$

With this substitution, Eqs. (1) and (2) may be easily reduced to a single equation as follows:

Let

$$X = r B_{1r} / [k B_z + (m/r) B_\theta],$$

then

$$\frac{d}{dr} F(r) \frac{dX}{dr} + G(r) X = 0, \quad (4)$$

where

$$F(r) = \frac{r}{m^2 + k^2 r^2} \left(k B_z + \frac{m}{r} B_\theta \right)^2 \quad (5a)$$

and

$$G(r) = \frac{2B_\theta}{r^2} \left(\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \right) - \frac{1}{r} \left(k B_z + \frac{m}{r} B_\theta \right)^2 \\ - \frac{(2mB_\theta)^2}{r^3(m^2 + k^2 r^2)} \\ - \frac{d}{dr} \left\{ \frac{2mB_\theta [k B_z + (m/r) B_\theta]}{r(m^2 + k^2 r^2)} \right\}. \quad (5b)$$

Since we are considering the marginal-stability problem, it follows that if the solution of Eq. (4) which is regular at the origin should vanish at the external conductor, then the equilibrium is neutrally stable to the perturbation being considered. If the solution crosses the axis before reaching the external conductor, this implies instability, since we should have to move the conductor in to stabilize the system.

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For the general equilibrium specified by arbitrary $B_\theta(r)$ and $B_z(r)$, Eq. (4) must, of course, be numerically integrated, although useful limits may be obtained from the variational expression^{5, 6} corresponding to Eq. (4). A code for the numerical investigation of Eq. (4) has been prepared. However, for the sharp-layer case, i.e., $\delta \ll r_0$, we may obtain analytic results using the following procedure:

For $r < r_0$ and $r > r_0 + \delta$, i.e., where there is no current, Eq. (4) may be solved explicitly in terms of Bessel functions. This gives us values for the logarithmic derivatives which must be joined on to the solution in the surface layer.

Within the surface layer we can neglect $1/r$ compared to $\partial/\partial r$ and rewrite Eq. (4) in the form

$$\frac{d}{dr}(u + H) = \frac{u^2}{r_0^2 F}, \quad (6)$$

where

$$u = \frac{r_0^2 F}{B_0^2 X} \frac{dX}{dr}$$

and

$$H = \frac{(k^2 r_0^2 - m^2) B_\theta^2 - 2mkrB_z B_\theta}{(m^2 + k^2 r^2) B_0^2}. \quad (7)$$

In the region of interest, B_θ and B_z are changing rapidly. It is apparent from Eq. (5) that for some ranges of k and m , depending on α_v , $F(r)$ will pass through zero, so that Eq. (6) develops a singularity. Parenthetically we may remark that because of this singularity there is some difficulty about the derivation of the magnetohydrodynamic Eq. (1). This question is receiving further study, but for the purpose of this paper we regard the conventional mode as valid.

For modes which have no singularity, the right-hand side is negligible over the small distance δ , and we find as the stability criterion

$$[u + H]_{r_0} > [u + H]_{r_0 + \delta}. \quad (8)$$

For reference purposes, the explicit forms of these functions are

$$\begin{aligned} [u + H]_{r_0} &= \alpha_p^2 (kr_0)^2 \frac{I_m(kr_0)}{(kr_0) I_m'(kr_0)} \\ [u + H]_{r_0 + \delta} &= (m + \alpha_v kr_0)^2 \\ &\times \frac{G_{\beta, m}(kr_0) \frac{I_m(kr_0)}{(kr_0) I_m'(kr_0)} - \frac{K_m(kr_0)}{(kr_0) K_m'(kr_0)}}{1 - G_{\beta, m}(kr_0)} \end{aligned}$$

where

$$G_{\beta, m}(kr_0) = \frac{K_m'(\beta kr_0)}{K_m'(kr_0)} \frac{I_m'(kr_0)}{I_m'(\beta kr_0)}. \quad (9)$$

Equation (8) is, of course, identical with the old stability calculation.

In discussing the singular case it is convenient to consider the situation as we integrate towards the

singularity. Until we get very close to the singularity—more specifically, until $F \sim \delta$ —the quantity $u + H$ will remain constant, as before. This determines the value of u as we approach the singularity. Very near the singularity we may neglect the term H' , so that in this region

$$\frac{1}{u} + \int \frac{dr}{F} = \text{constant}.$$

Hence, $1/u$ becomes infinite at the singularity as the integral diverges. Moreover, depending on the sign of u as it approaches the singularity, $1/u$ may pass through zero. This corresponds to X crossing the axis, i.e., to instability. In order to avoid instability in this case we must then require that

$$[u + H]_{r_0} - [H]_s > 0, \quad (10a)$$

$$[H]_s - [u + H]_{r_0 + \delta} > 0. \quad (10b)$$

Here, $[H]_s$ means the value of H at the singularity, which can easily be shown to be

$$[H]_s = [B_\theta^2]_s / B_0^2. \quad (11)$$

It may seem strange that two criteria must be satisfied for the stability of a single mode. The explanation is that the singularity in Eq. (6) is so strong that it completely separates the region interior to the singularity from the exterior. Thus, if we violate Eq. (10b), for example, the instability is essentially confined to the outside of the surface layer. The new stability criteria are considerably more complicated than the old one, since they depend on the structure of the surface layer. This structure may be described by giving B_θ^2 as a function of $\varphi = \tan^{-1} B_\theta/B_z$, the pitch of the field. Equation (10) then gives upper and lower limits for B_θ^2 . It will be noted that the value of φ at the singularity is simply related to k and m by $kr/m + \tan \varphi_s = 0$. The range of possible layer structures is further limited by the requirement that the plasma pressure be positive throughout. This gives

$$B_\theta^2 = B^2 \sin^2 \varphi < B_0^2 (1 + \alpha_v^2) \sin^2 \varphi. \quad (12)$$

These results are plotted in Fig. 1 for $\alpha_v = \pm 0.25$. As usual, we need consider only $m = 0, \pm 1$.

Curve I is the function $u + H$ divided by α_p^2 evaluated at r_0 . Curve II is $u + H$ evaluated at $r_0 + \delta$ with various β for $m = \pm 1$. The x 's are the $m = 0$ values of $u + H$ evaluated at $r_0 + \delta$, evaluated at $k = 0$, the most dangerous case. Equation (8), which must be satisfied for all k and m , hence all φ , and for both values of α_v , requires that Curve II lie above Curve I. This means that $\beta = 2.5$ is unstable.

The more stringent conditions (10) and (12) need only be applied to values of k and m so that a singularity develops within the layer. This means, for the case of positive α_v , the region to the right of $\bar{\varphi} = \tan^{-1} 1/|\alpha_v|$ and, for negative α_v , to the left of this point. Condition (10) requires that at

$$\bar{\varphi} = \tan^{-1} [-(\alpha_v/|\alpha_v|)(B_\theta/B_z)].$$

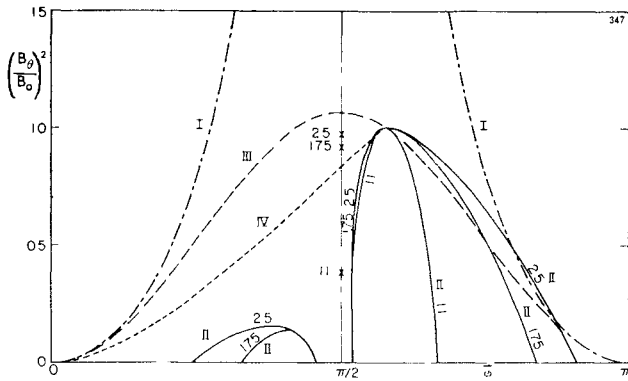


Figure 1. Stability diagram for $\alpha_v = \pm 0.25$ and $\beta = 1.1, 1.75, 2.5$

The shape of a thin surface layer is given by a graph of B_θ^2/B_0^2 vs φ . To insure stability, this shape must be between certain limits, shown in this figure for the case $\alpha_v = |\pm 0.25|$, $\beta = 1.1, 1.75, 2.5$. Previous^{1,2} stability criteria require only that curve I multiplied by α_p^2 must lie above curve II for all φ between 0 and π . In addition we must now require for the range of φ which occurs within the layer that the curve which defines the layer shape must itself lie between curve I (multiplied by α_p^2) and curve II and above the cross. All physical layers (with positive plasma pressure) must lie below curve III. It can be shown that curves III and II cross for all positive α_v so that no stable layer can be constructed

B_θ^2 must lie between Curves I and II and above the x . In addition, Eq. (12) requires B_θ^2 to be below Curve III. Hence, we deduce that there is no stable layer structure for $\alpha_v = +0.25$. Curve IV shows the type of layer which develops because of diffusion, as discussed below. This type of structure for $\alpha_v = -0.25$ is seen to be unstable against long-wavelength $m = 0$ modes, for the cases $\beta = 2.5$ and 1.75, and against $m = 1$ modes with pitch close to that of the external field.

It is, in fact, apparent that for positive α_v , i.e., the external field in the same direction as the internal field, no stable layers exist, since (10b) and (12) are incompatible. It appears that in most cases where Eq. (8) is satisfied and the field is reversed, some stable layers can exist. Moreover, their structure is not too improbable although it appears that the diffusion layer is likely to be unstable against long-wavelength $m = 0$ disturbances and $m = 1$ disturbances near the pitch of the external field. It will be noted that condition (10b) appears much more dangerous than (10a), so that the instabilities are limited to the outside regions of the plasma.

The importance of these instabilities is questionable. The bulk of the plasma is unaffected; moreover they are slow—the multiplication time is proportional to the surface layer width, and they cannot remain in the linear phase of growth for long. It appears likely that the surface will become unstable and then restabilize itself in a new equilibrium, perhaps helical rather than cylindrical. It is noteworthy that such helical equilibria tend to generate a reversed B_z , as required for stability. Eventually, as we have noted, diffusion will make the layer unstable and the cycle must be repeated.

While we have treated only the sharp-layer case, it is clear that the same qualitative features persist for thick layers. The most important practical question would seem to be under what conditions the instability can proceed far enough to drive a significant amount of plasma into the walls, thereby releasing impurities. Experimental work with a reversed B_z is in progress.

DISASSEMBLY AND HEATING

It is apparent that the stabilized pinch is not a true equilibrium when interparticle collisions are considered. These collisions imply a finite conductivity of the plasma,⁷ so that electric fields must be present to drive the currents. These fields imply a change of flux, i.e., a relaxation of the crossed fields. At the same time, ohmic heating must occur, and temperature gradients will lead to kinetic thermal conduction. As a consequence of these involved processes, the plasma is simultaneously disassembled and heated. We shall find that a proper choice of parameters leads to favorable conditions for a thermonuclear reaction with no external heating mechanism. Qualitatively, this is because the crossed-field configuration represents a higher energy state of the magnetic field than the state which will prevail after diffusion has occurred and the components of the fields have been mixed. The excess energy goes into heating the plasma. Of course, a quantitative analysis is required to estimate the magnitude of the effect and to compare the disassembly and heating rates.

We therefore study the evolution of the equilibrium situation described at the beginning of this paper. A detailed solution will be presented for the case with the initial conditions $\delta = 0$, a sharp layer; $\alpha_p \approx 1$, a low-temperature plasma; and $\alpha_v = 0$, a 90° angle between the fields. It is apparent from the diffusion character of the problem that early departure from the initial values is confined to the neighborhood of the surface. For analytic simplicity we shall therefore consider a plane problem, calling the initial surface-layer position $X = 0$ and letting the plasma extend $-\infty$ and the vacuum field to $+\infty$.

The equations of motion⁸ are: Maxwell's equations; the conservation of mass, momentum, and energy for the plasma; Ohm's Law; and the equation governing heat transport in the plasma.

These are

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (13)$$

$$\nabla \times \mathbf{B} = 4\pi \mathbf{j} \quad (14)$$

$$\frac{\delta \rho}{\delta t} + \nabla \cdot \rho \mathbf{v} = 0 \quad (15)$$

$$2\nabla(\rho kT) = \mathbf{j} \times \mathbf{B} \quad (16)$$

$$\frac{\partial}{\partial t} (3\rho kT + \nabla \cdot (\mathbf{Q} + 5\rho kT\mathbf{v}) - \mathbf{E} \cdot \mathbf{j}) = 0 \quad (17)$$

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma} - \frac{(\mathbf{j} \cdot \mathbf{B})\mathbf{B}}{2B^2\sigma} - \frac{3}{2} \frac{\rho k}{B^2\sigma} (\mathbf{B} \times \nabla T) \quad (18)$$

$$\mathbf{Q} = -\frac{\rho^2 T}{B^2\sigma} \left\{ \left(2 \frac{m_i}{m_e} \right)^{\frac{1}{2}} k \nabla T - \frac{3}{2\rho} (\mathbf{j} \times \mathbf{B}) \right\} \quad (19)$$

$$\sigma = \frac{3}{8} \sqrt{\frac{2}{\pi}} \frac{(kT)^{\frac{3}{2}}}{e^2 (m_e^{\frac{1}{2}}) \ln A} = \sigma (kT)^{\frac{3}{2}} \quad (20)$$

Here, $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, $\mathbf{j}$ the current density, ρ the number density of ions or electrons, k the Boltzmann constant, v the plasma velocity, m_i and m_e the ion and electron masses, and A the ratio of maximum to minimum impact parameter.

In Eq. (16) we have used the fact that the diffusion velocities are very subsonic, so that hydrostatic equilibrium is maintained. (Needless to say, we are not considering possible unstable motions.) It is assumed throughout that the distribution functions are close to Maxwellian, with equal temperatures for electrons and ions. This is easily shown to be the case after the diffusion wave has penetrated a distance large compared to an ion Larmor radius. Equations (18) through (20) are, of course, valid only within this limit. They have been derived⁸ from the transport equation by considering the small deviations from the Maxwell distribution which are necessary to compensate for the collision terms in the presence of field, density and temperature gradients. A convenient mathematical tool for this purpose is the Fokker-Planck Equation.⁹

For a plane problem with an initially sharp layer, the system of equation has a similarity solution of the variable $X/\sqrt{t}$. The resulting ordinary differential equations have been integrated on the General Atomic IBM-650 by Miss G. Roy. The solution is shown in terms of the appropriate dimensionless variables in Fig. 2.

The abscissa is the dimensionless similarity variable

$$\xi = \frac{1}{4} [\bar{\sigma} B_0^3 / (\pi \rho_0^3)^{\frac{1}{2}}]^{\frac{1}{2}} X / t^{\frac{1}{2}}.$$

Here, X is the Lagrangian coordinate representing the initial position of the mass point, B_0 and ρ_0 are the field strength and particle density in the undisturbed regions, ϕ is the angle the field makes with the original direction of the field in the plasma, and β is the dimensionless pressure, $\beta = 16\pi\rho kT/B_0^2$.

It will be seen that the plasma pressure reaches a maximum value of about 0.43 near $\xi = 1$. The field is about half uncrossed at this point. For $\xi < 1$ the solution is nearly isothermal; for $\xi > 1$ the density remains almost constant. In the following discussion we will refer to $\xi = 1$ as the depth of penetration of

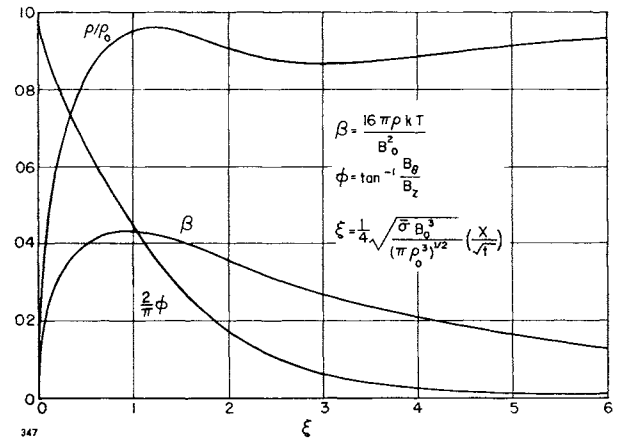


Figure 2. Solution of diffusion Equations 13-20 for the case $\alpha_p = 1, \alpha_v = 0$

The results of diffusion resulting from a case in which the plasma pressure is initially low compared with the magnetic pressure and in which the magnetic field in the vacuum is at right angles to the field in the plasma. Plasma pressure, plasma density, and magnetic field direction are plotted vs $X/\sqrt{t}$ where X is the Lagrangian distance from the original interface

the wave, and denote quantities at this point by the subscript p .

The following features of the solution seem of some significance.

1. The stabilized pinch needs only to be preheated to a temperature consistent with the recombination rate (about 100 eV). From this point on, the heating occurs naturally through intermixing of the fields. The plasma comes to a temperature

$$T_p \approx .5 B_0^2 / 16\pi\rho_0. \quad (21)$$

As mentioned earlier, this heating does not occur for uncrossed fields.⁸

2. The plasma-vacuum interface moves a distance of about $1.2 X_p$ into the space initially occupied by the vacuum field. Hence, in a typical pinch configuration no plasma will touch the wall until the wave has penetrated quite deeply.

3. The front of the wave is characterized by a nearly constant shear. As discussed in the first part of this paper, the resulting layer shape is linearly unstable against only a rather narrow region of wavelength perturbations if a negative B_z is provided.

4. The rate of penetration depends on the conductivity at the temperature of the heated plasma. In cgs units, with the temperature in kilovolts,

$$X_p = 18 t^{\frac{1}{2}} T_p^{-\frac{1}{2}}. \quad (22)$$

5. Similarly, the energy delivered to the plasma is

$$\frac{dE}{dt} = 25 \frac{B_0^2}{8\pi} \frac{1}{t^{\frac{1}{2}}} T_p^{-\frac{1}{2}} \text{ ergs/cm}^2 \text{ sec.} \quad (23)$$

The other important mechanisms in the energy balance of the burning pinch are fusion and radiative

loss; i.e., bremsstrahlung. Both of these produce energy according to

$$\frac{dE}{dt} = \rho_0^2 F(T) \text{ erg/cm}^3 \text{ sec.} \quad (24)$$

Thus, if the temperature is greater than a critical value, about 6 keV for D-T and 50 keV for D-D, the reaction is self-sustaining.¹⁰ It may be well to operate only slightly above this critical value to that disassembly is not substantially faster than that given by Eq. (24). At a fixed temperature, the disassembly time is proportional to r_0^2 , as may be seen from Eq. (22). Moreover, using Eqs. (24) and (21) we see that the burning time is proportional to B_0^{-2} . Hence, we should expect the efficiency to be a function of $B_0 r_0$ only, i.e., the current. In fact, it can be easily shown that if the quantities B_0^2/ρ_0 and $B_0 r_0$ are held fixed, the complete set of equations for disassembly, heating, and burning can be simply scaled to a change in r_0 . We may note that the currents must be at least 2×10^6 amp to contain the α particle resulting from the D-T reaction.

Finally, we indicate in Table 1 a possible D-T reactor design, taking an optimistic criterion for disassembly, i.e., $X_p = r_0$. The figures should be taken as only a rough indication.

Table 1. Possible Characteristics of a Diffusion-limited, Self-heated D-T Reactor

Quantity	Value	Scaling
Major radius of torus	30 cm (arbitrary)	r_0
Minor radius of torus	6 cm	r_0
Initial plasma radius (r_0)	1.5 cm	r_0
Current	3×10^6 amp	$(r_0)^0$
Pressure at wall	400 atm	r_0^{-2}
Initial pinched density (ρ_0) . . .	$1.3 \times 10^{17}/\text{cm}^3$	r_0^{-2}
Burning temperature (T_p)	6 keV	$(r_0)^0$
Disassembly time	0.15 sec	r_0^2
Total magnetic energy	4×10^6 joule	r_0
Losses (copper torus)	2.5×10^6 joule	r_0
Energy produced	3×10^7 joule	r_0
Temperature rise of copper surface due to radiation	500°C	r_0^{-2}
Efficiency of burning	10%	$(r_0)^0$

ACKNOWLEDGEMENTS

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The qualitative features of the intermixing heating were realized independently by S. A. Colgate and J. L. Tuck. It is a particular pleasure to acknowledge some stimulating discussions with Dr. Colgate and with A. N. Kaufman, who independently had studied some of these questions.⁸

GLOSSARY OF SYMBOLS

r_0	radius of pinch
B_θ	unperturbed azimuthal field
B_z	unperturbed longitudinal field
B_0	value of azimuthal field, B_θ , at r_0
α_p	ratio of internal longitudinal field, B_z , to B_0 .
α_v	ratio of external longitudinal field, B_z , to B_0 . α_v is defined as positive if the external field has the same sign as the internal one
β	(as used in first part) ratio of external conductor radius to r_0
p	plasma pressure
δ	thickness of surface layer
k	longitudinal wave number of perturbation
m	azimuthal wave number of perturbation
$\mathbf{B}_1$	perturbed magnetic field
X, F, G	defined in Eq. (4)
u, H	defined in Eq. (7)
$I(X), K(X)$	Bessel functions in Watson's notation
$[B_\theta^2]_s$	value of B_θ^2 at the radius where $F(r) = 0$
φ	$\tan^{-1}(B_\theta/B_z)$
σ	plasma conductivity
β	(as used in second part) $16\pi\sigma kT/B_0^2$
ξ	a dimensionless variable defining the depth of penetration
T_p	temperature at $\xi = 1$
ρ_0	undisturbed plasma density

Mr. Rosenbluth presented a survey, at the Conference, of Papers P/347 (above), P/354, P/1861, P/376 and P/2433:

Linear stability theory in a magnetohydrodynamic, collision-dominated fluid is a fairly well understood subject.¹¹ However, in a high-temperature plasma in which the particles interact only through the macroscopic fields, the situation is not so clear. Let us begin by discussing the types of waves characteristic of an infinite homogeneous plasma with constant magnetic field.

The situation is shown in Table 2. The magnetic field is taken to be in the Z direction, and we consider a wave propagating in the X, Z plane. The distribution function is an arbitrary function of the magnitude of velocity and its component parallel to the field. In general, four types of wave appear possible. One of these is a trivial mass flow in the direction of the field which causes no charge or

Table 2. Types of Plasma Waves

$$B = B_z; \mathbf{E} = \mathbf{E} \exp [i (\omega t + k_X X + k_Z Z)]; f = f(v^2, v_Z)$$

Type	$\mathbf{E}$	Character	Stability
Plasma oscillation . .	$\mathbf{E}_k$	Electrostatic $\omega \approx \omega_p = \sqrt{4\pi n e^2 / m_e}$ Landau damped	Overstability occurs if groups of electrons have a v_Z substantially exceeding mean thermal velocity.
Alfvén waves	$\mathbf{E}_x$	Transverse $\frac{\omega}{k_z} \approx \sqrt{\left(\frac{B^2}{4\pi Q}\right)}$ Undamped-Incompressible	Overstable for long wave-lengths if ion distribution is anisotropic.
Hydromagnetic waves	$\mathbf{E}_y$	$\frac{\omega}{k_X} \approx \sqrt{\left(\frac{B^2}{4\pi Q} + P\right)}$	Unstable if $\frac{P_{\perp}^2}{P_{\parallel}} > \frac{B^2}{8\pi}$ or $P_{\parallel} - P_{\perp} > \frac{B^2}{4\pi}$

current. The other three are indicated in Fig. 3 with a qualitative discussion of their properties. The comments are not inclusive or exact.

If the distribution function is isotropic and a decreasing function of energy, it is easy to show on general statistical grounds that the plasma must be stable.¹² However, it appears that even in the infinite homogeneous case, small deviations in the distribution function may lead to alarming instabilities.

For the plasma oscillations, overstability, i.e., growing oscillations, occurs for a wavelength such that $k_Z U = \omega_p$, where U is the velocity of a non-thermal group of electrons, or of electrons and ions relative to each other.¹³ Thus, these particles move in phase with the disturbance. It will be noted that the resonant particles see a non-oscillatory electric field and hence can move across magnetic field lines.

In the event that plasma heating is produced by electric fields parallel to the magnetic-field lines, we may expect that groups of high-energy electrons will be readily created because of the fall-off of cross section with energy. Thus, such a parallel field leads naturally to an unstable situation. The Alfvén-wave instability depends on a similar resonance for $\omega + k_Z v_Z = \text{Larmor frequency}$.¹³ It is particularly noteworthy since it occurs even for very small pressure anisotropies.

The hydromagnetic waves, which are the most nearly analogous to ordinary sound waves, become violently unstable for large pressure anisotropies.¹⁴ In particular, a shock perpendicular to the field lines creates a large transverse pressure, $P_{\perp}$. The resulting

instability may play an essential role in producing the entropy necessary for the existence of the shock. Thus we see that there are many unstable situations even for a simple infinite plasma.

When we consider a finite geometry, the situation is, of course, much more complicated. Progress to date has been made largely by assuming that the characteristic scale of inhomogeneities is large compared to Larmor radius and Debye length and that frequencies are small compared to orbital and plasma frequencies. As we have heard in the paper presented by M. D. Kruskal,¹⁵ a variational expression for stability with these approximations may be obtained which does not differ substantially from the fluid case. It should be noted that the plasma oscillation and Alfvén instabilities which we have discussed do not appear in this approximation. Thus, there has been no complete theoretical demonstration that any finite confined plasma can be stabilized.

However, even in this magnetohydrodynamic approximation it is difficult, although possible, to attain a stable static equilibrium. Perhaps the simplest situation to be studied is one in which the plasma contains no internal magnetic field. This constant-pressure plasma is then confined by an external magnetic field which must obey the condition that $B^2/8\pi = \text{pressure along the surface}$. It has been shown that a necessary and sufficient condition for stability is that the principal normal to the surface must at all points be directed into the plasma.¹⁶ This result is not restricted to the linear theory. The geometry is illustrated in Fig. 3. The shaded region in the diagram represents the plasma. On the left we have a convex unstable surface. This is the pinch. The cusp device on the right is concave at all parts and therefore stable. It appears to be the only possible confined stable equilibrium in which there is no field embedded in the plasma. Unfortunately, there is a finite rate of plasma loss through the cusps.

Another situation which has received a preliminary study is that of a plasma supported against gravity by a rotating magnetic field. The rotation slows down the instability but does not eliminate it.¹⁶

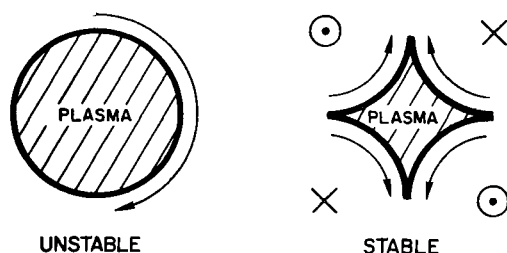


Figure 3

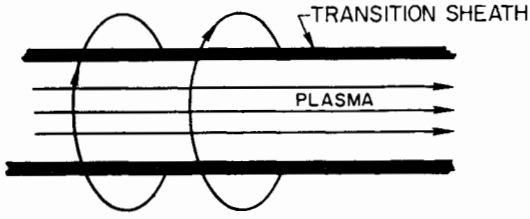


Figure 4. Stabilized pinch geometry

It is perhaps fair to say that the geometry which has received the most attention throughout the world is the stabilized pinch. It was realized early that the pinch was subject to the above-mentioned instability.¹⁷ In order to correct the situation, an axial magnetic field was introduced into the plasma. This situation is shown in Fig. 4. It was then shown by several authors that complete stability could be obtained by proper choice of the internal field and the position of an external conducting shell.¹⁸ However, these treatments neglected the structure of the current-carrying surface layer in which the field changes its direction. In practice, this sheath may be thick.

The equations of motion governing a perturbation of the form $e^{i(k_z + m\theta)}$ may be easily written. These equations develop a strong singularity at the radius where $kB_z + (m/r)B_\theta = 0$, that is to say, at such a radius that the pitch of the perturbation matches the spiral of the unperturbed magnetic field. This is perhaps not surprising since at this radius the perturbation will not bend the field lines appreciably and the plasma can flow freely along the lines.

The mathematical effect of the singularity is that the region exterior to the singular point is completely separated from the interior of the plasma. This brings into existence a class of surface instabilities which are not affected by the internal stabilizing field. The essential character of the instability is an azimuthal bunching of the parallel current filaments which exist at a given radius.

A useful necessary condition for stability¹⁹ is that at all points

$$\frac{B_z^2}{8\pi} \frac{r}{4} \left[\frac{d}{dr} \ln \frac{B_\theta}{rB_z} \right]^2 + \frac{d\phi}{dr} \geq 0.$$

Necessary and sufficient conditions have been found for the favorable case of a very thin transition layer.²⁰ It can be shown that only very special shapes of surface layer are stable. In particular, no stable surface can be found unless there is a longitudinal field outside the plasma which is opposite in direction to the internal field.

Let us now consider the diffusion and intermixing of the crossed magnetic fields due to interparticle collisions—without regard to stability. From the Rutherford cross section one can compute the various transport coefficients of the plasma—electrical and thermal conductivities and thermoelectric coefficients. Then the usual conservation equations—mass, momentum and energy—plus Maxwell's equations provide

a complete set of dynamical equations for the diffusion process.²⁰ In particular, we study the case of an initially sharp surface layer which separates a region containing low-pressure plasma and axial magnetic field from a vacuum region with azimuthal magnetic field; i.e., the stabilized pinch. For the early period of the diffusion, a plane approximation is adequate. In this case, the equations may be solved in terms of a similarity variable proportional to original distance from the interface divided by the square root of time.

The results are shown in Fig. 5. The abscissa is the dimensionless similarity variable; $\xi = 0$ is the initial interface. The scale is such that $\xi = 1$ is about the skin depth which one would estimate using the conductivity deduced from the temperature which exists at $\xi = 1$. ρ/ρ_0 is the plasma density relative to its initial value; ϕ is the pitch angle of the magnetic field; and β is the ratio of material pressure to magnetic pressure. The significant feature of the results is that most of the energy liberated by uncrossing the fields is delivered to the plasma,²¹ raising its pressure to about 0.43 of the initial magnetic pressure, regardless of the initial pressure. Hence, there exists a very efficient mechanism for creating very high temperature plasmas.

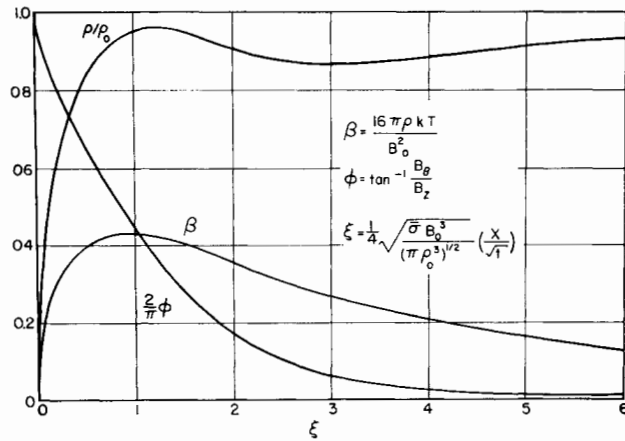


Figure 5. Solution of the dynamical equations for the plasma diffusion and field intermixing

Finally, we may inquire as to the eventual result of the diffusion process. If no external electric field is applied, the plasma will, of course, diffuse outward indefinitely until lost to the walls. On the other hand, an applied axial electric field is capable of causing a drift to balance out the collisional diffusion; and, in fact, detailed solutions have been found for a steady state of this type in collisional equilibrium away from the walls.²² Unfortunately, this equilibrium is very unstable hydrodynamically.

To sum up, recent theoretical work has shown that the stabilized pinch is a self-heating device. On the other hand, the existence of surface instabilities will require a careful programming of magnetic fields. In addition, the large electric fields used in pinch formation may well lead to the formation of unstable plasma waves.

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Plasma Radiation in a Magnetic Field

By B. A. Trubnikov and V. S. Kudryavtsev

ENERGY BALANCE OF A HIGH-TEMPERATURE PLASMA INCLUDING THE EFFECT OF MAGNETIC RADIATION

In order to study the possibilities of heating a plasma confined by an external magnetic field, it is necessary to take into account the presence of strong magnetic bremsstrahlung in the millimeter and sub-millimeter wavelength range. The existence of this radiation imposes a limit on the use of the D-D reaction to generate energy exceeding the input energy. In considering this effect, the self-absorption of this radiation should be taken into account.

The energy radiated per unit time by an electron moving along helical path in a magnetic field is given by the usual relation

$$d\mathcal{E}/dt = (2e^2/3c^3)\omega_c^2 v_\perp^2/(1 - \beta^2), \quad (1)$$

where $\omega_c \equiv |eH/m_0c|$, $\beta \equiv v/c$ and the other symbols have their usual meaning.

Assuming that electrons have a Maxwell distribution, the total radiated energy per unit time per unit volume is

$$Q_M = (4e^2/3c)(T_e/mc^2)\omega_c^2 n_e \times (1 + 5T_e/2mc^2 + \dots) \quad (2)$$

where n_e and T_e are the electron density and temperature, respectively.

Let us examine first the case when self-absorption may be neglected and all this radiation will escape the plasma. This escape of energy proves to be rather considerable at thermonuclear temperatures. In Fig. 1 we compare curves for the escape of magnetic radiation without self-absorption, the normal bremsstrahlung, and the energy release by D-D and D-T reactions, as functions of the temperature for the following conditions

$$\begin{aligned} T_e &= T_i \equiv T \\ n_e &= n_i \equiv n \\ \beta_H &\equiv 8\pi n(T_e + T_i)/H^2 = 1. \end{aligned} \quad (3)$$

We have used

$$\begin{aligned} F_M &= 0.005T^2(1 + 5T/2mc^2) \\ F_B &= 0.0532T^{3/2}\xi, \end{aligned} \quad (4)$$

where the quantity ξ is approximately unity.

One obtains the energy escape for the various processes in absolute units from the relation $Q = 10^{-22}n^2F(T)$, where T is in kev. Numerical values of F_{DD} and F_{DT} have been taken from Klimov and Zubarev (1952)* (Table 1).

At a temperature $T = 32$ kev, energy losses due to bremsstrahlung are equal to the nuclear energy released by the D-D reaction, $Q_B = Q_{DD}$. In this case the magnetic radiation proves to be 16 times as much, $Q_M = 16 Q_{DD}$. At a temperature $T = 5$ kev, these values are equal, $Q_B = Q_{DT} = Q_M$.

In a D-T reaction (deuterium and tritium mixture in one to one ratio), provided the conditions of Eq. (3) are satisfied, there exists a temperature range from 5 to 46 kev in which the nuclear energy released exceeds the energy loss due to magnetic radiation. For the D-D reaction, however, the magnetic radiation without self-absorption always proves to be many times greater than the production of nuclear energy. The conditions specified by Eq. (3) were adopted for the purpose of estimating the relative magnitude of these effects and may not always be satisfied in a practical situation.

If the energy loss due to electron radiation is compensated by the transfer of energy q^+ from ions to electrons via Coulomb collisions,

$$q^+ = 4(2\pi m)^{1/2} M^{-1} n^2 e^4 \lambda (T_i - T_e) T_e^{-3/2} \quad (5)$$

$\lambda \approx 20$

then at high ion temperatures $T_i \gg T_e$, so that $nT_i = H^2/8\pi$, and for Q_M equal to q^+ , we obtain an electron temperature

$$T_e = mc^2(3\lambda m/4(2\pi)^{1/2} M)^{2/5} = 39 \text{ kev} \quad (6)$$

that is independent of the temperature of the ions.

A more correct estimate of the electron temperature may be obtained if it is assumed that all the nuclear energy released is transferred from the ions to the electrons:

$$Q_{DD} = q^+, \quad T_e (\text{kev}) = 8.3[(T_i - T_e)/F_{DD}(T_i)]^{2/3}. \quad (7)$$

* *Ed. note*: D. N. Zubarev and V. N. Klimov, Stationary Regime of a Magnetic Thermonuclear Reactor, *Plasma Physics and Problems of Controlled Thermonuclear Reactions*, Vol. 1, p. 249 (in Russian), edited by M. A. Leontovich, Moscow (1958).

Table 1. Rate of D-D and D-T Energy Production as a Function of the Temperature

T (kev)	F_{DD}	F_{DT}
3	—	0.076
5	—	0.72
10	0.039	1.2
20	0.758	5.44
30	0.313	8.8
40	0.485	11.03
50	0.666	12.08
60	0.854	—
70	1.045	12.53
80	1.235	—
90	1.42	—
100	1.61	11.76
200	3.38	—
500	6.98	—
1000	10.7	—

For the D-D reaction, numerical results are given in Table 2.

However, the energy loss by magnetic radiation is always many times greater than the nuclear energy released in the D-D reaction. In contrast to this, we have for the D-T reaction with $T_e = T_i$ and $\beta_H = 1$,

$$Q_{DT} > Q_M + Q_B \text{ when } 7 \text{ kev} < T < 45 \text{ kev.} \quad (8)$$

However, the magnetic radiation from the individual electrons will be self-absorbed in the plasma itself in contrast to the bremsstrahlung, so that the power which escapes from a plasma with a volume V through the surface S is given by

$$W_M = W_1 S = K_L Q_M V. \quad (9)$$

Here W_1 is the energy emitted per second per unit area and K_L is the dimensionless transparency coefficient which indicates the fraction of the radiation from individual electrons that escapes from the plasma. Our aim is to determine the conditions under which the coefficient K_L will be small enough to provide the necessary degree of containment of the magnetic radiation inside the plasma. For the sake of simplicity, let us study a plane plasma layer of thickness L in which there is a magnetic field parallel to the layer boundary (Fig. 2).

It is evident that the transparency of an infinitely thin layer is unity,

$$\lim_{L \rightarrow 0} K_L = 1 \quad (10)$$

If the layer thickness is increased, the function K_L decreases monotonically and, for large L , behaves asymptotically like L^{-1} .

$$K_L \approx (2/L)(W_1)_{black} \\ (W_1)_{black} \equiv \sigma T^4. \quad (11)$$

Table 2. Ion and Electron Temperatures when the Energy from the D-D Reaction is Transferred to the Electrons

T_i kev	50	100	200	500	1000	2000
T_e kev	42.3	64.8	86	120	154	200

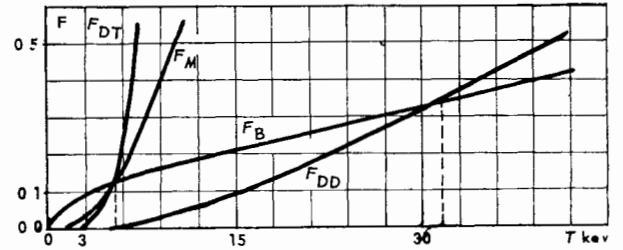


Figure 1

The factor 2 is included because two surfaces of the layer are radiating. However, for the conditions of interest, black-body conditions are never attained over the entire spectrum. Only the low frequencies, which have a spectral intensity described by the Rayleigh-Jeans radiation law

$$dI^i/d\omega d\Omega \equiv I_{RJ} = \omega^2 T_e / 8\pi^3 c^2, \quad (12)$$

where the superscript i denotes the polarization, are effectively trapped in the plasma. Then the flux per unit surface area (untrapped harmonics are neglected) will be equal to

$$W_1 \simeq 2 \int_0^{\omega^*} d\omega \int_0^{\pi/2} I_{RJ} \cos \phi d\Omega d\omega \\ = (\omega^*)^3 T_e / 12\pi^2 c^2. \quad (13)$$

Here ω^* is the maximum frequency for which the layer is opaque. To find this critical frequency it is necessary to compute the absorption coefficients $\alpha_{\omega^i}(\theta)$ for electromagnetic waves in a plasma situated in a magnetic field. Since ω^* is several times larger than ω_1 , it is sufficient for our purposes to calculate only the absorption coefficient for high frequencies where it is small and the index of refraction is nearly unity. This simplifies our calculations and the formulae given below are obtained in this fashion. For an equilibrium (Maxwell) velocity distribution of electrons, the $\alpha_{\omega^i}(\theta)$ are connected with the emission coefficients $\eta_{\omega^i}(\theta)$ by Kirchhoff's law

$$\eta_{\omega^i}(\theta) / \alpha_{\omega^i}(\theta) = I_{RJ}. \quad (14)$$

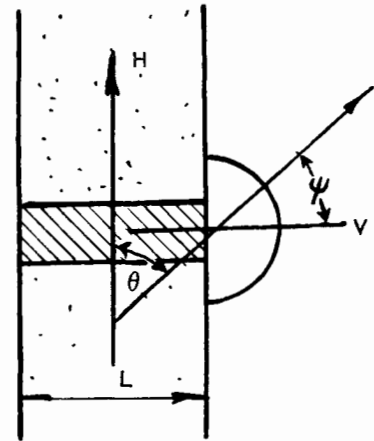


Figure 2

It is known that the spectrum of radiation due to the motion of an electron along a helical path in a magnetic field consists of discrete lines or harmonics, $I = \sum_{n=1}^{\infty} I_n$. The harmonic frequencies are integral multiples of the fundamental frequency,

$$\omega_1 = |eH/m_0c|/(1 - \beta^2)^{1/2}$$

with which the electron rotates in a magnetic field of strength H .

It may be shown that for circular motion of an electron, the ratio of the energy emitted at the fundamental frequency to the energy at all frequencies is ¹

$$I_1/I = I_1(\sum_{n=1}^{\infty} I_n)^{-1} \simeq 1 - 1.9\beta^2 + 1.05\beta^4 + \dots, \quad (15)$$

where $\beta = v/c$.

The temperatures which are of interest to us, and which are necessary for a self-sustaining D-D reaction ($T_e \sim 30$ to 100 kev), are weakly relativistic ($\beta \sim 0.5$) so that relativistic effects begin to play a role and a considerable fraction of the magnetic radiation is concentrated in frequencies with high harmonics. However, it should be noted that the radiation with the fundamental frequency is the most intense.

By integrating over the spectral distribution of the magnetic radiation of an individual electron, and averaging over a Maxwellian velocity distribution, the radiation power (emission coefficient) $\eta_{\omega}^i(\theta)$ may be determined. Then, from (14), the absorption coefficient, $\alpha_{\omega}^i(\theta)$ may be found. A plasma in a magnetic field is an anisotropic medium in which there is a possibility that two differently polarized waves may propagate. The fact that the *ordinary* and *extraordinary* waves are not radiated and absorbed in the same manner is unimportant for our present purpose.

Qualitatively, the emission coefficients and the absorption coefficients differ by factor ω^2 according to Eq. (14). The quantity $L\alpha_{\perp}$ is shown in Figure 3. The solid curve is obtained by adding together the radiation with different harmonic frequencies. The subscript $\perp$ denotes waves propagating normal to the direction of the magnetic field.

The line radiation in the higher harmonics is found to be Doppler broadened as a result of the thermal motion and merges into a continuous spectrum.

The values of the absorption coefficients, for several different temperatures, were determined by numerical computation and the spectral intensity was calculated. The radiation escaping in the direction normal to the surface of a plane layer is

$$I_{\perp} = I_{RJ}(2 - e^{-L\alpha^1} - e^{-L\alpha^2}).$$

Figures 4a and 4b show graphs of the function

$$f(X) = X^2(2 - e^{-L\alpha^1(X)} - e^{-L\alpha^2(X)}),$$

where $X \equiv \omega/\omega_c$, for two values of the parameter $\Lambda = L\omega_p^2/c\omega_c = 4\pi eLn/H$.

In those cases where the lines effectively merge into a continuous spectrum, a relatively simple

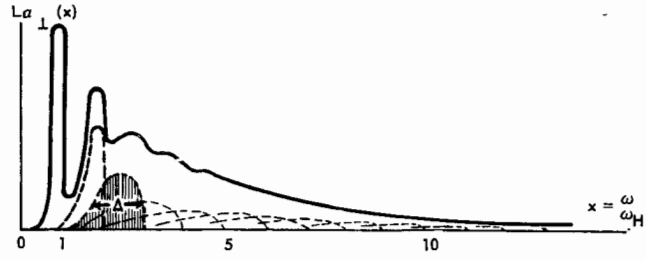


Figure 3. The dependence of $L\alpha_{\perp}(X)$ (the product of the absorption coefficient and the slab dimension) on the frequency measured in units of the fundamental, $X = \omega/\omega_c$

asymptotic formula may be obtained for the absorption coefficients

$$\alpha_{\omega}^{1,2} = \frac{\omega_p^2}{c\omega_c} \frac{3(\pi M)^{1/2}}{2\kappa} \exp \left[-\mu(\kappa^{1/2} - 1 + \frac{9}{20\kappa^{1/2}}) \right] \times \left\{ 1 + \frac{1}{\mu\kappa^{1/2}} \right\}, \quad (16)$$

where

$$\omega_p^2 \equiv 4\pi ne^2/m, \quad \mu \equiv mc^2/T_e, \quad \kappa \equiv 9\omega/2\mu\omega_c.$$

This formula is valid only when $\mu \gg 1$ and $\kappa \lesssim 1$. Since the absorption coefficients α_{ω} decrease drastically when ω is increased for the waves of both polarizations, the critical frequency ω^* (which determines where the Rayleigh-Jeans radiation should be cut off, see (13)) can be approximately determined from the equation

$$L\alpha_{\omega^*} \approx 1. \quad (17)$$

For frequencies below critical, $\omega < \omega^*$, the magnetic radiation is completely trapped and we have

$$L\alpha_{\omega} \gg 1, \quad (18)$$

and for the frequencies above critical, $\omega > \omega^*$, we have

$$L\alpha_{\omega} \ll 1, \quad (19)$$

when the plasma layer of thickness L is transparent. These super-critical harmonics escape freely from the plasma. However, their intensity, determined now directly from the formula for the radiation power η_{ω} , falls rapidly with increasing ω . For this reason, the emission of radiation with frequencies higher than ω^* may be neglected.

Substituting (16) into (17) yields

$$L\alpha_{\omega^*} \approx 1 = \Lambda \frac{3(\pi\mu)^{1/2}}{2\langle\kappa\rangle} \exp \left[-\mu(\langle\kappa\rangle^{1/2} - 1 + \frac{9}{20\langle\kappa\rangle^{1/2}}) \right], \quad (20)$$

where $\langle\kappa\rangle = 9X_{ef}/2\mu$ and $X_{ef} = \omega^*/\omega_c$ is the number of effectively trapped harmonics.

Substituting Q_M from (2) and W_1 from (13) into Eq. (10) for the *transparency coefficient* of the layer gives

$$\begin{aligned} K_L &= SW_1/Q_M V \\ &= 2W_1/LQ_M \\ &= X_{ef}^3/2\pi\Lambda(1 + 5T_e/2mc^2) \end{aligned} \quad (21)$$

where $1 + 5T_e/2mc^2$ is a correction factor close to

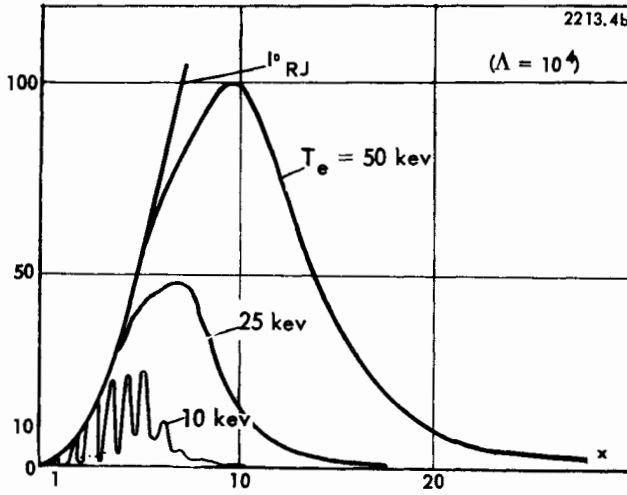


Figure 4a

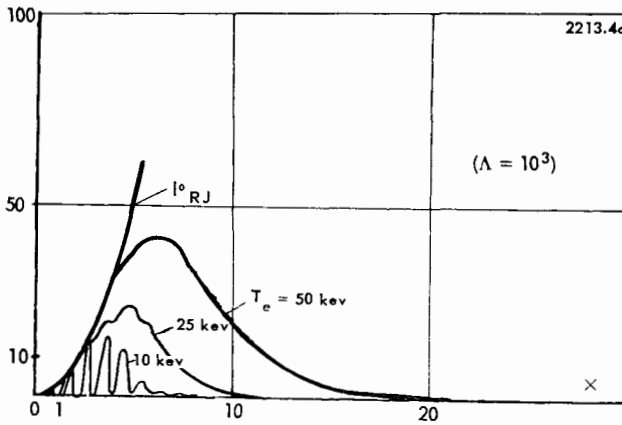


Figure 4b

unity. Thus K_L is a function of only two parameters, Λ and T_e :

$$K_L = \phi(\Lambda, T_e) \text{ where } \Lambda = 4\pi eLn/H. \quad (22)$$

In Fig. 5 the curves $K_L = \text{constant}$ are constructed from six calculated points with the aid of the asymptotic expression (20) in the range where it is valid.

If we assume that the bremsstrahlung Q_B escapes freely from the layer then the condition for a self-sustained thermonuclear reaction can be described by the inequality

$$VQ_{\text{nuc}} > K_L VQ_M + VQ_B. \quad (23)$$

As we have already pointed out, the trapping of the magnetic radiation for the D-T reaction is not a problem (see for example Eq. (8)).

For the D-D reaction, the minimum layer thickness L_{min} necessary for a self-sustained reaction may be found from the following relation (T in keV)

$$\frac{Q_M}{Q_{\text{DD}} - Q_B} = \frac{1}{\beta_H} \frac{2.5 \times 10^{-3} T_e (T_e + T_i) (1 + 5T_e/2mc^2)}{F_{\text{DD}}(T_i) - F_B(T_e)} = \frac{1}{K_L}. \quad (24)$$

When $T_e = T_i$ and $\beta_H = 1$ we have the following numerical results (Table 3).

The values of the parameter Λ , when T_e and K_L are known, can be obtained from Fig. 5 or from the asymptotic formulae (20) and (21). Assuming, for instance, that $H = 10^4$ gauss, we present in Table 4 $L_{\text{min}} = \Lambda H / 4\pi en$ for the same range of temperatures discussed in Table 3.

For $H = 10^5$ gauss, the densities n will be 100 times greater than in the previous example, while the dimension L_{min} will be 10 times less than the above-mentioned case.

Somewhat different results are obtained in cases where the electron temperature is less than the ion temperature. For this case we employ the equation $Q_{\text{DD}} = q^+$. Then with $\beta_H = 1$, we have the results tabulated in Table 5.

The values of L_{min} are determined here for $H = 10^4$ gauss and show the difficulties that are encountered in obtaining a self-sustained D-D reaction in an apparatus in which there is a large magnetic field inside a plasma where $H^2/8\pi \sim 2nT$. To diminish losses due to magnetic radiation definite measures must be taken. In particular, Budker has pointed out that effective reflectors which return energy back to the plasma may be used. It should also be noted that no magnetic radiation exists in systems in which plasma is confined by alternating external fields which do not penetrate inside the plasma.

ELECTRON DISTRIBUTION FUNCTION FOR A PLASMA IN A MAGNETIC FIELD

As has been indicated in the previous sections, the presence of a magnetic field in a limited volume of plasma results in a large radiation flux with a considerable part of the flux escaping from the plasma. Then the plasma is not in an equilibrium state and the electron velocity distribution function may be

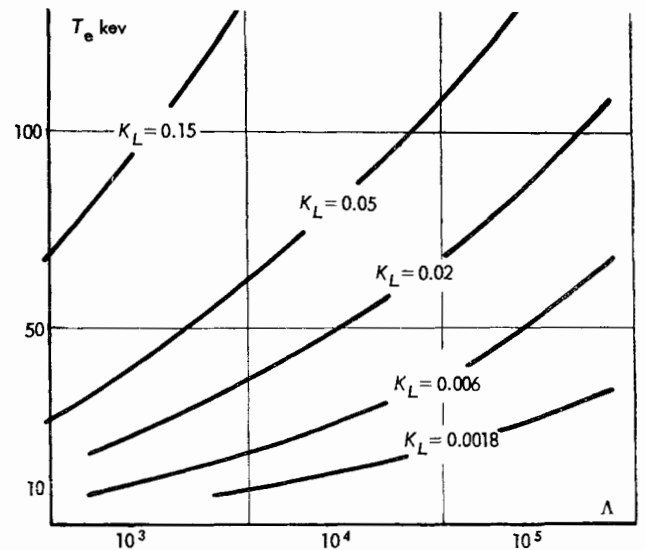
Figure 5. Transparency coefficient as a function of Λ and T_e

Table 3

T_e kev	32	35	40	50	60	70	80	90	100
$F_B(T_e)$	0.35	0.37	0.41	0.48	0.55	0.63	0.70	0.77	0.85
$Q_M/(Q_{DD}-Q_B)$.	∞	240	128	82	78	80	84	90	99
KL (%)	0.00	0.42	0.78	1.22	1.35	1.25	1.19	1.11	1.01
$\Lambda \times 10^{-4}$	∞	6.9	3.1	2.9	4.8	10	20	39	77.5
X_{ef}	∞	12	11.5	13	16	20	25	30	37

significantly different from a Maxwell distribution. Magnetic radiation is mainly due to the fast electrons and the relative number of these should therefore be less than in the equilibrium state. This could, in turn, lead to a decrease in magnetic radiation losses. In order to evaluate this effect, let us consider the following ideal process. Suppose that, as a result of Coulomb collisions, electrons receive energy from hot ions at a specified temperature T_i , and then lose this energy by radiation due to their motion in the magnetic field. The emitted radiation is not expected to be self-absorbed by the plasma.

In the stationary case, the kinetic equation for the electron distribution function f_e in a spatially uniform plasma takes the form †

$$\nabla_v \cdot (f_e/m) \mathbf{F} = \text{coll}(f, f) = -\nabla_v \cdot \mathbf{j} \quad (25)$$

where $\mathbf{j}$ is the particle flux in velocity space determined by collisions.² The force due to bremsstrahlung radiation should be substituted into (25) as the force acting on the electron

$$F_{\text{rad}} = -(2e^2/3c^3)\omega_c^2 \mathbf{v}_\perp \quad (26)$$

where $\mathbf{v}_\perp \perp \mathbf{H}$.

In order to simplify the problem an effective isotropic force $\mathbf{R}$ may be used instead of (26):

$$\mathbf{R} = \frac{2}{3}(2e^2/3c^3)\omega_c^2 \mathbf{v}. \quad (27)$$

It can be easily shown that these two expressions give the same total radiation intensity. Then the solution of the equation

$$\nabla_v \cdot [(f_e/m)\mathbf{R} + \mathbf{j}] = 0 \quad (28)$$

will yield a spherically symmetric electron distribution function $f_e = f_e(|\mathbf{v}|)$, which greatly reduces the complexity of the mathematical problem. In parti-

cular, an integral may be immediately obtained from Eq. (28):

$$(f_e/m)\mathbf{R} + \mathbf{j} = 0. \quad (29)$$

In the linear approximation, taking $x = mv^2/2T_e$, the distribution function can be written

$$f_e = f_0(1 + \phi); \quad |\phi| \ll 1, \quad f_0 = ce^{-x}. \quad (30)$$

Equation (29) is then transformed into an ordinary differential equation of the form

$$\phi'''(x) + (2F'/F - 1/2x)\phi''(x) = 0 \quad (31)$$

where

$$F = F(x) = \int_0^x \xi^{3/2} e^{-\xi} d\xi + x^{3/2} \int_x^\infty e^{-\xi} d\xi.$$

Finally, (31) may be reduced to the following form

$$\phi(x) \approx \alpha \pi^{-1/2} (-8x^{5/2}/15 + 7.1x + 2.2),$$

where

$$\alpha \equiv 2H^2(T_e/mc^2)^{3/2}/3(2\pi)^{1/2}\lambda nmc^2 \quad (32)$$

and λ is the usual Coulomb logarithm.

Thus we see that the Maxwell distribution is noticeably distorted starting with $x = x_{cr} \sim \alpha^{-2/5}$. For instance, for $H = 2 \times 10^4$ gauss, $n = 10^{14} \text{ cm}^{-3}$ and $T_e = 50$ kev, we obtain

$$\alpha = 1/475 \text{ for } \lambda = 20$$

$$\phi(x) = -1 \text{ for } x_c = 20. \quad (33)$$

We see that the distortion of the equilibrium distribution is important only at very high energies and does not lead, according to the calculations, to any significant decrease in the total magnetic radiation of the plasma layer. This distortion will prove to be still smaller if we consider the fact that under actual conditions the larger part of the magnetic radiation (as we have already seen) is absorbed by plasma. This contributes to a tendency toward thermodynamic equilibrium.

Finally the authors wish to express their gratitude to M. A. Leontovitch for his valuable suggestions and

† Ed. note: For the collision term $\text{coll}(f, f)$ there appears in the original $st(f, f)$, for stolknovitel'nii chlen.

Table 4

L_{min} m		32	17	19	38	93	210	460	1030
$n \times 10^{-13} \text{ cm}^{-3}$	3.9	3.6	3.1	2.5	2.1	1.8	1.6	1.4	1.2

Table 5

T_i	T_e	$Q_M / (Q_{DD} - Q_B)$	K_L	Λ	X_{ef}	L_{min}	$n \times 10^{-13}$
50	42.3	49	2%	$6.3 \cdot 10^3$	9.2	3.9 m	2.7
100	64.8	35	3%	$1.2 \cdot 10^4$	13	13 m	1.5
200	86	33	3%	$5 \cdot 10^4$	21	95 m	0.88
500	120	50	2%	$5 \cdot 10^5$	40	2.1 m	0.4

for discussions of the results of this work. The authors would also like to express their gratitude to A. E. Bazhanova, under whose guidance the numerical calculations in this work were carried out.

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High-frequency Plasma Oscillations

By **A. I. Akhiezer, Y. B. Fainberg, A. G. Sitenko, K. Stepanov, V. Kurilko, M. Gorbatenko and U. Kirochkin** *

It is well known that the electrical conductivity of a plasma, the ion-electron equilibration time, and the time required to heat the electron component of a plasma all increase greatly with increasing temperature. Consequently, the usual method of Joule heating a plasma may be difficult to apply in the region of high temperatures ($\gtrsim 10^6$ °K), especially if the plasma current alone, without any additional measures, is used to generate magnetic fields for the confinement of the plasma. Therefore, it is of interest to study methods of plasma heating that do not directly use Joule heat, especially methods by which energy is directly supplied to the ion component during the time between collisions. Some of these methods make use of ionic resonance as well as other resonance phenomena which can occur in a plasma in an external magnetic field. This paper deals with certain aspects of the theory of high-frequency plasma oscillations.

KINETIC THEORY OF THE OSCILLATIONS OF AN UNBOUNDED PLASMA IN A MAGNETIC FIELD

The high-frequency properties of a plasma may be studied most completely by means of the kinetic equation, in which it is possible to omit the collision operator. This equation may be expressed as follows for particles of the α species:^{1, 2}

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{e_\alpha}{m_\alpha} \mathbf{E} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \mp \omega_{c\alpha} \frac{\partial f_\alpha}{\partial \psi} = 0 \quad (1)$$

where $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ is the perturbation of the equilibrium distribution function, which we will denote as $f_{0\alpha}(v^2)$; e_α and m_α are the charge and mass of the α species; $\omega_{c\alpha} = |e_\alpha| H_0 / m_\alpha c$; and $\mathbf{H}_0$ is the external constant magnetic field intensity; $\mathbf{E}$ is the electric field intensity; the top and bottom sign in $\mp$ are for ions and electrons, respectively; and the angles are shown in Fig. 1.

It is not difficult to see that the electric field intensity $\mathbf{E}$ satisfies the following equation:

$$\nabla^2 \mathbf{E} - \nabla \nabla \cdot \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial}{\partial t} \sum_\alpha e_\alpha \int \mathbf{v} f_\alpha d^3 \mathbf{v}. \quad (2)$$

We will look for the quantities f_α and $\mathbf{E}$ in the form of plane waves

$$f_\alpha, \mathbf{E} \sim \exp i(\mathbf{k} \cdot \mathbf{r} - \omega' t), \quad \text{Im } \omega' > 0. \quad (3)$$

By substituting these forms into (1) and (2) we obtain

$$\sum_{k=1}^3 [n'^2 (\kappa_i \kappa_k - \delta_{ik}) + \varepsilon_{ik}] E_k = 0, \quad i = 1, 2, 3, \quad (4)$$

where

$$\varepsilon_{ik}(\omega', \mathbf{k}) = \delta_{ik} + \sum_\alpha \frac{4\pi i e_\alpha}{\omega' \omega_{c\alpha}} \int v_1 f'_{0\alpha} e^{ia_\alpha \sin \theta + ib_\alpha \theta} \times \int^\theta v_k e^{-ia_\alpha \sin \psi - ib_\alpha \psi} d\psi d\mathbf{v},$$

$$a_\alpha = \pm k v_\perp \sin \theta / \omega_\alpha, \quad b_\alpha = \pm (k v_3 \cos \theta - \omega') / \omega_\alpha,$$

$$n' = kc / \omega', \quad \mathbf{x} = \mathbf{k} / k, \quad f'_{0\alpha} = \frac{\partial f_{0\alpha}}{\partial \varepsilon}, \quad \varepsilon = m_\alpha v^2 / 2.$$

The quantities ε_{ik} form the dielectric tensor. We see that ε_{ik} depends on the wave vector as well as on the frequency. In other words, there is spatial dispersion in the plasma as well as time dispersion.

The following relation may be derived from Eq. (4):³

$$\det[n'^2 (\kappa_i \kappa_k - \delta_{ik}) + \varepsilon_{ik}] \equiv A n'^4 + B n'^2 + C = 0, \quad (5)$$

where

$$A \equiv \varepsilon_{11} \sin^2 \theta + \varepsilon_{33} \cos^2 \theta + 2\varepsilon_{13} \cos \theta \sin \theta,$$

$$B \equiv 2(\varepsilon_{12}\varepsilon_{23} - \varepsilon_{22}\varepsilon_{13}) \cos \theta \sin \theta + \varepsilon_{13}^2 - \varepsilon_{11}\varepsilon_{33} - (\varepsilon_{22}\varepsilon_{33} + \varepsilon_{23}^2) - (\varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2) \sin^2 \theta,$$

$$C \equiv \det(\varepsilon_{ik}).$$

(5) is used to determine the index of refraction of waves propagating in a plasma.

Electron Oscillations

If the ion motion is not taken into account in the above formulas, we obtain high-frequency electron oscillations.

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Oscillations at Zero Temperature

Let us consider, first of all, such oscillations when the plasma temperature T_e is equal to zero. The coefficients in the dispersion equation (5) do not depend on $\mathbf{k}$ when $T_e = 0$. Therefore the solution to Eq. (5) is

$$n_{\pm}^2 = [-B_0 \pm (B_0^2 - 4A_0C_0)^{1/2}]/2B_0 \quad (6)$$

where

$$\begin{aligned} A_0 &= 1 - u - v + uv \cos^2 \theta, \\ B_0 &= u(2 - v) - 2(1 - v)^2 - uv \cos^2 \theta, \\ C_0 &= (1 - v)^3 - u(1 - v), \\ u &= (\omega_{ce}/\omega)^2, \quad v = (\omega_{pe}/\omega)^2, \\ \omega_{pe}^2 &= 4\pi e^2 n/m_e. \end{aligned}$$

This equation determines the refraction indices for the ordinary and extraordinary waves in the "hydrodynamic" approximation.

Setting the coefficient A_0 equal to zero, we find the natural frequency for longitudinal plasma oscillations in a magnetic field with $T_e = 0$:⁶

$$\omega_{\pm}^2 = \frac{1}{2} (\omega_{pe}^2 + \omega_{ce}^2) \pm \frac{1}{2} [(\omega_{pe}^2 + \omega_{ce}^2)^2 - 4\omega_{pe}^2\omega_{ce}^2 \cos^2 \theta]^{1/2}. \quad (7)$$

Thermal Corrections

Now let us take into account the thermal corrections, assuming that $\omega_{ce} \gg kc_e$, where $c_e = (T_e/m_e)^{1/2}$ is the thermal velocity of the electrons. In this case, the dispersion equation becomes^{4, 5}

$$A_1 n^6 + (A_0 + B_1) n^4 + (B_0 + C_1) n^2 + C_0 = 0 \quad (8)$$

where

$$\begin{aligned} A_1 &= -\frac{c_e^2}{c^2} v \left\{ 3(1 - u) \cos^4 \theta \right. \\ &\quad \left. + \frac{6 - 3u + u^2}{(1 - u)^2} \cos^2 \theta \sin^2 \theta + \frac{3 \sin^4 \theta}{1 - 4u} \right\}, \\ B_1 &= \frac{c_e^2}{c^2} v \left\{ \frac{2(1 + u - v)}{1 - u} \cos^2 \theta \sin^2 \theta + (1 + \cos^2 \theta) \right. \\ &\quad \times \left[(1 - u - v) \left(3 \cos^2 \theta + \frac{\sin^2 \theta}{1 - u} \right) \right. \\ &\quad \left. + (1 - v) \left(\frac{1 + 3u}{1 - u} \cos^2 \theta + \frac{3 \sin^2 \theta}{1 - 4u} \right) \right] \\ &\quad \left. + \frac{2 \sin^2 \theta}{1 - u} \left[\frac{1 + 3u - v - uv}{1 - u} \cos^2 \theta \right. \right. \\ &\quad \left. \left. + \frac{2(1 - u)(1 + 2u - v)}{1 - 4u} \sin^2 \theta \right] \right\}, \\ C_1 &= -\frac{c_e^2}{c^2} v \left\{ \frac{2(1 - v)}{1 - u} \left[\frac{1 + 3u - v - uv}{1 - u} \cos^2 \theta \right. \right. \\ &\quad \left. \left. + \frac{2(1 - u)(1 + 2u - v)}{1 - 4u} \sin^2 \theta \right] \right. \\ &\quad \left. + [(1 - v)^2 - u] \left(3 \cos^2 \theta + \frac{\sin^2 \theta}{1 - u} \right) \right\}. \end{aligned}$$

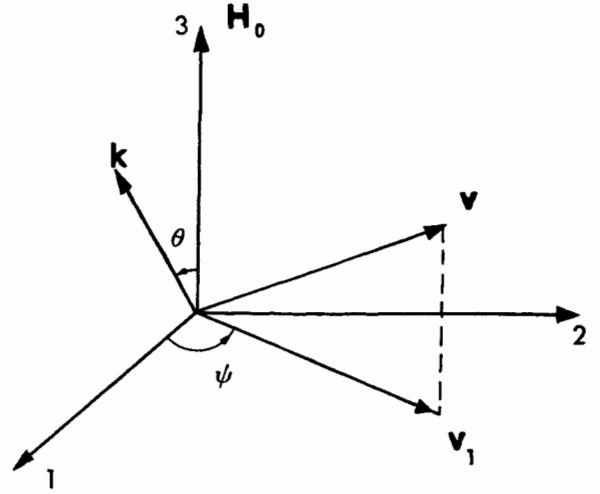


Figure 1

These equations have three roots n_1^2 , n_2^2 , n_3^2 , which determine the refraction indices of the ordinary, extraordinary and plasma waves, respectively. The first two roots are determined by

$$n_{1,2} = n_{\pm}(1 + \delta_{\pm}), \quad (9)$$

where

$$\delta_{\pm} = - (A_1 n_{\pm}^4 + B_1 n_{\pm}^2 + C_1) (2A_0 n_{\pm}^2 + B_0)^{-1}, \quad |\delta_{\pm}| \ll 1,$$

while the index of refraction n_3 for the plasma wave is determined by

$$n_3^2 = -A_0/A_1, \quad c_e/c \ll |A_0| \ll 1. \quad (10)$$

If the frequency is near ω_+ or ω_- , then³

$$\begin{aligned} n_1^2 &= -C_0/B_0, \\ n_{2,3}^2 &= [-A_0 \pm (\sqrt{A_0^2 - 4A_1B_0})]/2A_1. \end{aligned} \quad (11)$$

Let us consider the case of resonance, where $\omega \approx \omega_{ce}$. Here the indices of refraction for the ordinary and extraordinary waves are

$$n_{1,2}^2 = n_{\pm}^2(1 + \Delta_{\pm})$$

where

$$\begin{aligned} \Delta_{\pm} &= i \left(\frac{8}{\pi} \right)^{1/2} \frac{c_e \cos \theta}{c n_{\pm} v} \left\{ [1 - (\frac{1}{4} \sin^2 \theta + \cos^2 \theta) v] n_{\pm}^4 \right. \\ &\quad - [(1 - v)(1 - \frac{1}{4} v)(1 + \cos^2 \theta) \\ &\quad + (1 - \frac{1}{2} v) \sin^2 \theta] n_{\pm}^2 \\ &\quad \left. + (1 - v)(1 - \frac{1}{2} v) \right\} (2n_{\pm}^2 \sin^2 \theta \\ &\quad - 2v - 2 - \sin^2 \theta)^{-1}. \end{aligned} \quad (12)$$

We see that the resonance waves damp rapidly. The damping coefficient is of the same order as the quantity c_e/c ; it is much larger than the thermal corrections to the refraction indices for the ordinary and extraordinary waves.

If $\omega \approx m\omega_{ce}$, where $m = 2, 3, \dots$ (harmonic resonances), then

$$n_{1,2} = n_{\pm} + i\kappa_{\pm}, \quad (13)$$

where

$$\begin{aligned} \kappa_{\pm} &\sim (n_{\pm} c_e / c)^{2m-3} \sin^{2m-2} \theta \exp(-z_m^2), \\ z_m &= 2^{-\frac{1}{2}} (n_{\pm} c_e / c)^{-1} (1 - m\omega_{ce}/\omega). \end{aligned}$$

It should be noted that the damping is exponentially small far away from the resonance frequencies and is of the same order as the Landau damping:²

$$\gamma_L \sim \omega \exp(-\omega^2/2k^2c_e^2).$$

Let us now consider the longitudinal natural frequencies of the plasma. The electromagnetic waves in the plasma cannot be divided into strictly longitudinal and transverse waves in the presence of a magnetic field. However, for the limiting case of $n \rightarrow \infty$ the longitudinal plasma waves, which satisfy the condition $A = 0$, can be singled out. Taking the thermal correction into account, the roots of the equation $A = 0$ are given by^{3, 6}

$$\omega_{1,2}^2 = \omega_{\pm}^2 (1 + \varepsilon_{\pm}), \quad |\varepsilon_{\pm}| \ll 1, \quad (14)$$

where

$$\begin{aligned} \varepsilon_{\pm} &= \frac{k^2 c_e^2 v_{\pm} / \omega_{\pm}^2}{1 + v_{\pm} u_{\pm} (1 - u_{\pm})^{-2} \sin^2 \theta} \\ &\times \left\{ 3 \cos^4 \theta + \frac{6 - 3u_{\pm} + u_{\pm}^2}{(1 - u_{\pm})^3} \cos^2 \theta \sin^2 \theta \right. \\ &\left. + \frac{3 \sin^4 \theta}{(1 - u_{\pm})(1 - 4u_{\pm})} \right\}, \end{aligned}$$

$$v_{\pm} = (\omega_{pe}/\omega_{\pm})^2, \quad u_{\pm} = (\omega_{ce}/\omega_{\pm})^2.$$

The damping constant for the plasma oscillations is of the order of γ_L .

Let us consider the case where the frequency of the plasma oscillations ω_1 is close to $m\omega_{ce}$, $m = 2, 3, \dots$. If the angle between the direction of wave propagation and direction of the magnetic field is not near $\pi/2$, the resonant plasma oscillations will be strongly damped. The damping coefficient is³

$$\gamma_m = \frac{\pi^{\frac{1}{2}} m^4 \sin^{2m} \theta (kc_e/\omega_{ce})^{2m-4}}{2^{m+\frac{3}{2}} m! |\cos^3 \theta| [1 + m^4 (m^2 - 1)^{-2} \tan^2 \theta]} kc_e. \quad (15)$$

Plasma waves with frequencies in the interval $m\omega_{ce} - \varepsilon_m < \omega < m\omega_{ce} + \varepsilon_m$ cannot propagate perpendicular to the magnetic field. Here, the gap in the frequency spectrum is^{7, 3}

$$\varepsilon_m = (m^2 - 1)(2^{m+1} m!)^{-\frac{1}{2}} (kc_e/\omega_{ce})^{m-2} kc_e. \quad (16)$$

Ion-electron Oscillations

In the above analysis, we have considered plasma oscillations without taking the ion motion into account. Now we shall investigate low frequency plasma oscillations in which the ions as well as the electrons are in motion. These oscillations are usually described by means of the magnetohydrodynamic equations. This is valid only when the oscillation frequency is much smaller than the collision frequency ν . In reality, however, magnetohydro-

dynamic waves may exist for any relationship between ω and ν . It must only be assumed that the oscillation frequency is small compared with the ion cyclotron frequency ω_{ci} .^{8, 9}

It may be shown¹⁰ from Eqs. (1) and (2) that both an ordinary and an extraordinary magnetohydrodynamic wave exist for $\nu \ll \omega \ll \omega_{ci}$. The ordinary wave has the frequency

$$\omega_1 = kV_A \cos \theta; \quad V_A^2 = H_0^2/4\pi n_0 m_i^{\frac{1}{2}} \ll c, \quad (17)$$

and the extraordinary wave has the frequency

$$\omega_2 = kV_A. \quad (18)$$

When $\theta \sim 1$, the damping constant for the ordinary wave is given by:

$$\begin{aligned} (\gamma/\omega)_1 &\sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \frac{c_i}{V_A} \frac{\omega^2}{\omega_{ci}^2} \exp(-V_A^2/2c_e^2), \\ c_i &= (T_i/m_i)^{\frac{1}{2}} \ll V_A, \\ (\gamma/\omega)_1 &\sim \omega^2/\omega_{ci}^2, \quad c_i \sim V_A \\ (\gamma/\omega)_1 &\sim c_i^3 \omega^2/V_A^3 \omega_{ci}^2, \quad c_i \gg V_A. \end{aligned} \quad (19)$$

The damping constant for the extraordinary wave is given by

$$\begin{aligned} (\gamma/\omega)_2 &\sim (m_e/m_i)^{\frac{1}{2}} (c_i/V_A) \\ &\times \exp(-V_A^2/2c_e^2 \cos^2 \theta). \end{aligned} \quad (20)$$

These formulas, as well as Eqs. (12) and (15), do not include the damping caused by close collisions.

Formulas (19) and (20) may be used if the damping is small, i.e., $\gamma \ll \omega$. The rate of damping increases as the phase velocity falls. An ordinary magnetohydrodynamic wave is strongly damped ($\gamma_1 \sim \omega_1 \sim kV_A$), if $c_i^3 \omega_1^2 \gtrsim V_A^3 \omega_i^2$. An extraordinary wave begins to decay quickly if the Alfvén velocity V_A is comparable with the ion thermal velocity c_i .

Finally, let us consider low frequency longitudinal waves.¹³ In the absence of a magnetic field, the frequency of these waves is determined by the Tonks-Langmuir formula¹¹

$$\omega = \omega_0 = \omega_{pi} k a_e (1 + k^2 a_e^2)^{-\frac{1}{2}} \quad (21)$$

where

$$\omega_{pi}^2 = 4\pi e^2 n_0 / m_i; \quad a_e^2 = T_e / 4\pi e^2 n_0,$$

and the damping constant is¹²

$$\gamma = \gamma_0 = \omega_0 (\pi m_e / 8 m_i)^{\frac{1}{2}} - \omega_{pi} k a_e (1 + k^2 a_e^2)^{-2}. \quad (22)$$

Here it is assumed that $T_e \gg T_i$.

Equations (21) and (22) may also be used when there is a weak magnetic field that satisfies the condition $\omega_{ce} \ll kc_e$.

If there is a strong magnetic field, where $\omega_{ci} \gg kc_i$, two longitudinal waves may propagate in the plasma at frequencies ω_1 and ω_2 :

$$\begin{aligned} \omega_{1,2}^2 &= \frac{1}{2} \{ \omega_0^2 + \omega_{ci}^2 \pm [(\omega_0^2 + \omega_{ci}^2)^2 \\ &\quad - (2\omega_0 \omega_{ci} \cos \theta)^2]^{\frac{1}{2}} \}. \end{aligned} \quad (23)$$

These waves have the following damping coefficients

$$\gamma_{1,2} = (\pi/8)^{1/2} \omega_{1,2}^4 / k^3 a_e^3 |\cos \theta|^3 \\ \times [1 + \tan^2 \theta \omega_{1,2}^4 (\omega_{1,2}^2 - \omega_i^2)^{-2}] \omega_{pe}^2 \omega_{pi}^2. \quad (24)$$

In deriving (23) and (24) it was assumed that

$$|\omega - \omega_{ci}| \gg kc_i \cos \theta, \\ \omega \gg kc_i \cos \theta, \\ \omega \gg kc_e \cos \theta. \quad (25)$$

At intermediate magnetic field intensities, where $\omega_{ce} \gg kc_e$, $\omega_{ci} \ll kc_i$ and θ is not close to $\pi/2$, the frequency is obtained from (21) and the damping constant is

$$\gamma = \gamma_0 / |\cos \theta|. \quad (26)$$

WAVE GUIDE AND RESONANCE PROPERTIES OF A PLASMA CYLINDER IN A LONGITUDINAL MAGNETIC FIELD

In order to ascertain the possibility of using high frequencies to heat a plasma, let us look into the wave-guide and resonance properties of a plasma cylinder located in a magnetic field directed along the cylinder axis. It is necessary to consider wave propagation in a bounded plasma since the dispersive properties and the electromagnetic field distribution of an unbounded and bounded plasma may differ markedly.

Although only kinetic theory gives a complete picture of wave propagation, the basic features of the processes that are of interest to us can be found from a simpler set of equations, the two-fluid hydrodynamic equations

$$m_e \frac{d\mathbf{v}_e}{dt} = -e\mathbf{E} - \frac{e}{c} \mathbf{v}_e \times \mathbf{H}_0 + m_e \nu (\mathbf{v}_i - \mathbf{v}_e), \\ m_i \frac{d\mathbf{v}_i}{dt} = +e\mathbf{E} + \frac{e}{c} \mathbf{v}_i \times \mathbf{H}_0 - m_e \nu (\mathbf{v}_i - \mathbf{v}_e), \\ \nabla \times \mathbf{H} = + \frac{4\pi}{c} en(\mathbf{v}_i - \mathbf{v}_e) + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \\ \nabla \times \mathbf{E} = - \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \quad (27)$$

where $\mathbf{v}_e$ and $\mathbf{v}_i$ are the velocities of the electrons and ions, m_e and m_i are their masses, n is the equilibrium density of the plasma, ν is the effective collision frequency and $\mathbf{H}_0$ is the intensity of an external magnetic field parallel to the z axis.

Assuming that all the quantities are proportional to $\exp i(k_3 z - \omega t)$, the following equations may be derived for the longitudinal components of the electric and magnetic fields of axially symmetric waves

$$\Delta_{\perp}^2 E_z + 2p\Delta_{\perp} E_z + qE_z = 0, \\ \Delta_{\perp}^2 H_z + 2p\Delta_{\perp} H_z + qH_z = 0, \quad (28)$$

where

$$\Delta_{\perp} \equiv \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho}, \\ p = [(\epsilon_1^2 - \epsilon_2^2 + \epsilon_1 \epsilon_3)k^2 - (\epsilon_1 + \epsilon_3)k_3^2] / 2\epsilon_1, \\ q = \epsilon_3 \delta / \epsilon_1; \\ \delta = [(\epsilon_1 - \epsilon_2)k^2 - k_3^2][(\epsilon_1 + \epsilon_2)k^2 - k_3^2], \\ \epsilon_1 = 1 + \frac{\omega_{pe}^2(1+\mu)[\omega^2 - \omega_{ce}\omega_{ci} + i\omega\nu(1+\mu)]}{\omega^2\omega_{ce}^2(1-\mu)^2 - [\omega^2 - \omega_{ce}\omega_{ci} + i\omega\nu(1+\mu)]^2}, \\ \epsilon_2 = \frac{\omega\omega_{ce}\omega_{pe}^2(\mu^2 - 1)}{\omega^2\omega_{ce}^2(1-\mu)^2 - [\omega^2 - \omega_{ce}\omega_{ci} + i\omega\nu(1+\mu)]^2}, \\ \epsilon_3 = 1 - \frac{\omega_{pe}^2(1+\mu)}{\omega[\omega + i\nu(1+\mu)]}, \\ \omega_{ce} = \frac{eH_0}{m_e c}, \quad \omega_{ci} = \frac{eH_0}{m_i c}, \\ \omega_{pe}^2 = \frac{4\pi e^2 n}{m_e}, \quad \mu = \frac{m_e}{m_i}.$$

The quantities ϵ_1 , ϵ_2 , ϵ_3 form the dielectric tensor

$$\epsilon_{ik} = \begin{pmatrix} \epsilon_1 & i\epsilon_2 & 0 \\ -i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}.$$

The solution of Eq. (28) that is regular at the point $\rho = 0$ is

$$E_z = AJ_0(k_1 \rho) + BJ_0(k_2 \rho), \quad (29)$$

where

$$k_1^2 = p + \sqrt{(p^2 - q)}, \\ k_2^2 = p - \sqrt{(p^2 - q)}.$$

The remaining components of the fields inside the plasma cylinder are determined by Maxwell's equations:

$$E_{\rho} = - \frac{ik_1 \delta^{-1}}{k_3} HAJ_1(k_1 \rho) - \frac{ik_2 \delta^{-1}}{k_3} RBJ_1(k_2 \rho), \\ E_{\phi} = + \frac{k_1 \delta^{-1}}{\epsilon_2 k_3} QAJ_1(k_1 \rho) + \frac{k_2 \delta^{-1}}{\epsilon_2 k_3} SBJ_1(k_2 \rho), \\ H_z = - \frac{i\epsilon_1}{\epsilon_2 k k_3} MAJ_0(k_1 \rho) - \frac{i\epsilon_1}{\epsilon_2 k k_3} NBJ_0(k_2 \rho), \\ H_{\rho} = - \frac{k_1 \delta^{-1}}{\epsilon_2 k} QAJ_1(k_1 \rho) - \frac{k_2 \delta^{-1}}{\epsilon_2 k} SBJ_1(k_2 \rho), \\ H_{\phi} = - \frac{ik\epsilon_3}{k_1} AJ_1(k_1 \rho) - \frac{ik\epsilon_3}{k_2} BJ_1(k_2 \rho),$$

where

$$M = (\epsilon_3/\epsilon_1)(\epsilon_1 k^2 - k_3^2) - k_1^2, \\ N = (\epsilon_3/\epsilon_1)(\epsilon_1 k^2 - k_3^2) - k_2^2, \\ H = k_3^2(\epsilon_1 k^2 - k_3^2) + \epsilon_1 k^2 M, \\ R = k_3^2(\epsilon_1 k^2 - k_3^2) + \epsilon_1 k^2 N, \\ Q = \epsilon_2^2 k^2 k_3^2 + \epsilon_1(\epsilon_1 k^2 - k_3^2)M, \\ S = \epsilon_2^2 k^2 k_3^2 + \epsilon_1(\epsilon_1 k^2 - k_3^2)N.$$

Using the boundary conditions at the surface of the plasma cylinder we obtain the dispersion equation

$$\begin{aligned} & \frac{\varepsilon_3}{k_1 k_2} \frac{J_1(k_1 R_0) J_1(k_2 R_0)}{J_0(k_1 R_0) J_0(k_2 R_0)} + \frac{1}{\tilde{\kappa}^2} \frac{K_1^2(\tilde{\kappa} R_0)}{K_0^2(\tilde{\kappa} R_0)} \\ & + \frac{\varepsilon_3[(\varepsilon_1 k^2 - k_3^2)(\varepsilon_1 + 1) - \varepsilon_2^2 k^2]}{2\varepsilon_1 k_1^2 k_2^2} \frac{1}{\kappa} \frac{K_1(\tilde{\kappa} R_0)}{K_0(\tilde{\kappa} R_0)} \left[k_1 \frac{J_1(k_1 R_0)}{J_0(k_1 R_0)} + k_2 \frac{J_1(k_2 R_0)}{J_0(k_2 R_0)} \right] \\ & - \frac{\varepsilon_3 \{(\varepsilon_1 - 1)(\varepsilon_1 - \varepsilon_3)(\varepsilon_1 k^2 - k_3^2)^2 + \varepsilon_2^2 k^2 [\varepsilon_2^2 + \varepsilon_1 \varepsilon_3 - 2\varepsilon_1^2 + \varepsilon_1] + k_3^2(1 + 2\varepsilon_1 + \varepsilon_3)\}}{2\varepsilon_1^2 k_1^2 k_2^2 (k_1^2 - k_2^2)} \\ & \times \frac{1}{\tilde{\kappa}} \frac{K_1(\tilde{\kappa} R_0)}{K_0(\tilde{\kappa} R_0)} \left[k_1 \frac{J_1(k_1 R_0)}{J_0(k_1 R_0)} - k_2 \frac{J_1(k_2 R_0)}{J_0(k_2 R_0)} \right] = 0, \end{aligned} \quad (30)$$

where $\tilde{\kappa}^2 = k_3^2 - k^2$ and R_0 is the radius of the plasma cylinder.

The electron and ion velocities for the general case are given by

$$\begin{aligned} v_{e\rho} &= \frac{e}{m_e \Delta_1} \{ i\omega[\omega^2 - \omega_{ci}^2 + i\omega\nu(1 + \mu)]E_\rho + \omega_{ce}[\omega^2 - \omega_{ci}^2 + i\omega\nu\mu(1 + \mu)]E_\phi \}, \\ v_{e\phi} &= \frac{e}{m_e \Delta_1} \{ -\omega_{ce}[\omega^2 - \omega_{ci}^2 + i\omega\nu\mu(1 + \mu)]E_\rho + i\omega[\omega^2 - \omega_{ci}^2 + i\omega\nu(1 + \mu)]E_\phi \}, \\ v_{i\rho} &= \frac{e}{m_i \Delta_1} \{ -i\omega[\omega^2 - \omega_{ce}^2 + i\omega\nu(1 + \mu)]E_\rho + \omega_{ce}[\mu(\omega^2 - \omega_{ce}^2) + i\omega\nu(1 + \mu)]E_\phi \}, \\ v_{i\phi} &= \frac{e}{m_i \Delta_1} \{ -\omega_{ce}[\mu(\omega^2 - \omega_{ce}^2) + i\omega\nu(1 + \mu)]E_\rho - i\omega[\omega^2 - \omega_{ce}^2 + i\omega\nu(1 + \mu)]E_\phi \}, \end{aligned} \quad (31)$$

where $\Delta_1 \equiv [\omega\omega_{ce}(1 - \mu)]^2 - [\omega^2 - \omega_{ce}\omega_{ci} + i\omega\nu]^2$; $\nu^* = \nu(1 + \mu)$.

Let us consider Eq. (30) for several limiting cases.

If $\omega \ll \omega_{ci}$, $\omega\nu \ll \omega_{ce}\omega_{ci}$, then

$$\begin{aligned} \varepsilon_1 &= 1 + \omega_{pe}^2/\omega_{ce}\omega_{ci}, \\ \varepsilon_2 &= \omega\omega_{pe}^2/\omega_{ce}\omega_{ci}^2, \\ \varepsilon_3 &= 1 - \omega_{pe}^2/\omega(\omega + i\nu), \\ k_1^2 &= \varepsilon_1 k^2 - k_3^2, \\ k_2^2 &= \frac{\varepsilon_3}{\varepsilon_1} (\varepsilon_1 k^2 - k_3^2). \end{aligned} \quad (32)$$

Assuming $(k_1 R_0)^2 \ll 1$, $(k_2 R_0)^2 \ll 1$, $(\tilde{\kappa} R_0)^2 \gg 1$, and $\omega_{pe}^2 \gg \omega_{ce}\omega_{ci}$, we obtain the dispersion equation for magnetohydrodynamic waves in a bounded medium:

$$J_1(k_2 R_0) = 0, \quad (k_2 R_0)^2 = \lambda_p^2; \quad \lambda_p = 3.8, \dots, \dots, \quad (32)$$

from which we obtain a phase velocity

$$\begin{aligned} V_{ph} &= \omega_{ce}\omega_{ci}R_0^2/[\lambda_p^2(1 + i\gamma) + (\omega_{pe}^2 R_0^2/c^2)], \\ \gamma &= \nu/\omega. \end{aligned} \quad (33)$$

If the phase velocity is small ($V_{ph} = \omega/k_3 \ll c$), the solution to the dispersion equation close to the ion cyclotron frequency is

$$\omega = \omega_{ci} \left(1 - \beta_{ph}^2 \frac{\omega_{pe}^2}{\omega_{ce}\omega_{ci}} - \beta_{ph}^2 \frac{c^2 \lambda_p^2}{2\omega_{ce}\omega_{ci}R_0^2} \right). \quad (34)$$

$$\begin{aligned} \omega_{pe}^2 &\gg \omega_{ce}\omega_{ci}, \quad \beta_{ph}^2 \ll \omega_{ci}^2/\omega_{pe}^2, \\ \nu/\omega_{ci} &\ll (\omega_{ci} - \omega)/\omega_{ci} \ll 1, \end{aligned}$$

where $\beta_{ph} = V_{ph}/c$.

Here, the components of the electric and magnetic fields inside the plasma are

$$\begin{aligned} E_z &= E_0 \left\{ \frac{J_0(k_1 \varrho)}{J_0(k_1 R_0)} - \frac{1}{\varepsilon_3} \frac{I_0(k_3 \varrho)}{I_0(k_3 R_0)} \right\}, \\ H_z &= E_0 \left\{ -\frac{ik}{k_3} \left(1 + \frac{2a\omega_{ce}\omega_{ci}}{\omega_{pe}^2} - \frac{k_1^2}{k^2} \right) \frac{I_0(k_1 \varrho)}{I_0(k_1 R_0)} \right. \\ &\quad \left. + \frac{ik_3 I_0(k_3 \varrho)}{\varepsilon_3 k I_0(k_3 R_0)} \right\}, \\ H_\phi &= -i\varepsilon_3 E_0 \left\{ \frac{k J_1(k_1 \varrho)}{k_1 J_0(k_1 R_0)} - \frac{k}{\varepsilon_3 k_3} \frac{I_1(k_3 \varrho)}{I_0(k_3 R_0)} \right\}, \\ E_\rho &= E_0 \left\{ \frac{i\varepsilon_3 k^2}{k_3 k_1} \left(1 + \frac{2a\omega_{ce}\omega_{ci}}{\omega_{pe}^2} \right) \frac{J_1(k_1 \varrho)}{J_0(k_1 R_0)} \right. \\ &\quad \left. + \frac{2ik^2}{k_1^2} \left(1 + \frac{a\omega_{ce}\omega_{ci}}{\omega_{pe}^2} \right) \frac{I_1(k_3 \varrho)}{I_0(k_3 R_0)} \right\}, \\ E_\phi &= E_0 \left\{ -\frac{\varepsilon_3 k^2 J_1(k_1 \varrho)}{k_1 k_3 J_0(k_1 R_0)} - 2 \left(1 + \frac{a\omega_{ce}\omega_{ci}}{\omega_{pe}^2} \right) \right. \\ &\quad \left. \times \frac{k^2 I_1(k_3 \varrho)}{\varepsilon_3 k_1^2 I_0(k_3 R_0)} \right\}, \end{aligned} \quad (35)$$

where $a = (\omega - \omega_{ci})/\omega_{ci}\beta_{ph}^2$.

For $\mu \rightarrow 0$, $\nu = 0$ ($\varepsilon_3 \rightarrow \infty$), Eqs. (30) and (34) go over into the results obtained by Stix.¹⁴

In the case of very fast waves ($\beta_{ph} \rightarrow \infty$), the dispersion equation (30) breaks down into two

equations corresponding to two different types of waves

$$\frac{1}{k_1} \frac{J_1(k_1 \varrho)}{J_0(k_1 \varrho)} \mp \frac{1}{ik} \frac{K_1(\mp ikR_0)}{K_0(\mp ikR_0)} = 0,$$

$$\frac{k_2 J_1(k_2 R_0)}{J_0(k_2 R_0)} \mp \frac{k}{i} \frac{K_1(\mp ikR_0)}{K_0(\mp ikR_0)} = 0. \quad (36)$$

where

$$k_1^2 = \varepsilon_{\perp} k^2, \quad k_2^2 = \varepsilon_{\parallel} k^2;$$

$$\varepsilon_{\perp} = (\varepsilon_1^2 - \varepsilon_2^2)/\varepsilon_1, \quad \varepsilon_{\parallel} = \varepsilon_3.$$

Waves with phase velocity c cannot propagate in the plasma cylinder since the radiation condition at infinity is not satisfied in this case. However, if the plasma cylinder is surrounded by a metal casing, of radius R_0 , these waves may propagate and the dispersion equation is

$$\frac{1}{2} \delta \varepsilon_1 (R_0^2 - R_0^2) (k_2^2 - k_1^2) + k_1 R_0 \varrho \frac{J_1(k_1 R_0)}{J_0(k_1 R_0)} - k_2 R_0 S \frac{J_1(k_2 R_0)}{J_0(k_2 R_0)} = 0. \quad (37)$$

The corresponding components of the electric and magnetic fields are, for $R_0 \geq \varrho \geq R_0$,

$$E_z = \text{const} = 0,$$

$$H_z = \text{const} = H_0,$$

$$E_{\phi} = -H_{\rho} = -ikH_0(R_0^2 - \varrho^2)/2\varrho,$$

$$E_{\rho} = H_{\phi} = H_{\phi}(R_0)(R_0/\varrho),$$

$$H_{\phi}(R_0) = \frac{\varepsilon_2 \varepsilon_3}{\varepsilon_1} \frac{k^2 H_0}{k_2^2 - k_1^2} \left[\frac{k}{k_1} \frac{J_1(k_1 R_0)}{J_0(k_1 R_0)} - \frac{k}{k_2} \frac{J_1(k_2 R_0)}{J_0(k_2 R_0)} \right]; \quad (37a)$$

and for $R_0 \geq \varrho \geq 0$,

$$E_z = \frac{i\varepsilon_2}{\varepsilon_1} \frac{k^2 H_0}{k_2^2 - k_1^2} \left\{ \frac{J_0(k_1 \varrho)}{J_0(k_1 R_0)} - \frac{J_0(k_2 \varrho)}{J_0(k_2 R_0)} \right\},$$

$$E_{\rho} = \frac{\varepsilon_2}{\varepsilon_1} \frac{k^2 H_0}{k_2^2 - k_1^2} \left\{ \frac{k_1 H}{k \delta} \frac{J_1(k_1 \varrho)}{J_0(k_1 R_0)} - \frac{k_2 R}{k \delta} \frac{J_1(k_2 \varrho)}{J_0(k_2 R_0)} \right\},$$

$$H_{\phi} = \frac{\varepsilon_2 \varepsilon_3}{\varepsilon_1} \frac{k^2 H_0}{k_2^2 - k_1^2} \left\{ \frac{k}{k_1} \frac{J_1(k_1 \varrho)}{J_0(k_1 R_0)} - \frac{k}{k_2} \frac{J_1(k_2 \varrho)}{J_0(k_2 R_0)} \right\},$$

$$H_z = \frac{H_0}{k_2^2 - k_1^2} \left\{ M \frac{J_0(k_1 \varrho)}{J_0(k_1 R_0)} - N \frac{J_0(k_2 \varrho)}{J_0(k_2 R_0)} \right\},$$

$$E_{\phi} = -\frac{ik^2 H_0}{k_2^2 - k_1^2} \left\{ \frac{k_1 Q}{k \delta \varepsilon_1} \frac{J_1(k_1 \varrho)}{J_0(k_1 R_0)} - \frac{k_2 S}{k \delta \varepsilon_1} \frac{J_1(k_2 \varrho)}{J_0(k_2 R_0)} \right\},$$

$$H_{\rho} = -E_{\phi}. \quad (37b)$$

If the frequencies of the waves propagated in the plasma cylinder are large ($\omega \gg (\omega_e \omega_i)^{1/2}$), the ion motion may be neglected and the dispersion equation (30) reduces to¹⁵

$$\frac{\varepsilon_3}{k_1 k_2} \frac{J_1(k_1 R_0) J_1(k_2 R_0)}{J_0(k_1 R_0) J_0(k_2 R_0)} + \frac{1}{\tilde{\kappa}^2} \frac{K_1^2(\tilde{\kappa} R_0)}{K_0^2(\tilde{\kappa} R_0)} - \frac{\varepsilon_2^2 \varepsilon_3 (k^2 + k_3^2) (\varepsilon_3 k^2 + k_3^2)}{2 \varepsilon_1^2 k_1^2 k_2^2 (k_1^2 - k_2^2)} \frac{1}{\tilde{\kappa}} \frac{K_1(\tilde{\kappa} R_0)}{K_0(\tilde{\kappa} R_0)} \left[k_1 \frac{J_1(k_1 R_0)}{J_0(k_1 R_0)} - k_2 \frac{J_1(k_2 R_0)}{J_0(k_2 R_0)} \right] + \frac{\varepsilon_3 [(\varepsilon_1 k^2 - k_3^2) (\varepsilon_1 + 1) - \varepsilon_2^2 k^2]}{2 \varepsilon_1 k_1^2 k_2^2} \frac{1}{\tilde{\kappa}} \frac{K_1(\tilde{\kappa} R_0)}{K_0(\tilde{\kappa} R_0)} \left[k_1 \frac{J_1(k_1 R_0)}{J_0(k_1 R_0)} + k_2 \frac{J_1(k_2 R_0)}{J_0(k_2 R_0)} \right] = 0, \quad (38)$$

where

$$\varepsilon_1 = 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2},$$

$$\varepsilon_2 = -\frac{\omega_{pe}^2 \omega_{ce}}{\omega (\omega_{ce}^2 - \omega^2)},$$

$$\varepsilon_3 = 1 - \frac{\omega_{pe}^2}{\omega^2}.$$

Thus, slow and fast waves, as well as waves with phase velocity c , may propagate in a plasma cylinder located in a magnetic field.

Equations (28), (29), (29a) describe the penetration and distribution of the field in the plasma cylinder. It may be seen from these relationships and from numerical calculations, that electromagnetic waves penetrate quite deeply into the plasma if the plasma density is such that $\omega^2 \lesssim \omega_{ce}^2 \ll \omega_{pe}^2$. This is connected with the gyrotropic properties of a plasma cylinder. From Eq. (29), it follows that the radial distribution of the field depends on ε_1 and ε_2 as well

as on ε_3 . Therefore the field penetrates the plasma even when $\varepsilon_3 \approx 1 - (\omega_{pe}/\omega)^2 < 0$.

Let us now discuss the energy obtained by the particles in a high frequency electromagnetic field near resonance.

The resonance conditions for a dense plasma depend on the plasma density and geometry, in contrast to the resonance conditions for free electrons and ions, whose cyclotron resonant frequency does not depend on these two factors. This is due to a displacement of the resonant frequencies and a change in the way the field penetrates the plasma.

Here we shall limit ourselves to the two most interesting cases, i.e., when $k_3 \rightarrow \infty$ and when $k_3 \rightarrow 0$. In the first case, the phase velocity of the wave approaches zero and the corresponding oscillation frequency coincides with the ion or electron cyclotron resonance frequency. When $\omega \rightarrow \omega_{ei}$, assuming

$$\nu/\omega_{ce} \ll |(\omega_{ci} - \omega)/\omega_{ci}| \ll 1$$

$$\omega_{pe}^2/\omega_{ce} \omega_{ci} \gg 1,$$

the electron and ion velocities are equal to

$$v_{e\rho} = \frac{eE_\phi}{m_i\omega_{ci}}, \quad v_{i\rho} = \frac{eE_\phi}{m_i\omega_{ci}} \cdot \frac{\omega_{ci}}{\omega_{ci} - \omega}$$

$$v_{e\phi} = -iv_{e\rho}, \quad v_{i\phi} = -iv_{i\rho}. \quad (39)$$

We see that the velocities of the ions are much greater than those of the electrons. Let us note that in the case considered, the resonance frequency depends only slightly on the plasma density and the geometry of the cylinder (see Eq. (34)) and is the same as the cyclotron frequency for a free ion.

When $\omega \rightarrow \omega_{ce}$, assuming

$$\nu/\omega_{ce} \ll |(\omega_{ce} - \omega)/\omega_{ce}| \ll 1$$

$$(\omega_{pe}/\omega_{ce})^2 \gg 1,$$

the velocities of the particles are equal to

$$v_{e\rho} = \frac{eE_\phi}{m_e(\omega_{ce} - \omega)}, \quad v_{i\rho} = \frac{eE_\phi}{m_i\omega_{ci}},$$

$$v_{e\phi} = iv_{e\rho}, \quad v_{i\phi} = iv_{i\rho}. \quad (40)$$

In this case the electron velocities are much greater than the ion velocities.

Let us now consider the second limiting case $k_3 \rightarrow 0$, which corresponds to oscillation of the plasma cylinder as a whole. Here the electromagnetic fields do not depend on the z coordinate. The dispersion equation (30), with $k_3 = 0$, reduces to two equations (see Eq. (36)). The dielectric constant $\varepsilon_\perp$ corresponding to a purely transverse oscillation is¹⁶

$$\varepsilon_\perp = \frac{(\omega_{ce}^2 - \omega^2)(\omega^2 - \omega_{ci}^2) - \omega_{pe}^4(1 + \mu)^2 + 2(1 + \mu)(\omega^2 - \omega_{ce}\omega_{ci})(\omega_{pe}^2 - i\omega\nu) + \omega^2\nu^2(1 + \mu)^2}{(\omega_{ce}^2 - \omega^2)(\omega^2 - \omega_{ci}^2) + (1 + \mu)(\omega^2 - \omega_{ce}\omega_{ci})(\omega_{pe}^2 - 2i\omega\nu) + i\omega\nu(1 + \mu)^2(\omega_{pe}^2 - i\omega\nu)} \quad (41)$$

With $\nu = 0$, the zeros and poles of $\varepsilon_\perp$ will be located at the following frequencies:

Zeros:

$$\omega_\pm = [\frac{1}{2}(\omega_{ce}^2 + \omega_{ci}^2 + 2\omega_{pe}^2) \pm (\frac{1}{4}(\omega_{ce}^2 + \omega_{ci}^2 + 2\omega_{pe}^2)^2 - (\omega_{ce}\omega_{ci} + \omega_{pe}^2)^2)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad (42a)$$

$$\omega_+ = \omega_{ce} \left(1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} \right), \quad \omega_{pe}^2 \ll \omega_{ce}^2, \quad \omega_- = \omega_{ci} \left(1 + \frac{\omega_{pe}^2}{\omega_{ce}\omega_{ci}} \right), \quad \omega_{pe}^2 \ll \omega_{ce}\omega_{ci},$$

$$= \omega_{pe} + \frac{1}{2} \omega_{ce}, \quad \omega_{pe}^2 \gg \omega_{ce}^2, \quad = \omega_{pe} - \frac{1}{2} \omega_{ce}, \quad \omega_{pe}^2 \gg \omega_{ce}^2.$$

Poles:

$$\omega_\pm = [\frac{1}{2}(\omega_{ce}^2 + \omega_{ci}^2 + \omega_{pe}^2) \pm (\frac{1}{4}(\omega_{ce}^2 + \omega_{ci}^2 + \omega_{pe}^2)^2 - (\omega_{pe}^2 + \omega_{ce}\omega_{ci})\omega_{ce}\omega_{ci})^{\frac{1}{2}}]^{\frac{1}{2}}, \quad (42b)$$

$$\omega_+ = \omega_{ce} \left(1 + \frac{1}{2} \frac{\omega_{pe}^2}{\omega_{ce}^2} \right), \quad \omega_{pe}^2 \ll \omega_{ce}^2, \quad \omega_- = \omega_{ci} \left(1 + \frac{1}{2} \frac{\omega_{pe}^2}{\omega_{ce}\omega_{ci}} \right), \quad \omega_{pe}^2 \ll \omega_{ce}\omega_{ci},$$

$$= \omega_{pe} \left(1 + \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}^2} \right), \quad \omega_{pe}^2 \gg \omega_{ce}^2, \quad = (\omega_{ce}\omega_{ci})^{\frac{1}{2}} \left(1 - \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}^2} \right), \quad \omega_{pe}^2 \gg \omega_{ce}^2$$

The ion and electron velocities and the corresponding penetration depths are expressed by the following relations:

$$\omega = \omega_{ci} \left(1 + \frac{1}{2} \frac{\omega_{pe}^2}{\omega_{ce}\omega_{ci}} \right), \quad \omega_{pe}^2 \ll \omega_{ce}\omega_{ci}, \quad \nu \ll \omega_{ce}: \quad (43a)$$

$$v_{e\rho} = \frac{eE_\phi}{m_i\omega_{ci}} \frac{i\omega_{ce}\nu - \omega_{pe}^2 - \omega_{ci}^2}{2i\omega_{ce}\nu}, \quad v_{i\rho} = \frac{eE_\phi}{m_i\omega_{ci}} \frac{i\omega_{ce}}{2\nu}, \quad v_{e\phi} = \frac{eE_\phi}{m_i\omega_{ci}} \frac{i\omega_{ci}\nu - \omega_{pe}^2}{2\omega_{ci}\nu}, \quad v_{i\phi} = \frac{eE_\phi}{m_i\omega_{ci}} \frac{\omega_{ce}}{2\nu}$$

$$\delta = \frac{c}{\omega_{pe}} \left(\frac{2\nu}{\omega_{ci}} \right)^{\frac{1}{2}}.$$

$$\omega = \omega_{ce} \left(1 + \frac{1}{2} \frac{\omega_{pe}^2}{\omega_{ce}^2} \right), \quad \omega_{pe}^2 \ll \omega_{ce}^2, \quad \nu \ll \omega_{ce} \quad (43b)$$

$$v_{e\rho} = \frac{ieE_\phi}{m_i\omega_{ci}} \frac{\omega_{ce}}{2\nu}, \quad v_{i\rho} = \mu \frac{eE_\phi}{m_i\omega_{ci}} \frac{\omega_{pe}^2 - i\omega_{ce}\nu}{2i\omega_{ce}\nu}, \quad v_{e\phi} = -\frac{eE_\phi}{m_i\omega_{ci}} \frac{\omega_{ce}}{2\nu}, \quad v_{i\phi} = \mu \frac{eE_\phi}{m_i\omega_{ci}} \frac{\mu\omega_{pe}^2 + i\omega_{ce}\nu}{2\omega_{ce}\nu},$$

$$\delta = \frac{c}{\omega_{pe}} \left(\frac{\nu^2}{\omega_{ce}\omega_{ci}} \right)^{\frac{1}{2}}.$$

$$\omega^2 = \omega_{ce}\omega_{ci}(1 - \omega_{ce}^2/\omega_{pe}^2), \quad \omega_{pe}^2 \gg \omega_{ce}^2, \quad \nu \ll \omega_{ce}: \quad (43c)$$

$$\begin{aligned}
v_{e\rho} &= \frac{ieE_\phi}{m_i\omega_{ci}} \frac{(\omega_{ce}\omega_{ci})^{\frac{1}{2}}}{\nu}, & v_{i\rho} &= \frac{ieE_\phi}{m_i\omega_{ci}} \frac{(\omega_{ce}\omega_{ci})^{\frac{1}{2}}}{\nu}, & v_{e\phi} &= -\frac{eE_\phi}{m_i\omega_{ci}} \frac{\omega_{ce}}{\nu}, & v_{i\phi} &= \frac{eE_\phi}{m_i\omega_{ci}} \frac{\omega_{ci}}{\nu}, \\
\delta &= \frac{c}{\omega_{pe}} \left(\frac{\nu^2}{\omega_{ce}\omega_{ci}} \right)^{\frac{1}{2}}, \\
\omega^2 &= \omega_{pe}^2 + \omega_{ce}^2, \quad \omega_{pe}^2 \gg \omega_{ce}^2, \quad \nu \ll \omega_{ce}: \\
v_{e\rho} &= -\frac{ieE_\phi}{m_i\omega_{ci}} \frac{\omega_{ce}^2}{\nu\omega_{pe}}, & v_{i\rho} &= -\frac{ieE_\phi}{m_i\omega_{ci}} \frac{\omega_{ce}\omega_{ci}}{\nu\omega_{pe}}, & v_{e\phi} &= \frac{eE_\phi}{m_i\omega_{ci}} \frac{\omega_{ce}}{\omega_{pe}}, & v_{i\phi} &= \frac{eE_\phi}{m_i\omega_{ci}} \frac{\omega_{ci}}{\omega_{pe}}, \\
\delta &= \frac{c}{\omega_{ce}} \left(\frac{\nu}{\omega_{pe}} \right)^{\frac{1}{2}}, & & \times \frac{i\omega_{pe}\nu + \omega_{ce}^2}{\nu\omega_{pe}} & & \times \frac{\omega_{ce}\omega_{ci} - i\nu\omega_{pe}}{\nu\omega_{pe}}
\end{aligned} \tag{43d}$$

where E_ϕ is given on the boundary.

When the external electric field E_ρ is given on the boundary of the cylinder, the electron and ion velocities and the corresponding penetration depths are given by the following expressions:

$$\begin{aligned}
\omega &= \omega_{ci} \left(1 + \frac{\omega_{pe}^2}{\omega_{ce}\omega_{ci}} \right), \quad \omega_{pe}^2 \ll \omega_{ce}\omega_{ci}, \quad \nu \ll \omega_{ce}: \\
v_{e\rho} &= -\frac{eE_\rho(\omega_{pe}^4 + 3i\nu\omega_{ci}\omega_{pe}^2 - 2\nu^2\omega_{ci}^2)}{2m_i\nu\omega_{ci}^2\omega_{pe}^2}, & v_{i\rho} &= \frac{eE_\rho(\omega_{pe}^2 + 2i\omega_{ci}\nu)}{2m_e\nu\omega_{pe}^2}, \\
v_{e\phi} &= \frac{ieE_\rho(\omega_{pe}^4 + i\omega_{ci}\nu\omega_{pe}^2 + 2\nu^2\omega_{ci}^2)}{2m_e\omega_{ce}\omega_{ci}\omega_{pe}^2\nu}, & v_{i\phi} &= \frac{eE_\rho(\omega_{pe}^2 + 2i\omega_{ci}\nu)}{2im_e\nu\omega_{pe}^2}, & \delta &= \frac{c}{\omega_{ci}} \left(\frac{\omega_{pe}^2}{2\nu\omega_{ci}} \right)^{\frac{1}{2}}, \\
\omega &= \omega_{ce} \left(1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} \right), \quad \omega_{pe}^2 \ll \omega_{ce}^2, \quad \nu \ll \omega_{ce}: \\
v_{e\rho} &= -\frac{eE_\rho(\omega_{pe}^2 + 2i\omega_{ci}\nu)}{2m_e\omega_{ce}^2\nu}, & v_{i\rho} &= \frac{eE_\rho(\omega_{pe}^4 - i\nu\omega_{ce}\omega_{pe}^2 + 2\omega_{ce}\omega_{ci}\nu^2)}{2m_i\nu\omega_{ce}^2\omega_{pe}^2}, \\
v_{e\phi} &= \frac{eE_\rho(\omega_{pe}^2 + 2i\omega_{ci}\nu)}{2m_e\omega_{pe}^2\nu}, & v_{i\phi} &= \frac{ieE_\rho(\omega_{pe}^4 + i\nu\omega_{ce}\omega_{pe}^2 - 2\omega_{ce}\omega_{ci}\nu^2)}{2m_i\nu\omega_{ce}^2\omega_{pe}^2}, & \delta &= \frac{c}{\omega_{ce}} \left(\frac{\omega_{pe}^2}{2\nu\omega_{ce}} \right)^{\frac{1}{2}}.
\end{aligned} \tag{44a}$$

$$\begin{aligned}
\omega &= \omega_{pe} + \frac{1}{2}\omega_{ce}, \quad \omega_{ce} \ll \omega_{pe}, \quad \nu \ll \omega_{ce}: \\
v_{e\rho} &= -\frac{e\omega_{ce}E_\rho}{2m_e\nu\omega_{pe}}, & v_{i\rho} &= \frac{e\omega_{ce}E_\rho}{2m_i\nu\omega_{pe}}, \\
v_{e\phi} &= -\frac{ie\omega_{ce}E_\rho}{2m_e\nu\omega_{pe}}, \quad \left(\mu \ll \frac{\omega_{ce}}{\omega_{pe}} \ll 1 \right), & v_{i\phi} &= \frac{ie\omega_{ce}E_\rho}{2m_i\nu\omega_{pe}}, \quad \left(\mu \ll \frac{\omega_{ce}}{\omega_{pe}} \ll 1 \right), & \delta &= \frac{c}{\omega_{pe}} \left(\frac{\omega_{ce}}{2\nu} \right)^{\frac{1}{2}}. \\
\omega &= \omega_{pe} - \frac{1}{2}\omega_{ce}, \quad \omega_{ce} \ll \omega_{pe}, \quad \nu \ll \omega_{ce}: \\
v_{e\rho} &= \frac{e\omega_{ce}E_\rho}{2m_e\nu\omega_{pe}}, & v_{i\rho} &= -\frac{e\omega_{ce}E_\rho}{2m_i\nu\omega_{pe}}, \\
v_{e\phi} &= -\frac{ie\omega_{ce}E_\rho}{2m_e\nu\omega_{pe}}, \quad \left(\mu \ll \frac{\omega_{ce}}{\omega_{pe}} \ll 1 \right), & v_{i\phi} &= \frac{ie\omega_{ce}E_\rho}{2m_i\nu\omega_{pe}}, \quad \left(\mu \ll \frac{\omega_{ce}}{\omega_{pe}} \ll 1 \right), & \delta &= \frac{c}{\omega_{pe}} \left(\frac{\omega_{ce}}{2\nu} \right)^{\frac{1}{2}}.
\end{aligned} \tag{44c}$$

$$\begin{aligned}
\omega &= \omega_{pe} + \frac{1}{2}\omega_{ce}, \quad \omega_{ce} \ll \omega_{pe}, \quad \nu \ll \omega_{ce}: \\
v_{e\rho} &= -\frac{e\omega_{ce}E_\rho}{2m_e\nu\omega_{pe}}, & v_{i\rho} &= \frac{e\omega_{ce}E_\rho}{2m_i\nu\omega_{pe}}, \\
v_{e\phi} &= -\frac{ie\omega_{ce}E_\rho}{2m_e\nu\omega_{pe}}, \quad \left(\mu \ll \frac{\omega_{ce}}{\omega_{pe}} \ll 1 \right), & v_{i\phi} &= \frac{ie\omega_{ce}E_\rho}{2m_i\nu\omega_{pe}}, \quad \left(\mu \ll \frac{\omega_{ce}}{\omega_{pe}} \ll 1 \right), & \delta &= \frac{c}{\omega_{pe}} \left(\frac{\omega_{ce}}{2\nu} \right)^{\frac{1}{2}}. \\
\omega &= \omega_{pe} - \frac{1}{2}\omega_{ce}, \quad \omega_{ce} \ll \omega_{pe}, \quad \nu \ll \omega_{ce}: \\
v_{e\rho} &= \frac{e\omega_{ce}E_\rho}{2m_e\nu\omega_{pe}}, & v_{i\rho} &= -\frac{e\omega_{ce}E_\rho}{2m_i\nu\omega_{pe}}, \\
v_{e\phi} &= -\frac{ie\omega_{ce}E_\rho}{2m_e\nu\omega_{pe}}, \quad \left(\mu \ll \frac{\omega_{ce}}{\omega_{pe}} \ll 1 \right), & v_{i\phi} &= \frac{ie\omega_{ce}E_\rho}{2m_i\nu\omega_{pe}}, \quad \left(\mu \ll \frac{\omega_{ce}}{\omega_{pe}} \ll 1 \right), & \delta &= \frac{c}{\omega_{pe}} \left(\frac{\omega_{ce}}{2\nu} \right)^{\frac{1}{2}}.
\end{aligned} \tag{44d}$$

EXCITATION OF WAVES IN A PLASMA

We shall now consider the problem of wave excitation in a plasma by means of external currents. Let us begin with the simplest problem of exciting hydromagnetic waves in a fluid of infinite conductivity. The state of the fluid is described by the usual equations

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \frac{1}{c} \mathbf{j} \times \mathbf{H} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{45}$$

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{1}{s} \frac{\partial \mathbf{H}}{\partial t}, \\ \nabla \cdot \mathbf{H} &= 0, \\ \nabla \times \mathbf{H} &= \frac{4\pi}{c} (\mathbf{j} + \mathbf{j}_0),\end{aligned}\quad (46)$$

where $\mathbf{j}_0$ is the external current density. Assuming the current to be sufficiently small, we may linearize the set of equations (45). As a result, an equation which determines the velocity is obtained:

$$\begin{aligned}\frac{\partial^2 \mathbf{v}}{\partial t^2} - c_s^2 \nabla \nabla \cdot \mathbf{v} - \mathbf{V}_A \times \{ \nabla \times [\nabla \times (\mathbf{V}_A \times \mathbf{v})] \} \\ = \frac{1}{\rho_0 c} \mathbf{H}_0 \times \frac{\partial \mathbf{j}_0}{\partial t}\end{aligned}\quad (47)$$

where ρ_0 is the equilibrium density of the fluid, c_s is the speed of sound and $\mathbf{V}_A = \mathbf{H}_0 / (4\pi\rho_0)^{1/2}$.

The alternating magnetic field $\mathbf{h} = \mathbf{H} - \mathbf{H}_0$ and the change in density caused by the wave are determined by

$$\begin{aligned}\frac{\partial \mathbf{h}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{H}_0), \\ \frac{\partial \rho}{\partial t} &= \rho_0 \nabla \cdot \mathbf{v}.\end{aligned}\quad (48)$$

The change in the total energy of the medium per unit time is

$$J = (1/c) \int \mathbf{v} \cdot (\mathbf{H}_0 \times \mathbf{j}_0) d\mathbf{r}. \quad (49)$$

The Fourier component $\mathbf{v}(\mathbf{k}, \omega)$ of the velocity is obtained from

$$\begin{aligned}[\omega^2 - (\mathbf{k} \cdot \mathbf{V}_A)^2] \mathbf{v} - [(c_s^2 + V_A^2) \mathbf{k} - \mathbf{V}_A (\mathbf{k} \cdot \mathbf{V}_A)] \mathbf{k} \cdot \mathbf{v} \\ + \mathbf{k} \mathbf{k} \cdot \mathbf{V}_A \mathbf{V}_A \cdot \mathbf{v} = \frac{i\omega}{c\rho_0} \mathbf{H}_0 \times \mathbf{j}_0\end{aligned}\quad (50)$$

where $\mathbf{j}_0(\mathbf{k}, \omega)$ is the Fourier component of the external current density.

Putting the determinant of Eq. (50) equal to zero, we obtain the dispersion equation for the free oscillations of an infinitely conducting fluid located in a magnetic field. After obtaining $\mathbf{v}$ from Eq. (50), we can derive the following general equation for the radiation intensity of three types of waves, one hydrodynamic wave and two magnetosonic:

$$\begin{aligned}dJ = 8\pi^5 \frac{V_A^2 \omega^2}{c^2} \left\{ \left| j_{\perp} \left(\frac{\omega}{u_1}, \theta, \phi \right) \right|^2 \right. \\ + \frac{u_2^2 - c_s^2 \cos^2 \theta}{u_2^2 - u_3^2} \left| j_{\perp} \left(\frac{\omega}{u_2}, \theta, \phi \right) \right|^2 \frac{\sin^2 \phi}{u_2^3} \\ \left. + \frac{c_s^2 \cos^2 \theta - u_3^2}{u_2^2 - u_3^2} \left| j_{\perp} \left(\frac{\omega}{u_3}, \theta, \phi \right) \right|^2 \frac{\sin^2 \phi}{u_3^3} \right\} d\omega \quad (51)\end{aligned}$$

where u_1 , u_2 and u_3 are the phase velocities of the hydromagnetic and magnetosonic waves:

$$\begin{aligned}u_1^2 &= (V_A \cos \theta)^2, \\ u_{2,3}^2 &= \frac{1}{2} \{ c_s^2 + V_A^2 \\ &\quad \pm [(c_s^2 + V_A^2) - (2c_s V_A \cos \theta)^2]^{1/2} \},\end{aligned}\quad (52)$$

θ is the angle between the direction of wave propagation and the external magnetic field and ϕ is the angle between $\mathbf{j}_0 \times \mathbf{H}_0$ and $\mathbf{k} \times \mathbf{H}_0$. The external current is considered to be a harmonic function of time.

For a surface current $\mathbf{j}$,

$$\mathbf{j} = \mathbf{j}_s \delta(z) \exp -i\omega t,$$

the total radiation intensity of the hydromagnetic waves is

$$J_s = \pi V_A j_s^2 / c^2. \quad (53)$$

This quantity does not depend on the frequency of the current.

For a line current the radiation intensity is

$$J_l = \pi \omega j_l^2 / 2c^2. \quad (54)$$

Equations (44) and (45) may only be used to describe oscillations at a frequency much less than the ion cyclotron frequency.

To determine the intensity of excited waves near the ion cyclotron frequency, we may use (27) or the simpler set ¹⁷

$$\rho \frac{\partial \mathbf{v}}{\partial t} = c^{-1} \mathbf{j} \times \mathbf{H}_0,$$

$$\begin{aligned}\mathbf{E} + c^{-1} \mathbf{v} \times \mathbf{H}_0 - (m_i / \rho e c) \mathbf{j} \times \mathbf{H}_0 &= m_i m_e / \rho e^2 \partial \mathbf{j} / \partial t, \\ \nabla \times (\nabla \times \mathbf{E}) &= -c^{-2} \partial^2 \mathbf{E} / \partial t^2 - 4\pi c^{-1} \partial (\mathbf{j} + \mathbf{j}_0) / \partial t.\end{aligned}\quad (55)$$

This set of equations is already linearized. The second equation takes the place here of the relation $\mathbf{E} + (\mathbf{v} \times \mathbf{H}_0) / C = 0$, which is Ohm's law for an infinitely conducting medium. Collisions are not taken into account in Eq. (55) for the sake of simplicity.

It may be shown that the Fourier components of the electric field are determined by

$$\begin{aligned}E_x &= \frac{4\pi i}{\omega \Delta} j_0 \{ \beta_A^4 [(1 - \xi_i - \xi_e)^2 - \xi_i^2] [1 - \beta_A^2 \xi_i \xi_e (1 - n^2)] \\ &\quad \times n^2 \sin^2 \theta \sin \phi \cos \phi \\ &\quad + i \beta_A^2 \xi_i [1 - \beta_A^2 \xi_i \xi_e (1 - n^2 \sin^2 \theta)] \} \\ E_y &= \frac{4\pi i}{\omega \Delta} j_0 \{ \beta_A^4 [(1 - \xi_i \xi_e) - \xi_i^2] [1 - n^2] 1 - \sin^2 \theta \cos^2 \phi \\ &\quad - \beta_A^2 \xi_i \xi_e (1 - n^2) (1 - n^2 \sin^2 \theta \sin^2 \phi) \\ &\quad + \beta_A^2 (1 - \xi_i \xi_e) [1 - \beta_A^2 \xi_i \xi_e (1 - n^2 \sin^2 \theta)] \},\end{aligned}\quad (56)$$

where

$$\begin{aligned}\Delta &\equiv A n^4 + B n^2 + C, \quad n \equiv kc / \omega, \\ \beta_A &\equiv V_A / c, \quad \xi_i \equiv \omega / \omega_{ci}, \quad \xi_e \equiv \omega / \omega_{ce} \\ A &\equiv \beta_A^4 \{ (1 - \beta_A^2 \xi_i \xi_e) [(1 - \xi_i \xi_e)^2 - \xi_i^2] \\ &\quad - (1 - \xi_i \xi_e - \xi_i^2) \sin^2 \theta \}, \\ B &\equiv -\beta_A^2 \{ 2(1 - \beta_A^2 \xi_i \xi_e) [1 - \xi_i \xi_e \\ &\quad + \beta_A^2 (1 - \xi_i \xi_e - \xi_i^2) \sin^2 \theta] \\ &\quad - [1 + \beta_A^2 (1 - \xi_i \xi_e - \xi_i^2)] \sin^2 \theta \}, \\ C &\equiv (1 - \beta_A^2 \xi_i \xi_e) \{ \beta_A^4 [(1 - \xi_i \xi_e)^2 - \xi_i^2] \\ &\quad + 2\beta_A^2 (1 - \xi_i \xi_e) + 1 \},\end{aligned}\quad (57)$$

where the directions of the axes are given in Fig. 2.

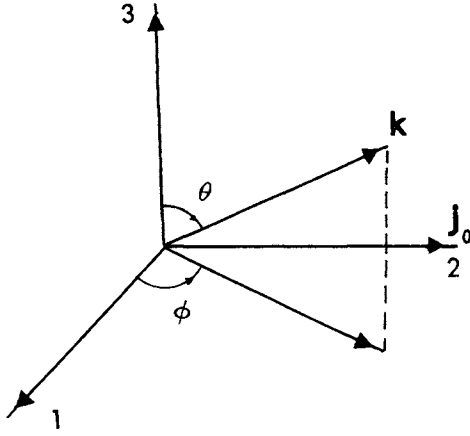


Figure 2

Taking Δ to be zero, we obtain the indices of refraction for waves that can propagate in the medium (pressure effects are neglected here):

$$n_{1,2}^2 = 1 + \left\{ 1 - \xi_i \xi_e - \frac{1 - (2 - \beta_A^2)(1 - \xi_i \xi_e - \xi_i^2)}{2(1 - \beta_A^2 \xi_i \xi_e)} \sin^2 \theta + \xi_i \left(1 - \frac{\sin^2 \theta}{1 - \beta_A^2 \xi_i \xi_e} + \left[\frac{1 + \beta_A^2(1 - \xi_i \xi_e - \xi_i^2)}{2\xi_i(1 - \beta_A^2 \xi_i \xi_e)} \right]^2 \sin^4 \theta \right)^{\frac{1}{2}} \right\} \times \beta_A^{-2} \left\{ (1 - \xi_i \xi_e)^2 - \xi_i^2 - \frac{1 - \xi_i \xi_e - \xi_i^2}{1 - \beta_A^2 \xi_i \xi_e} \sin^2 \theta \right\}^{-1}. \quad (58)$$

The total radiation intensity $\mathbf{J}$ is determined by Eq. (49). Substituting E_y from Eq. (56) into (49), we obtain the following general equation for J :

$$J = (2\pi)^4 \frac{\omega^2}{c^3} \text{Rei} \beta_A^2 \int \left\{ \beta_A^2 [1 - \beta_A^2 \xi_i \xi_e - n^2(1 - \beta_A^2 \xi_i \xi_e) + n^2(1 + \xi_i \xi_e \beta_A^2) \sin^2 \theta \cos^2 \phi - n^4 \beta_A^2 \xi_i \xi_e \sin^2 \theta \sin^2 \phi] [(1 - \xi_i \xi_e)^2 - \xi_i^2] + (1 - \xi_i \xi_e)(1 - \beta_A^2 \xi_i \xi_e) + n^2 \beta_A^2 \xi_i \xi_e (1 - \xi_i \xi_e \sin^2 \theta) \right\} \times \left(\frac{1}{n^2 - n_1^2} - \frac{1}{n^2 - n_2^2} \right) \frac{|\mathbf{j}_0(\mathbf{n})|^2}{(B^2 - 4AC)^{\frac{1}{4}}} d\mathbf{n}, \quad (59)$$

where $\mathbf{n} = \mathbf{k}c/\omega$.

Let us consider in more detail the excitation of waves by the surface current $\mathbf{j}_0(\mathbf{r}) = \mathbf{j}_0 \delta(z) \exp -i\omega t$. In this case, the intensity of radiation per unit surface area is

$$J = \frac{\pi j_0^2}{c(n_1 + n_2)} \left\{ \frac{1}{n_1 n_2} \left[1 + \frac{1 - \xi_i \xi_e}{\beta_A^2 [(1 - \xi_i \xi_e)^2 - \xi_i^2]} \right] + 1 \right\} \quad (60)$$

where

$$n_{1,2}^2 = 1 + [\beta_A^2(1 - \xi_i \xi_e \mp \xi_i)]^{-1}.$$

If $\xi_i(\xi_e \pm 1) = (1 + \beta_A^2)/\beta_A^2$, the resonance condition is satisfied and J tends to infinity. The resonant frequencies for the case of surface current excitation are

$$\omega = \frac{1}{2}\omega_{ce} + (\frac{1}{4}\omega_{ce}^2 + \omega_{ce}\omega_{ci} + \omega_{pe}^2)^{\frac{1}{2}} \quad (61)$$

where

$$\omega_{pe}^2 = 4\pi q e^2 / m_e m_1.$$

Let us now consider several limiting cases. If $\xi_e \ll 1$, then

$$J = \frac{\pi j_0^2}{c(n_1 + n_2)} \left\{ \frac{1}{n_1 n_2} \frac{1 + \beta_A^2(1 - \xi_i^2) - \xi_i \xi_e}{\beta_A^2(1 - \xi_i)} + 1 \right\}. \quad (62)$$

In this event if $\xi_i^2 \ll 1$, then

$$n_{1,2}^2 = (1 + \beta_A^2)/\beta_A^2, \quad J = \pi \beta_A j_0^2 / c(1 + \beta_A^2)^{\frac{1}{2}}. \quad (63)$$

If $\xi_i^2 \sim 1$, then

$$n_1^2 = 1/[\beta_A^2(1 - \xi_i)], \quad n_2^2 = (1 + 2\beta_A^2)/2\beta_A^2, \quad J = \pi \beta_A j_0^2 / c(4\beta_A^2 + 2)^{\frac{1}{2}}. \quad (64)$$

If $\xi_i^2 \gg 1$, then

$$n_1^2 = 1 - 1/\beta_A^2 \xi_i, \quad n_2^2 = 1 + 1/\beta_A^2 \xi_i,$$

$$J = \frac{\pi j_0^2}{c} \frac{1}{\sqrt{\left(1 - \frac{1}{\beta_A^2 \xi_i}\right)} + \sqrt{\left(1 - \frac{1}{\beta_A^2 \xi_e}\right)}} \times \left\{ 1 + \frac{1 - \frac{1}{\beta_A^2 \xi_i} + \frac{\xi_e}{\beta_A^2 \xi_i}}{\sqrt{\left(1 - \frac{1}{\beta_A^4 \xi_i^2}\right)}} \right\} \quad (65)$$

Let us now consider the limiting case $\xi_e \sim 1$ ($\xi_i \gg 1$). Here $n_1^2 = 1 - (2\beta_A^2 \xi_i)^{-1}$, $n_2^2 = (1 + \beta_A^2)/\beta_A^2$ and

$$J = \frac{\pi j_0^2}{c} \frac{1}{\sqrt{\left(1 - \frac{1}{2\beta_A^2 \xi_i}\right)} + \sqrt{\left(\frac{1 + \beta_A^2}{\beta_A^2}\right)}} \times \left\{ 1 + \frac{1 + \frac{1}{2\beta_A^2}}{\sqrt{\left[\left(1 - \frac{1}{2\beta_A^2 \xi_i}\right)\left(\frac{1 + \beta_A^2}{\beta_A^2}\right)\right]}} \right\} \quad (66)$$

Finally, if $\xi_e \gg 1$, then

$$n_1^2 = n_2^2 = 1 - (\omega_{pe}/\omega)^2, \quad J = \frac{\pi j_0^2}{c} \frac{1}{[1 - (\omega_{pe}/\omega)^2]^{\frac{1}{2}}}, \quad (67)$$

If the plasma is excited by a harmonic line current, we obtain Eq. (54) for the radiation intensity per unit length.

ENERGY LOSSES AND COHERENCE EFFECTS†

The possible existence of slow electromagnetic waves in confined and unconfined plasmas is significant not only from the standpoint of the absorption and propagation of external electromagnetic radiation, but also because slow waves in a plasma can increase the effectiveness of interaction between the charged particles and the plasma, for both the ordered and disordered motion of these particles. The interaction of individual particles, bunches of particles, or beams of particles with slow waves may turn out to be responsible for a number of physical phenomena which occur in the plasma. The elementary processes of interaction between the charged particles and the fields excited by them result in composite Cherenkov and Doppler effects and also in polarization losses. It is also necessary to point out a peculiar radiation effect which may take place when the velocity of a uniformly moving particle satisfies the inequality $v < V_{ph}$ but the medium is bounded in the direction of motion of the particle. For this radiation it is essential that the width of the dielectric layer through which the particle moves be less than $\lambda/2$. This

effect was demonstrated by S. I. Vavilov, but it is very difficult to observe in the visible region of the spectrum. When there is an interaction between a charged particle and a bounded plasma it may be of considerable importance.‡

However, it should be pointed out that the energy losses of an individual charged particle by the excitation of various plasma oscillations are very small, owing to the comparatively low plasma density. For example, when the particle moves uniformly through a plasma in a magnetic field, these losses are as follows:^{19, 20}

For weak magnetic fields, the energy loss per unit distance is

$$\frac{dW}{dx} = -\frac{q^2 \omega_{pe}^2}{2v^2} \ln \frac{v^2}{(\omega_0^2 + \omega_{ce}^2)b^2},$$

where q is the charge of the particle, v its speed, and b is a transverse dimension of the plasma.

For strong magnetic fields,

$$\frac{dW}{dx} = -\frac{q^2 \omega_{pe}^2}{2v^2} \left\{ \ln \frac{k_m^2 v^2}{\omega_{ce}^2 (1 - \beta^2)} - 1 \right\}.$$

For plasma with a density $n = 10^{10}$ electrons cm^{-3} and with an energy of 50 keV the loss is

$$\frac{dW}{dx} \approx -10^{-7} \text{ eV/cm}.$$

The energy lost by a particle moving in a plasma wave guide without a magnetic field has been calculated:^{21, 22}

$$\frac{dW}{dx} = -\frac{q^2 \omega^2 (1 - \beta^2 \epsilon_3)}{v^2 \epsilon_3 (1 - \epsilon_3)} \times \left[\frac{\epsilon_3 K_1(\sigma a) K_0(\kappa a) - (\sigma/\kappa) K_1(\kappa a) K_0(\sigma a)}{\kappa a I_1(\sigma a) K_1(\kappa a) - \kappa^2 \sigma a^3 K_0(\kappa a) I_0(\sigma a) + (2/1 - \beta^2 \epsilon_3) I_1(\sigma a) K_0(\kappa a)} \right]$$

where

$$K_1(\kappa a) I_0(\sigma a) = -(\epsilon_3 \kappa / \sigma) K_0(\kappa a) I_1(\sigma a),$$

$$\kappa^2 = k_3^2 - k^2, \text{ and } \sigma = k_3^2 - \epsilon_3 k^2. \text{ This agrees with the measured loss in order of magnitude.}$$

When a particle moves in a spatially periodic plasma (e.g., in plasma in which there are sausage-type coagula) a parametric Cherenkov effect takes place. In this case the energy loss is:²³

Due to Cherenkov loss,

$$\left(\frac{dW}{dx} \right)_{\text{Cher}} = -\frac{q^2 \omega_{0e}^2 b}{2v^2 L} \ln \left(\frac{L k_m^2 v_m^2}{b 4\pi n e^2} \right).$$

Due to polarization loss,

$$\left(\frac{dW}{dx} \right)_{\text{pol}} = -\frac{q^2 \omega_{0e}^2 b}{v^2 L} \ln \frac{k_m b}{7.4}.$$

It should be noted that in the case of anisotropic or gyrotropic plasma, Cherenkov radiation occurs even when $v \rightarrow 0$ since the condition for radiation is²⁴

$$\frac{k_m^2 v^2}{\omega^2} > \frac{\epsilon_z}{\epsilon_r} (\beta^2 \epsilon_r - 1) > 0.$$

If longitudinal electron oscillations are excited in the plasma, when $v > V_T$ the loss is²⁵

$$\frac{dW}{dx} = -\frac{q^2 \omega_{pe}^2}{v^2} \ln \left(\frac{\gamma m v^3}{q e \omega_{pe}} - \frac{V_T^2}{2v^2} \right).$$

For plasma with a density $n \approx 10^{12} - 10^{14} \text{ cm}^{-3}$, the energy loss of the charged particles is small. However the magnitude of this energy loss (or, alternatively, of the energy gain of the high-frequency field) may be substantially increased by utilizing the coherence effect; it was Veksler²⁶ who first pointed out the possibility of using this for particle acceleration. If the wavelength of the wave with which a bunch of charged particles interacts is greater than the dimensions of the bunch, the energy loss

† Cf. K. D. Sinelnikov *et al.*, P/2211, this Volume of these Proceedings, for related experimental information.

‡ For the case of interaction between particles and a resonant cavity the question was examined by Sinelnikov, Akhiezer and Fainberg. Cf. P/2211, this Volume of these Proceedings.

of an individual particle increases N times, where N is the number of particles in the bunch. This follows formally from the fact that the loss $\sim q^2$, where $q = Ne$ is the charge of the bunch.

Taking into account the conditions of coherence, it can be shown that the maximum value of the field which can be excited by a bunch is $E \approx Ne/L_{\max}$, where $L_{\max}$ is the maximum dimension of the bunch. It follows that

$$dW/dz \approx Ne^2/L_{\max}.$$

Thus even in very rare plasma ($n \sim 10^{10} \text{ cm}^{-3}$) the energy loss of the particles increases 10^8 – 10^{10} times when the number of particles in the bunch is $N \sim 10^9$. It should be noted that in many cases there is no need for initial modulation of the beam in order to realize the conditions of coherence. Thus, for example, when a beam of charged particles moves through a plasma with a velocity $v_0 > V_T$, it becomes unstable, the resulting density and velocity fluctuations produce increasing fields, and the beam is modulated at the same time. The rate of growth of these oscillations is ²⁷

$$\gamma_{\max} = \frac{3^{\frac{1}{2}}}{2^{\frac{1}{2}}} \frac{\omega_0}{V_{Te}} \left(\frac{\omega_{pe, \text{beam}}}{\omega_0} \right)^{\frac{2}{3}}$$

When a beam excites longitudinal oscillations of a plasma in a magnetic field,²²

$$\begin{aligned} \gamma_{\pm} \omega &= 3^{\frac{1}{2}} 2^{-\frac{1}{2}} \\ &\times [\omega_{pe, \text{beam}}^2 \cos^2 \theta \omega_{\pm} (\omega_{\pm}^2 - \omega_{ce}^2) \\ &\times (2\omega_{\pm}^2 - \omega_{ce}^2 - \omega_{pe}^2)^{-1}]^{\frac{1}{2}} \end{aligned}$$

where the frequency ω is given by

$$\begin{aligned} \omega^2 &\approx k^2 v^2 \cos^2 \theta \approx \frac{1}{2} (\omega_{ce}^2 + \omega_{pe}^2) \\ &\pm \frac{1}{2} [(\omega_{ce}^2 + \omega_{pe}^2)^2 - 4\omega_{ce}^2 \omega_{pe}^2 \cos^2 \theta]^{\frac{1}{2}}. \end{aligned}$$

Similar phenomena occur when there is an interaction between several beams. For example, in the case of two beams with Maxwell distributions there is an oscillation growth rate ²⁹

$$\gamma_{\max} = \frac{1}{2} \frac{W_0}{v} \left(1 + \frac{V_T^2 \omega^2}{v^2 \omega_0^2} \right)^{-\frac{1}{2}},$$

where v is the average velocity $(v_1 + v_2)/2$ of the beams, ω_0 is their Langmuir frequency, $\omega_{opt}^2 \delta^2 = \frac{3}{4} \omega_0^2 v^2$, and $\delta = v_1 - v_2$.

If the densities of the beams are different and if the temperature of the exciting beam is high, instability occurs at the Doppler-shifted frequency ²³

$$\omega = \omega_{20}/[1 - (v_2/v_1)]$$

where ω_{20} is the Langmuir frequency of the denser beam 2 and v_1 and v_2 are the velocities of the beams.

In a number of cases (for spatially periodic plasma, non-uniform plasmas or plasma in the first stage of instability) coherence is realized not because of a modulation of the beam, but because of a modulation (spatial non-uniformity) of the medium.

In any case, when the conditions for Cherenkov radiation, Doppler radiation, or polarization losses are fulfilled for the individual particles, the beam of particles and the plasma through which it moves become unstable. This follows from the circumstance that the growth rate of the oscillations is greatest for frequencies satisfying the conditions of Cherenkov or Doppler radiation. For example, when the beam passes through a plasma, $\gamma = \gamma_{\max}$ for $\omega = \omega_{pe} (1 - V_T^2/v^2)^{-\frac{1}{2}}$.

These mechanisms of collective interaction can explain instabilities in different types of discharges and the associated oscillations in the plasma. They can also account for the short relaxation time of the Maxwell distribution. Finally, the increase in energy loss of bunches of charged particles can be utilized for injection into magnetic traps and for measurement of plasma parameters. These new methods are based on the fact that the energy loss of the particles of the coherent bunch can be made very large when the energy loss of particles of an unmodulated beam are very small. For example, with $n = 10^{10} \text{ cm}^{-3}$, $N = 10^9$, and the electron energy $W = 50 \text{ kev}$, $dW/dx = 100 \text{ ev/cm}$ when for the unmodulated beam $dW/dx = 10^{-7} \text{ ev/cm}$. The energy loss may be further increased if coherence between bunches is utilized. In this case the distance between bunches must be $L = \beta \lambda$ where $\beta = v_{\text{bunch}}/c$ and λ is the wavelength of the excited wave. It should be noted that to facilitate coherent losses it is necessary to turn to longer waves (different types of ion oscillations). By using coherent interaction it is also possible to excite a desired frequency in systems with many frequencies. For this the conditions of coherency must be fulfilled: the dimensions of the bunches should satisfy the condition $\lambda_1 > a > \lambda_2$ where λ_1 is the wavelength of the wave to be excited.

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Heating of a Confined Plasma by Oscillating Electromagnetic Fields

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We wish to consider the heating of a fully ionized, infinitely long, cylindrical plasma confined by an axial magnetic field. The method of heating originally suggested by Lyman Spitzer, Jr. (1953) involves the application of an external oscillating electromagnetic field over a region of extent L in the axial direction. The oscillating electric field vector in the plasma is perpendicular to the lines of force of the main confining field. Since this oscillating electric field is produced by changing the main magnetic field in time, this method of heating is often denoted by the term "magnetic pumping". Since we are interested in heating the ions to thermonuclear temperatures in a controlled fusion reactor we shall not consider electron heating in the following but it should be noted that some of the methods considered can also be utilized to heat the electrons directly.

It is found that simple analytical results can be obtained in four limiting cases. These cases are classified in terms of the sizes of four characteristic times. These times are: (1) $\tau_{\text{coll}} = 1/\nu$, the time for a 90° deflection of an ion due to many small angle encounters with electrons and ions, (2) $\tau_f = 2\pi/\omega$, the period of the oscillating applied field, (3) τ_{tr} , the time of transit of a typical ion through the length L and (4) $\tau_{\text{ci}} = 2\pi/\omega_{\text{ci}}$, the cyclotron period of an ion. The four cases which have been analyzed are:

- (1) collisional heating, $\tau_{\text{ci}} \ll \tau_{\text{coll}} \sim \tau_f \ll \tau_{\text{tr}}$
- (2) transit-time heating, $\tau_{\text{ci}} \ll \tau_{\text{tr}} \sim \tau_f \ll \tau_{\text{coll}}$
- (3) acoustic heating, $\tau_{\text{ci}} \ll \tau_{\text{coll}} \ll \tau_f \sim \tau_{\text{tr}}$
- (4) ion cyclotron resonance heating, $\tau_{\text{ci}} \sim \tau_f \ll \tau_{\text{tr}} \ll \tau_{\text{coll}}$

In the following four sections we treat the four cases outlined above. In cases (2) and (4), where the collision time is very long compared with the other characteristic times, the question arises as to the time necessary for actual heating in the sense of achieving

randomization of the ordered energy delivered to the system. Finally we discuss a mechanism which can act to thermalize the ordered energy by phase or "fine-scale" mixing.

A further question which may be raised is, why is the plasma not cooled by the oscillating field? In fact, in transit time heating, for certain initial velocities, particles lose energy in traversing the oscillating field region even after an average over the initial phases of the field relative to the particles has been performed. Although it is intuitively obvious that heating will occur, a simple thermodynamic argument is given here which shows that after averaging over a Maxwellian velocity distribution the energy transfer cannot be negative.

Assume that the initial state i is in kinetic equilibrium; i.e., that the velocity distribution is Maxwellian and the spatial distribution is uniform. Denote by f the state of the gas at some future time after the oscillating field has been applied. The transition from state i to state f is adiabatic: i.e., no heat has been added to the plasma ($dQ = 0$). Now let collisions act to bring the plasma to the equilibrium state f' . Clearly the internal energy of the plasma is the same for states f and f' . Since the entire transition from state i to state f' is adiabatic, by the second law of thermodynamics the entropy of state f' , $S(f')$, is greater than $S(i)$. Since the energy of an equilibrium state is a monotonically increasing function of entropy for fixed confining volume, the energy of state f' , $E(f')$, is greater than $E(i)$. Therefore, energy has been added to the plasma by the oscillating field. This argument assumes perfect confinement of the plasma, since it applies only when i , f , and f' contain the same number of particles.

COLLISIONAL HEATING 1-3

It is known that the magnetic moment $\mu = mw_{\perp}^2/2B$ is an adiabatic invariant for a particle in a magnetic field which is sufficiently slowly varying in space and time.⁴ Here m is the particle mass and $w_{\perp}$ the magnitude of the particle velocity perpendicular to the magnetic field B . Thus, in the absence of collisions, a time-varying field will only cause a propor-

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tionate variation in the perpendicular energy $E_{\perp}$. If we imagine a square-wave pulse field superimposed on the main field and further assume that a collision occurs during the time the pulsed field is non-zero, energy will be transferred from $E_{\perp}$ to $E_{\parallel}$. After the field has returned to zero we see that a net energy transfer to the gas has taken place.

Taking into account the two assumptions that the magnetic moment is a constant and the departure from equipartition during a collision time is small, we obtain an approximate set of equations governing this effect:

$$\begin{aligned} \frac{dE_{\perp}}{dt} &= \frac{E_{\perp}}{B} \frac{dB}{dt} + \nu(E_{\parallel} - \frac{1}{2}E_{\perp}) \\ \frac{dE_{\parallel}}{dt} &= -\nu(E_{\parallel} - \frac{1}{2}E_{\perp}), \end{aligned} \quad (1)$$

where ν is the collision frequency. Note that because of our assumption that $\tau_{tr} \gg \tau_{coll}$, changes in ν or B with time resulting from motion in the axial direction can be neglected. If we assume that B is given by

$$B = B_0(1 + \varepsilon \cos \omega t), \quad (2)$$

and treat ε as small compared to unity we can solve Eq. (1) by a perturbation treatment. A simple application of the Floquet theory for differential equations with periodic coefficients leads to the increase in energy ΔE , in a time $2\pi/\omega$,

$$\Delta E = \frac{\pi}{3} \varepsilon^2 E_0 \omega \nu / (9\nu^2/4 + \omega^2). \quad (3)$$

The difference Eq. (3) can be replaced by the differential equation

$$\frac{dE}{dt} = [\frac{1}{3}\varepsilon^2 \omega^2 \nu / (9\nu^2/4 + \omega^2)] E \equiv \alpha E. \quad (4)$$

In investigating the rate of heating from (4), note that the energy dependence of ν is

$$\nu \sim E^{-1/2}. \quad (5)$$

TRANSIT-TIME HEATING ²

In the limiting case of transit-time heating, the assumption is made that the particles pass through the heating section sufficiently fast so that they suffer no collisions during the transit time. It is also assumed that the transit is slow enough for the adiabatic approximation to hold. The equation of motion of an ion moving in the z (axial) direction is then

$$m\ddot{z} = -\mu \partial B / \partial z. \quad (6)$$

We assume for this treatment that B is given by

$$B = B_0 + B_1 f(z) \cos(\omega t - \alpha), \quad (7)$$

where B_0 and B_1 are constants, $f(z)$ describes the field shape, and α is a phase angle specifying the time the ion enters into the field region. Equation (6) then becomes

$$\ddot{z} = -(\mu B_1/m)(df(z)/dz) \cos(\omega t - \alpha). \quad (8)$$

Because of the constancy of the magnetic moment μ , it is clear that in this mode of heating all the energy changes will be due to the parallel velocity changes. That the parallel velocity will change is easily seen if Eq. (8) is interpreted as representing a mechanical system in which a particle passes through a time-varying potential well. It is not so clear, however, what the net effect of the oscillating field on a distribution of particles will be.

Equation (8) can be easily solved by perturbation theory assuming B_1/B_0 to be small. The energy change of a particle $\Delta E = \frac{1}{2}m(w_{\parallel f}^2 - w_{\parallel i}^2)$, where $w_{\parallel i}$ and $w_{\parallel f}$ are the initial and final parallel velocities respectively, can then be obtained and averaged over values of the phase α . In this calculation it is necessary to compute $w_{\parallel f}$ to second order in B_1/B_0 . The result is ²

$$\begin{aligned} \langle \Delta E \rangle_{\alpha} &= -\frac{(\mu B_1)^2 \omega}{4mw^4} \frac{d}{d\omega} \\ &\times [\omega^2 \int_{-\infty}^{\infty} dz f(z) e^{i\omega z l w_{\parallel i}^2}], \end{aligned} \quad (9)$$

where $w_{\parallel i}$ is the zeroth-order parallel velocity. To enable us to analyze this further we have chosen the specific field shape $f(z) = \exp -(z/a)^2$. With this choice (9) becomes

$$\begin{aligned} \langle \Delta E \rangle_{\alpha} &= \frac{\pi}{m} \left(\frac{\mu B_1 a \omega}{2w_{\parallel i}^2} \right)^2 \left[\left(\frac{\omega a}{w_{\parallel i}^2} \right)^2 - 2 \right] \\ &\times \exp \left[-\frac{1}{2}(a\omega/w_{\parallel i})^2 \right]. \end{aligned} \quad (10)$$

Equation (10) clearly shows that particles whose transit time is somewhat greater than the field period are heated, while those for which the converse is true are, in fact, cooled.

A quantity of greater interest is the flux of energy of heated particles passing out of the field section. We obtain this by multiplying $\langle \Delta E \rangle_{\alpha}$ by $w_{\parallel i} n$, where n is the particle density, and then averaging over both the perpendicular and parallel velocities. Thus, we find that the flux F is

$$\begin{aligned} F &= \langle n w_{\parallel i} \langle \Delta E \rangle_{\alpha} \rangle \\ &= (\pi/2)^{1/2} (B_1/B_0)^2 n k T (kT/m)^{1/2} G(x) \end{aligned} \quad (11)$$

where

$$G(x) \equiv -x^2/2 [(1/x) K_1(x) + K_1'(x)] \quad (12)$$

and

$$x = a\omega (m/kT)^{1/2} \sim \tau_{tr}/\tau_f. \quad (13)$$

$G(x)$ is a function with a single maximum at $x = 1.55$ where $G(1.55) = 0.24$. The limiting forms are given by

$$G(x) = \begin{cases} \frac{x^2}{2} \ln \frac{1}{x} & x \ll 1 \\ \left(\frac{x}{2} \right)^{3/2} \pi^{1/2} e^{-x} & x \gg 1 \end{cases}. \quad (14)$$

When this calculation was originally made a different average from that given by (11) was computed and therefore the results given here do not agree with those of Ref. 2.

Simple inequalities for the validity of the second order theory are easy to obtain from the integrated form of Eq. (6). These are:

$$\begin{aligned} \frac{w_{\perp}^2}{w_{\parallel}^2} \frac{B_1}{B_0} &\ll 1 \quad \text{if} \quad \frac{a\omega}{w_{\parallel}} \ll 1, \\ \frac{w_{\perp}^2}{w_{\parallel}^2} \frac{B_1}{B_0} \frac{w_{\parallel}}{a\omega} &\ll 1 \quad \text{if} \quad \frac{a\omega}{w_{\parallel}} \gg 1. \end{aligned} \quad (15)$$

One effect which has ostensibly been neglected in the above treatment is the reflection of some of the particles before a complete transit of the oscillating field region has been made. Necessary conditions for nonreflection can be written down which are satisfied if Eq. (15) is satisfied. Note, however, that in the course of integrating the particle flux over the Maxwell distribution conditions (15) may be violated. Therefore, Eq. (11) should properly be regarded only as an estimate of the flux.

There is one further effect which a more sophisticated treatment of transit-time heating should take into account; namely, that in the analogous situation of mechanical particles passing through a time-varying potential well there will be bunching some distance from the well. In this case, where the particles are charged, it is possible that positive ion waves will be set up in the plasma regions outside the oscillating field section. Under such a circumstance, however, it seems quite likely that either collisions between positive ions of different e/m or the fine-scale mixing phenomena discussed in section (6) would eventually thermalize the energy in the positive ion waves.

ACOUSTIC HEATING ²

If the effective mean-free path is small, the oscillating field will produce density variations which will result in the propagation of sound waves in the plasma. The small effective mean-free path may be due to high densities or to cooperative phenomena which are at present little understood. We consider here the production of energy in the form of sound waves but do not treat the problem of their absorption by the plasma. The following assumptions are made in this calculation: (1) the plasma is treated as an ideal gas satisfying an adiabatic equation of state, (2) the particles do not diffuse appreciably during the period of oscillation, and (3) there is no appreciable mass motion out of the ends of the field region. Further, only those waves are treated which propagate in the axial direction. Under these approximations, the effect of the oscillating field is to produce a radial mass velocity. Let us assume that the plasma has uniform density within the radius r , and zero density outside. Since the material follows the lines of force, the total magnetic flux through the cross section πr^2 is constant. Thus

$$v_r = dr/dt \cong (r/2B_0) dB_1/dt, \quad (16)$$

where we have again assumed that $B_1/B_0 \equiv \epsilon$ is small compared to unity.

The usual equations of motion for small amplitude sound waves are:

$$\begin{aligned} \rho_0 \nabla \cdot \mathbf{v} &= -\partial \rho / \partial t \\ \rho_0 \partial v_z / \partial t &= -\partial p / \partial z \\ p &= C^2 \rho, \end{aligned} \quad (17)$$

where p , ρ , and $C^2 = \gamma kT/m$ are the pressure, density and square of sound velocity, respectively. We further assume that the field is given by

$$B = B_0 \{1 + \epsilon \exp - [(z/a)^2 + i\omega t]\}. \quad (18)$$

Noting that we are only interested in the forced oscillations, we obtain the solution to Eq. (17) in the form

$$\begin{aligned} p &= -\frac{1}{2} i \rho_0 C \omega \epsilon \exp + i\omega t \\ &\times \left\{ \int_{-\infty}^z \exp[-(z'/a)^2 - ik(z-z')] dz' \right. \\ &\left. + \int_z^{\infty} \exp[-(z'/a)^2 + ik(z-z')] dz \right\} \end{aligned} \quad (19)$$

where $k = \omega/C$. However, since it is sufficient for our purposes to consider the traveling wave solutions outside of the forcing region, we let $z \rightarrow +\infty$ in the above expression. The result is

$$p = p_0 k \epsilon (\pi a^2/4)^{1/2} \sin(kz - \omega t) \exp - (ak/2)^2. \quad (20)$$

The flux of energy is $p v_z$ and, therefore, we easily obtain the average flux

$$\begin{aligned} F &= (\pi \gamma^{1/2}/8) \\ &\times (B_1/B_0)^2 n k T (kT/m)^{1/2} x^2 \exp(-x^2/2\gamma), \end{aligned} \quad (21)$$

where x is given by (13).

Upon comparison of this expression for the flux with Eq. (11) for the flux obtained for transit-time heating, it is seen that there is good agreement although these two cases represent extremes as far as the approximations are concerned.

ION CYCLOTRON HEATING ^{5, 6}

In this case the external field is set up in the same way as in the other cases but its frequency is near the ion cyclotron frequency and thus the magnetic moment of the ions need not be constant but increases rapidly. The transit time of the ions across the heating section is long compared to the ion cyclotron period so that the ions undergo many gyrations in their passage through the heating section when collisions are neglected entirely. The heating of ions by this method is very efficient. The ions gain large energies during a single passage through the heating section since the ion particle velocities are always in phase with the electric field and the ions are continually accelerated. Further, the motions of all the separate ions are in phase, being correlated by the external field, so that large currents are developed in the plasma. Therefore, we must consider the field produced by the plasma currents in calculating the heating of the ions.

The ions will, in general, have a radial motion relative to the magnetic lines of force due to their correlated motion and their larger Larmor radii.

Since the electrons are tied to these lines, a space charge will be set up producing a radial electric field. It can be seen that this radial electric field when combined with the externally produced azimuthal electric field will produce a circularly polarized electric field whose sense is just opposite to the motion of the ions. This circularly polarized component produces no energy increase of the ions. Thus, in some sense we may say that the plasma has shielded itself from the circularly polarized component of the external field which rotates in the correct sense to increase the energy of the ions.

The way around this difficulty was pointed out by Stix⁶ who suggested that the space charge could be cancelled by the electrons if the electric field did not have a uniform phase along the axial direction. Thus, in some regions, the electric field has an opposite sign to the field in adjacent regions. In these regions a positive space charge is built up while simultaneously in adjacent sections a negative space charge is built up. Electrons must then flow along the lines of force to cancel these space charges. If the electrons are able to cancel out the space charge totally, one would have the total externally applied field acting to heat the ions. However, the actual current of the electrons oscillating at the cyclotron frequency produces a back emf on the electrons. Since the total electric field along the lines of force must be zero, the total space charge cannot be completely cancelled and there must be a residual space charge to produce a field along the lines to cancel this back emf. This space charge also produces a radial field which combines with the applied electromagnetic field so that even in this case the circularly polarized field, in the favorable sense, is somewhat shielded. Stix⁷ has considered this situation under slightly different circumstances and has established that the favorably polarized component is able to penetrate the plasma appreciably.

In analyzing this problem, the electromagnetic fields are not known in advance but must be solved for in a self-consistent manner. With complex notation their behavior was assumed to be given by

$$\epsilon = E_r + iE_\theta = (\epsilon_+ e^{i\omega t} + \epsilon_- e^{-i\omega t}) f(z), \quad (22)$$

with the shape factor $f(z)$ given by

$$f(z) = \begin{cases} \sin kz & -a \leq z \leq a \\ 0 & |z| > a \end{cases} \quad (23)$$

with $ka = \pi N$, N being an integer. ϵ_- and ϵ_+ represent the two circularly polarized components of the electric field in the favorable and unfavorable senses, respectively. The assumed form is then justified by the self-consistent calculation and the forms of ϵ_+ and ϵ_- determined. It should be emphasized that, due to the form of f , the ions will not see the frequency of the impressed electric field but a Doppler shifted frequency $\omega \pm kv_\parallel$, so that for ω different from ω_{ci} some ions will always be heated.

The motion of the ions is solved in the field (22) neglecting variations of $\epsilon_\pm(r)$ across the orbit, neglect-

ing the effect of the oscillating magnetic field, and assuming that the axial velocity is uniform. Upon summing over particles with all possible axial velocities, one finds that to a good approximation the ion current may be given by

$$j_r + ij_\theta \cong (e^2 n / mk V_T) \epsilon_-(r) I_0(\alpha) \sin kz e^{-i\omega t}, \quad (24)$$

where n is the ion number density, V_T the mean axial velocity, $\alpha = \omega - \omega_{ci} / k V_T$, and $I_0(\alpha)$ is a nearly constant complex quantity of order unity. The transverse electron current is negligible and the axial current is chosen to make $\nabla \cdot \mathbf{j} = 0$ since the space charge must be small compared to $\nabla \cdot \mathbf{j}_\perp$. Since $E_\parallel = 0$ due to the mobility of electrons along the lines of force, we may solve

$$\nabla \times (\nabla \times \mathbf{E}) = (4\pi/c) \partial \mathbf{j} / \partial t \quad (25)$$

which results from Maxwell's equations, to find

$$\epsilon_+^* + \epsilon_- = i\gamma I_0 \epsilon_- \quad (26)$$

where

$$\gamma = (4\pi e^2 n / m) (1/c^2 k^2) \omega / k V_T \quad (27)$$

and ϵ_+^* is the complex conjugate to ϵ_+ . For high densities γ is large and (26) says that the field produced by the plasma must be very large compared to the favorably circularly polarized component which is driving the ions. Thus it is clear that the ϵ_+ component must predominate over ϵ_- . On imposing the boundary conditions, it is found that ϵ_+ is nearly that expected from the impressed field in the absence of plasma so that ϵ_- is shielded by the plasma. On the other hand, if γ is much less than unity, the back reaction of the plasma may be neglected.

The total energy flux of the plasma passing out the ends is

$$F = \frac{\pi^{3/2} N n e^2}{2mk^2 V_T} \left| \frac{\epsilon_+}{1 + i\gamma I_0} \right|^2 e^{-\alpha^2}. \quad (28)$$

This result is in agreement with that obtained by Stix.⁶

F seems to be a maximum for $\alpha = 0$ but a closer examination of the particles shows that those with small parallel velocities are heated to large energies while those with moderate velocities are not heated at all. All the particles would be heated to a somewhat smaller total energy if $\alpha \approx 1$, which corresponds to that frequency at which particles with the mean axial velocity see the cyclotron frequency.

FINE SCALE MIXING AS A MEANS FOR RANDOMIZING ORGANIZED MOTION

In collisional and acoustic heating the possibility exists that the energy transferred to the plasma by the electromagnetic field goes initially into organized motion. It is important to know the rate at which oscillations produced by such pumping are converted into random motion of the ions. If this rate is too low, then the pumping mechanism is inadequate for practical purposes. Of course, collisional damping

will always occur, but for the temperatures of interest this damping mechanism is relatively slow.

However, there is another damping mechanism which is very effective. It is similar to the damping found by Landau⁸ and Van Kampen⁹ for electrostatic plasma oscillations. This mechanism depends on phase mixing of the wave due to thermal motions in the plasma. It is this process which will be discussed in this section.

The physical picture for this type of damping is the following. Originally one has an organized wave in the plasma and because of thermal motion, ions which are in one region of the wave move into adjoining regions where the phase of the wave is different. Thus the various parts of the wave become mixed and the organized motion disappears. This process does not lead to a Maxwell distribution in general, but it does lead to random motions of the ions. Once this mixing has occurred, collisions become much more effective and probably produce a Maxwell distribution in a short time. Even if this does not happen, however, the random motion is equivalent to a temperature.

To illustrate this mechanism we shall consider the case of ion cyclotron heating. A simple model of the plasma which is used is the following: the plasma is taken to consist of a number of cold beams of particles traveling along the magnetic field. This is an approximation to the thermal motion along the field but does not take into account thermal motions perpendicular to the field. This is done since thermal motion perpendicular to the field has little influence on the effect under consideration.

The linearized equations of motion for this case are

$$\begin{aligned} m_i (\partial \mathbf{v}_i / \partial t + \mathbf{v}_i \cdot \nabla \mathbf{v}_i) &= [e_i / c] (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}_0 + \mathbf{V}_i \times \mathbf{B}) \\ \nabla \times \mathbf{E} &= - \frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{E} &= 4\pi \sum_i e_i (n_i - n_{i0}) \\ \nabla \times \mathbf{B} &= \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + 4\pi \sum_i n_{i0} e_i \mathbf{v}_i \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned} \quad (29)$$

The quantities appearing in the above equations have the following meanings:

- $m_i \equiv$ mass of the particles in the i th beam
- $e_i \equiv$ charge of the particles in the i th beam
- $\mathbf{v}_i \equiv$ the velocity perturbation of particles in the i th beam
- $\mathbf{V}_i \equiv$ the zeroth-order velocity of the i th beam, with $\mathbf{V}_i = V_i \mathbf{e}_z$.

$\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively due to the disturbance, n_i and n_{i0} are the perturbed and zeroth-order number densities in the i th beam respectively, and $\mathbf{B}_0$ is the zeroth-order magnetic field with $\mathbf{B}_0 = B_0 \mathbf{e}_z$. The beams may be composed of either ions or electrons.

In investigating this problem, the simplest type of waves which show the effect were considered. These

were transverse, circularly polarized waves traveling along the magnetic field. It should be mentioned that Gershman¹⁰ has treated a closely-related problem. He solved the initial value problem for hydromagnetic waves traveling along the magnetic field. His method of treatment was the same as that used by Landau for treating the electrostatic oscillation problem. The approach presented here is closely akin to van Kampen's treatment of electrostatic oscillations. Our problem is also different in that we are interested in the propagation of waves out of a pumping region rather than in the time behavior of an initial perturbation. Our treatment covers waves with frequencies near the cyclotron frequency as well as hydromagnetic waves.

The various quantities entering in the calculation were taken to be of the form

$$\begin{aligned} \mathbf{v}_i &\perp \mathbf{B}_0, \quad \mathbf{B} \perp \mathbf{B}_0, \quad \mathbf{E} \perp \mathbf{B}_0 \\ \mathbf{v}_i &= \mathbf{v}_i^0 e^{i(\omega t - kz)} \\ \mathbf{E} &= \mathbf{E}^0 e^{i(\omega t - kz)} \\ \mathbf{B} &= \mathbf{B}^0 e^{i(\omega t - kz)}. \end{aligned} \quad (30)$$

When these expressions are substituted in the equations of motion the following relations are found

$$(ck)^2 = \omega^2 - \sum_i \frac{\omega_{pi}^2 [\omega - k_i V_i]}{\omega - k V_i \pm \omega_{ci}}, \quad (31)$$

$$v_i^0 = \frac{B_{ci}}{m_i ck} \frac{\omega - k V_i}{\omega - k V_i \pm \omega_{ci}} \quad (32)$$

where

$$\begin{aligned} \omega_{pi}^2 &= 4\pi e^2 n_{i0} / m_i, \\ \omega_{ci} &= |e_i| B_0 / m_i c. \end{aligned} \quad (33)$$

The first equation is the dispersion relation, the second gives the perturbation velocity in terms of the magnetic field. The plus and minus sign depends on the polarization and the charge on the particles. If the positive sign is used for ions then the negative sign is used for electrons. Observe that for certain values of ω and k the denominator vanishes in the expression for v_i^0 . This occurs for those particles which see their cyclotron frequency when the polarization of the wave is appropriate. It is these particles which are close to resonance which are most strongly perturbed.

The dispersion relation is most easily analyzed by making a plot like that shown in Fig. 1. This figure is obtained by fixing ω and plotting the left- and right-hand sides of the dispersion relation. The left-hand side is the parabola $(ck)^2$ and the right-hand side is the more complicated curve. Every place one of the denominators in the dispersion relation goes to zero, the right-hand side goes to infinity. The two curves cross at a number of points. Each crossing gives a possible k for the given ω . Thus the plasma can transmit many waves with the frequency ω .

When the plasma is pumped at the frequency ω , each of these waves will be excited to some degree. The amplitude of each wave will depend on the details of the pumping mechanism. In the region where the

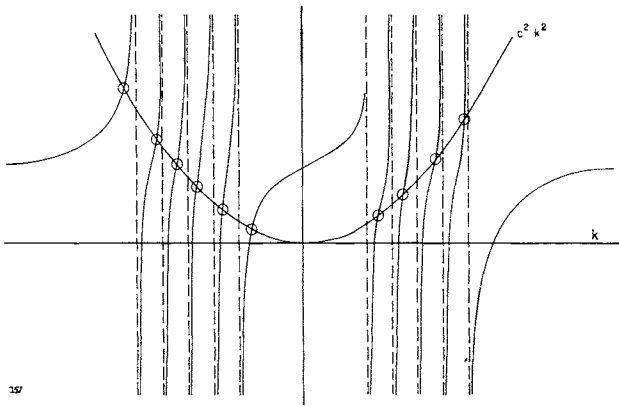


Figure 1. Typical dispersion current for cyclotron waves

pumping occurs the various waves will be forced to be in phase. As one leaves this region the various waves will get out of phase with each other. If the pumping mechanism is such as to excite a large number of waves almost equally, then this phase mixing will result in the disappearance of all macroscopic quantities. This is not true, however, for the microscopic motion. Each beam is excited mainly by one k ; that k which is closest to resonance for it. This can be seen from the expression for v_i . Thus, what was originally an organized motion is converted into effectively random motion of the ions.

More detailed calculations bear out this result. It is found that as the plasma is pumped so that a hydromagnetic wave is produced, then little or no damping results. This is the case if the predominant k is far from resonance for most of the particles. Going to the limit of an infinite number of beams so as to represent a continuous velocity distribution function $F(V)$, normalized to unity, we find that the damping distance for hydromagnetic waves is

$$(kc)^2/\omega_{pi}^2 \omega_{ci} F[(\omega + \omega_{ci})/k].$$

Stix⁶ has obtained a similar result by another method. If the predominant k falls in a region where many beams are near resonance, the damping is fast and occurs in less than a wave length. If one again goes to the limit of an infinite number of beams, one finds that in this case no coherent wave leaves the pumping section.

CONCLUSION

In summary, it is of interest to examine the temperature dependence of the rates of heating of the various types of magnetic pumping. The collisional heating is more effective at lower temperatures, the rate of heating dropping off as $1/T^{1/2}$ at higher temperatures since the collision cross section becomes small as $1/T^2$. However, the transit time heating and acoustic heating improve at higher temperatures; the rate of heating varying as $T^{3/2}$ if the frequency is optimized at each temperature. In ion cyclotron heating the electromagnetic shielding is given by the factor γ which decreases at higher temperatures. Thus ion cyclotron heating increases as $T^{1/2}$ for γ greater than one at higher temperatures and then decreases as $1/T^{1/2}$ for γ less than one. It should be remarked, however, that this type of heating is rather sudden; the particles can be heated from near zero temperature to extremely high temperature on passing through the heating section. Finally, it should be mentioned that the randomization of organized motion by fine scale mixing is more effective at higher temperatures.

In conclusion the authors must express their deepest gratitude to Lyman Spitzer, Jr. for not only initiating the basic ideas involved in this paper but for constant encouragement and advice in the carrying out of their development.

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Absorption of High-frequency Electromagnetic Energy in a High-temperature Plasma

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In a plasma of high temperature, the collision frequency is very low and there is a decrease in the effectiveness of known heating mechanisms which are based on collisions. The problem then arises of *noncollisional* plasma heating with high-frequency electromagnetic fields. By the term *high-frequency field*, we mean electromagnetic oscillations whose period is much smaller than the characteristic confinement times of the hot plasma.

The interaction of a high-frequency field with a plasma can also be used as a probe for studying its properties.

The solution of these two theoretical problems is associated with an investigation of the penetration of electromagnetic waves into a plasma, i.e., with an analysis of the electromagnetic absorption spectrum caused by various mechanisms and with the investigation of the rôle which these mechanisms play in heating of plasma.

In this paper an analysis of the cyclotron and Cherenkov mechanisms is given. These are two fundamental mechanisms for noncollisional absorption of electromagnetic radiation by a plasma in a magnetic field. The expressions for the dielectric permeability tensor, $\epsilon_{\alpha\beta}$, for a plasma with a nonisotropic temperature distribution in a magnetic field, are obtained by integrating the kinetic equation with Lagrangian particle co-ordinates in a form suitable to allow a comprehensive physical interpretation of the absorption mechanisms.

Using the tensor $\epsilon_{\alpha\beta}$ and taking into account the thermal motion, the oscillations of a plasma column stabilized by a longitudinal field have been analyzed. For a uniform plasma, the frequency spectrum has been obtained together with the direction of electromagnetic wave propagation when both the cyclotron and Cherenkov absorption mechanisms take place. The influence of nonlinear effects on the electromagnetic wave absorption and the part which cyclotron and Cherenkov absorption play in plasma heating have also been investigated.

Any mechanism responsible for the absorption of electromagnetic energy involves the accumulation

of energy by a charge in an electric field and the subsequent redistribution of this energy among the various degrees of freedom in the random motion. Absorption mechanisms can be physically classified as noncollisional or resonant processes and collisional or nonresonant processes.

A continuous accumulation of energy by a charge is possible in a static or quasi-stationary field if there is no dissipation. But if the charge is in a high-frequency field, it is continually accumulating and losing energy alternately. If plasma heating is to be effective, it is necessary that the characteristic collision time required for energy dissipation be comparable with the oscillation period. The corresponding absorption mechanism, which is associated with frequent collisions, will be called here the non-resonant or collisional mechanism. It includes ordinary Joule heating as well as *betatron* or *gyrorelaxational* heating proposed by Budker in 1951 and independently by Schlüter.¹

If the collision time is much longer than the oscillation period, the continuous effective energy accumulation by a charged particle will be possible only under resonance conditions when the direction of the electric field coincides with the direction of increasing velocity. This mechanism will be called resonant or noncollisional absorption.

In principle, resonances are possible in the following cases.

(A) If a charge performs a periodic motion there will be a resonance when the field frequency coincides with the motional frequency.

(B) For a charge in a magnetic field there is an ion or electron cyclotron resonance. If the particle velocity along a line of force is v_z , the frequency spectrum of the absorbed radiation is determined by the Doppler effect according to the relation

$$\omega - k_z v_z = \omega_{ce}, \quad (1)$$

where k_z is the z component of the wave vector, $\omega_{ce} = eH/mc$ and the other symbols have their usual meaning.

(C) Resonance can also take place in the absence of a periodic motion provided that the charged particle

Original language: Russian.

velocity is equal to the phase velocity of the electromagnetic wave. This is, of course, possible only for slow waves.

(D) If there is no magnetic field, the resonant accumulation of energy by charged particles takes place only for longitudinal electromagnetic plasma oscillations. The corresponding absorption caused by electron oscillations is known as Landau damping. Damping of a similar type takes place when there are ion (sonic) oscillations.

(E) For a sufficiently dense plasma in a magnetic field, resonance for the transverse branch of the electromagnetic oscillations is also possible. The index of refraction is of the order $N \sim \omega_{pe}/\omega_{ce}$, where $\omega_{pe}^2 \equiv 4\pi e^2 n/m$, so that the electromagnetic phase velocity is $v_{ph} = (H^2/4\pi m)^{1/2}$. The ratio of the phase velocity to the thermal velocity of the charged particle is $v_{ph}/v_T = (H^2/8\pi P)^{1/2} \equiv \beta^{-1/2}$. Therefore, when the plasma pressure P is approximately equal to the magnetic pressure, the wave velocity and the average thermal velocity are of the same order. Under these conditions, the inverse Cherenkov absorption mechanism becomes substantial.² A charge moving along a magnetic field is in resonance with the wave field where the following condition is satisfied

$$\omega - k_z v_z = 0. \quad (2)$$

If $\omega/k_z < c$, this condition, as well as condition (1), determines a velocity of the charges which will absorb energy.

Because of the spread of thermal velocities, the frequency range for which absorption takes place may become very broad for both cyclotron and Cherenkov absorption.

There is one fundamental difference between collisional and non-collisional heating. In the first case, when collisions are important, the velocity distribution during the heating process is nearly Maxwellian while the temperature is rising. In the second case, when only the charges moving in resonance accumulate energy, the velocity distribution during heating will differ greatly from the Maxwellian distribution. In this case, the term *heating* will mean an increase in the average random energy of the electrons and ions.

The interaction of an electromagnetic field with a plasma is determined, in principle, by the Maxwell self-consistent equations together with the kinetic equations for the electron and ion plasma. Under the assumption that the velocity associated with the regular motion of the charges in a wave field is smaller than the average thermal velocity, these equations are linear in the field quantities. In this case, many problems concerning plasma oscillations can be suitably solved by using the dielectric permeability tensor $\epsilon_{\alpha\beta} = \epsilon'_{\alpha\beta} + i\epsilon''_{\alpha\beta}$, whose antihermitian part ($i\epsilon''_{\alpha\beta} = (\epsilon_{\alpha\beta} - \epsilon_{\beta\alpha}^*)/2$) determines the absorption. When there are no collisions, the general form of the dielectric tensor can be expressed explicitly provided that the kinetic equations are integrated in Lagrangian

coordinates. The correction to the distribution function which one obtains in this way is given by

$$f_1 = -e \int_{-\infty}^t \{ \mathbf{E}(\mathbf{r}(t'), t') + c^{-1} \mathbf{v}(t') \times \mathbf{H}(\mathbf{r}(t'), t') \} \\ \times \partial f_0 / \partial \mathbf{p}(t') dt'. \quad (3)$$

If the fields vary harmonically in space and time, i.e., as $\exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$, the corresponding tensor $\epsilon_{\alpha\beta}$ will be

$$\epsilon_{\alpha\beta} = \delta_{\alpha\beta} + i \sum \int_0^\infty K_{\alpha\beta}(t) dt, \quad (4)$$

with real and imaginary parts

$$\epsilon'_{\alpha\beta} = \delta_{\alpha\beta} + (i/2) \sum \int_{-\infty}^{+\infty} K_{\alpha\beta}(t) dt \\ \epsilon''_{\alpha\beta} = \frac{1}{2} \sum \int_{-\infty}^{+\infty} K_{\alpha\beta}(t) dt.$$

The summation is carried out with respect to the different species of charged particles. The function $K_{\alpha\beta}(t)$ for each species, is

$$K_{\alpha\beta}(t) = \frac{4\pi e^2 n}{\omega} \int v_\alpha(t) \left\{ \left(1 - \frac{\mathbf{k} \cdot \mathbf{v}}{\omega} \right) \frac{\partial f_0}{\partial p_\beta} + \frac{v_\beta}{\omega} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{p}} \right\} \\ \times [\exp i(\omega t - \mathbf{k} \cdot \int_0^t \mathbf{v}(t') dt')], \quad (5)$$

where $\mathbf{v}(t)$ is the particle velocity in the unperturbed plasma expressed in terms of the initial velocity v and the time t . The integration is carried out with respect to the initial momenta. For the particular case of the Maxwellian distribution, the expression in brackets reduces to the simple expression $^2 v_\beta f_0/T$.

For a plasma with no collisions in a uniform magnetic field directed along the z axis, we have

$$v_x(t) = v_x \cos \omega_{ce} t - v_y \sin \omega_{ce} t \\ v_y(t) = v_x \sin \omega_{ce} t + v_y \cos \omega_{ce} t \\ v_z(t) = v_z. \quad (6)$$

The quantity in the exponent of (5) can then be expressed in terms of (6)

$$\omega t - \mathbf{k} \cdot \int_0^t \mathbf{v} dt' = (\omega - k_z v_z) t - (k_x v_x / \omega_{ce}) \sin \omega_{ce} t \\ + (k_x v_y / \omega_{ce}) (1 - \cos \omega_{ce} t), \quad (7)$$

where $\mathbf{k} = (k_x, 0, k_z)$.

If the unperturbed distribution function, f_0 , is a product of Maxwellian distributions with different temperatures corresponding to particle motions parallel and perpendicular to $\mathbf{H}_0$ ($T_\parallel$ and $T_\perp$), the x, y components of the expression in the brackets of (5) will be

$$v_{x,y} \left[\frac{1}{T_\perp} - \frac{k_z v_z}{\omega_{ce}} \left(\frac{1}{T_\perp} - \frac{1}{T_\parallel} \right) \right] f_0$$

and the z component is

$$v_z \left[\frac{1}{T_\parallel} + \frac{k_x v_x}{\omega_{ce}} \left(\frac{1}{T_\perp} - \frac{1}{T_\parallel} \right) \right] f_0.$$

By substituting these expressions, together with (6) and (7), into (5) and (4), closed formulae are readily obtained for the tensor $\varepsilon_{\alpha\beta}$.

In a strong magnetic field and for relatively low frequencies, $\omega \lesssim \omega_{ce}$, the expression for $\varepsilon_{\alpha\beta}$ is greatly simplified by expanding the exponent in powers of $k_x v_{\perp}/\omega_{ce}$. In the zero-order approximation, we have

$$\begin{aligned}\varepsilon &= 1 + \frac{i}{2} \sum \frac{\omega_p^2}{\omega} \left\langle \left[1 + \frac{k_z v_z}{\omega} \left(1 - \frac{T_{\perp}}{T_{\parallel}} \right) \right] \delta_+ (\omega - \omega_c - k_z v_z) + \left[1 - \frac{k_z v_z}{\omega} \left(1 - \frac{T_{\perp}}{T_{\parallel}} \right) \right] \delta_+ (\omega + \omega_c - k_z v_z) \right\rangle, \\ g &= \frac{i}{2} \sum \frac{\omega_p^2}{\omega} \left\langle \left[1 - \frac{k_z v_z}{\omega} \left(1 - \frac{T_{\perp}}{T_{\parallel}} \right) \right] \delta_+ (\omega - \omega_c - k_z v_z) - \left[1 + \frac{k_z v_z}{\omega} \left(1 - \frac{T_{\perp}}{T_{\parallel}} \right) \right] \delta_+ (\omega + \omega_c - k_z v_z) \right\rangle, \\ \eta &= 1 + i\pi \sum \frac{\omega_p^2}{\omega} \left\langle \frac{m v_z^2}{T_{\parallel}} \delta_+ (\omega - k_z v_z) \right\rangle,\end{aligned}\quad (9)$$

where the possibility of anisotropic unperturbed temperatures is taken into account. The brackets $\langle \rangle$ denote an average over a Maxwellian distribution along the magnetic field and the quantity $\delta_+(x) = i/x + \pi\delta(x)$ is defined in terms of the Dirac delta function. For given ω, k , the argument of the delta functions determine the particle velocities for which these particles are resonant with the wave field.

$$\begin{aligned}\varepsilon &= 1 + \sum \frac{1}{2} \left(\frac{\omega_0}{\omega} \right)^2 \left\{ 1 - \frac{T_{\perp}}{T_{\parallel}} + \left[1 - \left(1 - \frac{T_{\perp}}{T_{\parallel}} \right) \left(1 - \frac{\omega_H}{\omega} \right) \right] \frac{i\pi^{\frac{1}{2}} \omega}{k_z v_{\parallel}} W \left(\frac{\omega - \omega_H}{k_z v_{\parallel}} \right) \right. \\ &\quad \times \left. 1 - \frac{T_{\perp}}{T_{\parallel}} + \left[1 - \left(1 - \frac{T_{\perp}}{T_{\parallel}} \right) \left(1 + \frac{\omega_H}{\omega} \right) \right] \frac{i\pi^{\frac{1}{2}} \omega}{k_z v_{\parallel}} W \left(\frac{\omega + \omega_H}{k_z v_{\parallel}} \right) \right\} \\ g &= \sum \frac{1}{2} \left(\frac{\omega_0}{\omega} \right)^2 \left\{ \left[1 - \left(1 - \frac{T_{\perp}}{T_{\parallel}} \right) \left(1 - \frac{\omega_H}{\omega} \right) \right] \frac{i\pi^{\frac{1}{2}} \omega}{k_z v_{\parallel}} W \left(\frac{\omega - \omega_H}{k_z v_{\parallel}} \right) \right. \\ &\quad \left. - \left[1 - \left(1 - \frac{T_{\perp}}{T_{\parallel}} \right) \left(1 + \frac{\omega_H}{\omega} \right) \right] \frac{i\pi^{\frac{1}{2}} \omega}{k_z v_{\parallel}} W \left(\frac{\omega + \omega_H}{k_z v_{\parallel}} \right) \right\} \\ \eta &= 1 + \sum 2 \left(\frac{\omega_0}{k_z v_{\parallel}} \right)^2 \left\{ \frac{i\pi^{\frac{1}{2}} \omega}{k_z v_{\parallel}} W \left(\frac{\omega}{k_z v_{\parallel}} \right) + 1 \right\}.\end{aligned}\quad (10)$$

where $v_{\parallel} = (2T_{\parallel}/m)^{\frac{1}{2}}$ and

$$W(z) \equiv e^{-z^2} (1 + 2i\pi^{\frac{1}{2}} \int^z e^{-t^2} dt)$$

is the probability integral with a complex argument.³ The approximation $k_{\perp} v_{\perp}/\omega_{ce} \ll 1$ made in (10) is valid provided that the Larmor radius is small compared with the characteristic transverse length in the problem. In this approximation it is easy to obtain a solution to several problems associated with plasma oscillations in a uniform magnetic field even when the extent of the plasma is limited in the direction perpendicular to the field.

Let us consider the oscillations in a pinched plasma stabilized by a longitudinal magnetic field, taking into account the thermal motion of the charged particles. The plasma pressure can be counterbalanced by the magnetic pressure of the longitudinal and azimuthal external fields. The azimuthal field is produced by

$\omega t - \mathbf{k} \cdot \mathbf{v} dt = (\omega - k_z v_z)t$ and the tensor $\varepsilon_{\alpha\beta}$ will be

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} \varepsilon & ig & 0 \\ -ig & \varepsilon & 0 \\ 0 & 0 & \eta \end{pmatrix}. \quad (8)$$

The components ε, g and η are given by the expressions

A comparison with (1) and (2) shows that the cyclotron absorption resonance is determined by the imaginary part of ε and g , the Doppler effect being taken into account; the Cherenkov absorption, associated with particle motion along the lines of magnetic force, is determined by the imaginary part of η . Averaging over the longitudinal velocities yields the components of the tensor $\varepsilon_{\alpha\beta}$

a longitudinal current flowing on the surface of the plasma. The oscillations are taken to be of the form $\exp i(kz + m\phi - \omega t)$. The basic equations are the Maxwell equations

$$\nabla(\nabla \cdot \mathbf{D}) - \Delta \mathbf{E} = (\omega/c)^2 \mathbf{D} \quad (11)$$

where $D_r = \varepsilon E_r + ig E_{\phi}$, $D_{\phi} = -ig E_r + \varepsilon E_{\phi}$ and $D_z = \eta E_z$. The boundary conditions are:

(1) The tangential components of the electric field on the surface of the pinch are continuous and vanish on the surface of an infinite conductivity (metal) cylinder which surrounds the plasma column.

(2) The normal component of the pressure tensor is continuous at the surface of the perturbed plasma cylinder.

Solutions of the equations for the electric field are expressed in terms of Bessel functions whose argument is $\kappa_{1,2} r$, where

$$\kappa_{1,2}^2 \equiv [\eta(\varepsilon\omega^2/c^2 - k^2) + ig(ig - \varepsilon\lambda_{1,2})(\omega/c)^2]/\varepsilon$$

and the variables λ_1 and λ_2 are roots of the equation

$$\lambda^2 + i\lambda[g^2 + [(kc/\omega)^2 - \varepsilon](\eta - \varepsilon)](g\varepsilon)^{-1} + [(kc/\omega)^2 + \eta - \varepsilon] = 0.$$

Neglecting the thermal motion with $k = 0$, a dispersion equation first deduced by Körper⁴ can be obtained.

Let us now investigate cyclotron absorption by ions. For frequencies satisfying the condition

$$kc, \omega_c \gg \omega \gg kv_{\parallel}, kv_A$$

(where v_A is the Alfvén velocity), an expansion in η^{-1} can be made. If the plasma pressure at the pinch boundary is also neglected in comparison with the magnetic pressure, the dispersion equation can be written as follows for the $m = 0$ mode

$$\begin{aligned} \alpha a h_i^2 \frac{I_0(\alpha a)}{I_1(\alpha a)} \\ = 1 - h_e^2 k a \frac{K_0(ka)I_1(kb) + I_0(ka)K_1(kb)}{K_0(ka)I_1(kb) - I_0(ka)K_1(kb)}, \end{aligned} \quad (12)$$

where

$$\begin{aligned} h_i &\equiv (H_z)_i/H_\phi(a), \quad h_e \equiv (H_z)_e/H_\phi(a), \\ \alpha^2 &= \{[k^2 - (\omega/c)^2\varepsilon]^2 - (g\omega/c^2)^2\}/[(\omega/c)^2\varepsilon - k^2], \end{aligned}$$

and a, b are the radii of unperturbed pinch and external conductor, respectively. With $H_\phi = 0$, $b = \infty$, $(H_z)_e = (H_z)_i$, and neglecting the thermal motion in ε and g , this formula agrees with a result deduced by Stix⁵ for the oscillations of a cold plasma cylinder in a longitudinal magnetic field.

In contrast to other work,^{4,5} the influence of the thermal motion on noncollisional cyclotron damping is taken into account. The dispersion equation permits us to find the imaginary part of the frequency which determines the rate of cyclotron energy absorption by the plasma. Then, knowing the expressions for the electromagnetic field, one can determine the amount of energy absorbed by the plasma.

There is no Cherenkov absorption in Eq. (12). To take this into account, the higher-order terms in the η^{-1} and $k_{\perp}v_{\perp}/\omega_H$ expansions should be evaluated. The Cherenkov absorption of low-frequency waves in a pinched plasma is discussed elsewhere.⁶

The analysis of the complete dispersion equation for a plasma cylinder is a rather complicated procedure. It is of interest to investigate first the absorption spectrum for the simpler case of a uniform plasma. In this case, the wave damping is determined by the imaginary part of the index of refraction, $N = p(1 + i\kappa)$. With the above assumption, namely $k_{\perp}v_{\perp}/\omega_c \ll 1$, the indices of refraction of both the extraordinary and ordinary waves, N_1 and N_2 , are expressed in terms of $\varepsilon_{\alpha\beta}$ by the same formulae which were employed when thermal motion was not taken into account. Therefore, for the propagation across the magnetic field of a linearly polarized wave with its electric vector perpendicular to the constant

magnetic field, one finds $N_1^2 = (\varepsilon^2 - g^2)/\varepsilon$; while $N_2^2 = \eta$ for waves with the electric vector along the direction of the magnetic field.

In the case of transverse wave propagation, $k_z = 0$, there is no absorption and the formulae for $\varepsilon_{\alpha\beta}$ take the same form as when the thermal motion is not taken into account. However, even for purely transverse wave propagation, there exist absorption regions in the plasma that are beyond the range of validity of the assumptions made in order to obtain the expression (10) for the dielectric tensor. For example, the calculation of the relativistic Doppler effect shows that there is cyclotron absorption with higher harmonics.

For frequencies substantially greater than ω_{ce} and in a frequency range where $\text{Re } N \gg 1$, Cherenkov absorption will also occur when the thermal motion of charged particles across the magnetic field is taken into account. Formally, this result follows from the fact that with $\omega \gg \omega_{ce}$ it can be assumed in Eq. (7) that $\sin \omega_{ce}t \simeq \omega_{ce}t$ and, as a consequence, one obtains $\omega = k_{\perp}v_{\perp}$ for the resonant frequency. This is just the condition for Cherenkov radiation and the corresponding Cherenkov absorption. For waves propagating in the direction of the magnetic field the refraction index squared is $N_{1,2}^2 = \varepsilon \pm g$ and, therefore, has no component η which can give rise to Cherenkov absorption. The absence of Cherenkov absorption for both longitudinal and transverse waves when $\omega \lesssim \omega_{ce}$ is connected with the fact that according to the theory of Cherenkov radiation, there can be no (Cherenkov) radiation in these directions from the motion of a charged particle along a line of force. For frequencies $\omega \gg \omega_{ce}$, the conception that the charge is "attached" to a line of force is not valid and, therefore, transverse Cherenkov radiation is also possible.

Let us now determine the absorption edges where $\text{Im } N \ll \text{Re } N$ and $\text{Im } \varepsilon \ll \text{Re } \varepsilon$; then one can make an expansion in terms of the imaginary parts of ε, g and η and obtain the index of refraction in the form $N = p(1 + i\kappa)$. For longitudinal wave propagation, when $\kappa \ll 1$, the cyclotron absorption by electrons is described by the following expressions²

$$\begin{aligned} p^2 &= 1 + \frac{2c}{v_{\parallel}p} \left(\frac{\omega_{pe}}{\omega} \right)^2 e^{-(\tilde{\omega}c/v_{\parallel}p)^2} \int_0^{\tilde{\omega}c/v_{\parallel}p} e^{t^2} dt \\ \kappa &= \frac{\pi^{1/2}c}{2p^3v_{\parallel}} \left(\frac{\omega_{pe}}{\omega} \right)^2 e^{-(\tilde{\omega}c/v_{\parallel}p)^2}, \end{aligned}$$

where

$$\tilde{\omega} \equiv (\omega - \omega_{ce})/\omega. \quad (13)$$

For oblique wave propagation, the absorption edge is caused by Cherenkov absorption by electrons with $\omega \ll \omega_{ce}$, where the cyclotron resonance absorption is no longer significant. With the assumption that

$$\omega_{pe}/\omega_{ce}^2 \gg 1$$

and

$$\sin^4 \theta / \cos^2 \theta \ll 4\omega_{pe}^4/\omega_{ce}^2\omega^2,$$

we have

$$N^2 = \frac{\omega_{pe}^2}{\omega[\omega_{ce} \cos \theta - \omega]} \times \left\{ 1 + i \frac{\pi^{\frac{1}{2}}}{2} \left(\frac{c}{v_{\parallel}} \right)^3 \frac{\omega^{\frac{1}{2}}}{\omega_{pe} [\omega_{ce} \cos \theta - \omega]^{\frac{1}{2}}} \times \frac{\sin^2 \theta}{\cos \theta} \exp - \frac{\omega[\omega_{ce} \cos \theta - \omega]}{\omega_{pe}^2 \cos^2 \theta} \left(\frac{c}{v_{\parallel}} \right)^2 \right\}. \quad (14)$$

For ion cyclotron resonance, when $|\theta - \pi/2| \gg (m/M)^{\frac{1}{2}}$, we have the relations

$$\begin{aligned} p^2 &= \frac{\omega_{pi}}{\omega_{ci}^2 - \omega^2} \frac{\omega_{ci}(1 + \cos^2 \theta) \pm [\omega_{ci}^2 \sin^4 \theta + 4\omega^2 \cos^2 \theta]^{\frac{1}{2}}}{2\omega_{ci} \cos^2 \theta} \\ \kappa &= \frac{\pi^{\frac{1}{2}}}{4} \left(\frac{\omega_{pi}}{\omega} \right)^2 \frac{c}{v_{\parallel} p^3 \cos \theta} \left[\exp - \left(\frac{c[\omega - \omega_{ci}]}{pv_{\parallel} \omega} \right)^2 \right] \\ &\quad \times \frac{(1 + \cos^2 \theta)[\omega_{ci}^2 \sin^4 \theta + 4\omega^2 \cos^2 \theta]^{\frac{1}{2}} \pm [\omega_{ci} \sin^4 \theta + 4\omega \cos^2 \theta]}{2 \cos^2 \theta [\omega_{ci}^2 \sin^4 \theta + 4\omega^2 \cos^2 \theta]^{\frac{1}{2}}}. \end{aligned} \quad (15)$$

For oblique wave propagation in the region $\omega \ll (\omega_H)_i$, there are also noncollisional absorptions which have been studied by Braginsky and Kasantzev.* Unity can be dropped from the expression for N^2 in sufficiently dense plasmas when $(\omega_0/\omega_H)^2 \gg 1$; the absorption then depends only on the ratio of the electron or ion pressure to the magnetic pressure

$$\begin{aligned} \beta_e &\equiv 8\pi n T_e / H^2 \\ \beta_i &\equiv 8\pi n T_i / H^2, \end{aligned}$$

for $n_e = n_i \equiv n$. (16)

If we now introduce the variables $x \equiv \omega/\omega_H$ and $z \equiv [(\omega - \omega_H)/\omega p](c/v_{\parallel})$, then Eq. (13) has the form

$$\begin{aligned} \frac{1}{3}(1 - x^2)^3/x &= 2\beta_e z^3 e^{-z^2} \int_0^z e^{-t^2} dt, \\ pv_{\parallel}/c &= (1 - x)/zx, \\ \kappa &= \frac{1}{2}\pi^{\frac{1}{2}}\beta_e [x^2/(1 - x)^3] z^3 e^{-z^2} = \frac{1}{4}\pi^{\frac{1}{2}} \left(\int_0^z e^{-t^2} dt \right)^{-1}. \end{aligned} \quad (13a)$$

The absorption described by the above formulae covers a rather wide frequency range, even when the plasma pressure is small compared to the magnetic pressure. As an example, for an appropriately polarized electromagnetic wave propagating along a uniform magnetic field, the absorption edges for ion and electron cyclotron resonance depend on β_i and β_e in the manner shown in Fig. 1. It can be seen that the relative absorption width, $\Delta\omega/\omega_c$, is quite large even when the temperature is comparatively low. Therefore, when $\beta_e = 10^{-2}$, the absorption at ten wavelengths occurs in the range $\Delta\omega/\omega_c = 0.3$ (the "plasma wavelength" is $\lambda = 2\pi c/\omega_p$). It can also be seen from an examination of Eqs. (14) and (15) that there is a broad frequency range where ion and electron cyclotron absorption can occur for oblique wave propagation.

It seems inviting to use the specific *dielectric* properties of a high-temperature plasma in a magnetic field for actively or passively probing the plasma. For example, black-body intensity measurements would constitute a useful passive probe, since it can be seen from the formulae and graphs presented here that there are extensive "black" regions of the spectrum associated with ion and electron cyclotron resonance radiation. It follows from the expressions for $\epsilon_{\alpha\beta}$ (see Eq. (10)) and N^2 that "black" regions of the spectrum also exist for oblique waves. In this case, even when thermal motion is not taken into account, absorption occurs since the index of refraction greatly exceeds unity because of Cherenkov radiation. A black spectral region can also be expected in the vicinity of $N^2 = 0$, where the longitudinal plasma oscillations are transformed into transverse electromagnetic waves. The various mechanisms which give rise to relationships between the transverse and longitudinal waves at a plasma boundary are reviewed elsewhere.⁹

It is possible, in principle, to develop a probe based on an investigation of the shape of the absorption spectrum. Since the width of the absorption resonance depends on the parameters β_i and β_e , this kind of investigation could be of use in measuring the electron or ion pressure. In practice, however, the nonuniformity of the magnetic field which is always present will complicate such probing.

It has been demonstrated that electromagnetic energy will be effectively absorbed by a plasma in spite of the absence of collisions between particles. However, our linear approximations fail to give an answer to the question of how this energy is distributed between the plasma ions and electrons. Evidently, in noncollisional absorption an increase of the energy of the plasma cannot be immediately identified with a temperature rise. To solve this problem one must resort to nonlinear considerations.

When one investigates wave propagation along the field it is possible to slightly improve the linear considerations which have been employed previously in analyzing cyclotron resonance absorption. Let us consider frequencies close to the electron Larmor

* *Ed. note*: See S. I. Braginsky and A. P. Kasantzev, *Magnetohydrodynamic Waves in Rarefied Plasmas, Plasma Physics and Problems of Controlled Thermonuclear Reactions*, AN SSSR, Vol. 4, p. 24 (in Russian), Edited by M. A. Leontovich, Moscow (1958).

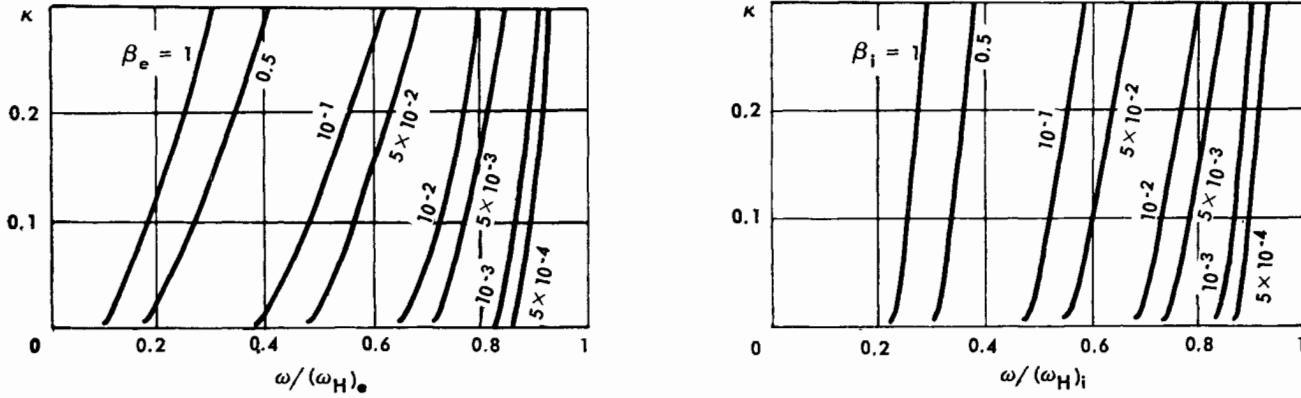


Figure 1. Dependence of electron and ion cyclotron resonance absorption edges on β_e and β_i , respectively

frequency, $(\omega_H)_e = eH/m_e c$, when the ions can be regarded as being at rest. In the kinetic equation for electrons we shall neglect the term containing the magnetic field. This approximation can be partially justified by the fact that in the ordinary linear approximation, with an unperturbed isotropic Maxwellian distribution, a corresponding term is absent; namely, $\mathbf{v} \times \mathbf{H} \partial f_0 / \partial \mathbf{v} = 0$ with $f_0(\mathbf{v}) = f_0(v^2)$. If all the variables depend only on one spatial coordinate directed along the constant magnetic field, then a solution to the problem with the initial conditions $f(\mathbf{v}, z, 0) = f_0(v_x^2 + v_y^2, v_z^2)$, $E_x(z, 0) = E_x^0(z)$, $E_y(z, 0) = E_y^0(z)$, $E_z(z, 0) = 0$ can be obtained for transverse waves by "integrating along trajectories". The appropriate kinetic equation and wave equation are

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial z} + (e\mathbf{E}/m) \cdot \partial f / \partial \mathbf{v} + (e/mc) \mathbf{v} \times \mathbf{H} \cdot \partial f / \partial \mathbf{v} = 0 \quad (17)$$

$$-\nabla \times (\nabla \times \mathbf{E}) = (1/c^2) \partial^2 \mathbf{E} / \partial t^2 = (4\pi e/c^2) \partial (f \mathbf{v} / d\mathbf{v}) / \partial t. \quad (18)$$

The following result for the distribution function is obtained from these equations

$$f(v_x, v_y, v_z, z, t) = f_0([v_x - v_1(t, z)]^2 + [v_y - v_2(t, z)]^2, v_z^2) \quad (19)$$

where

$$v_1(t, z) = \frac{e}{m} \cos \omega_c t \int_0^t [E_x(t', z') \cos \omega_H t' - E_y(t', z') \sin \omega_H t'] dt' + \frac{e}{m} \sin \omega_c t \int_0^t [E_y(t', z') \cos \omega_H t' + E_x(t', z') \sin \omega_H t'] dt', \quad (20)$$

$$z' \equiv z - v_z(t - t')$$

and $v_2(t, z)$ has a similar form.

Equation (18) is linear in the field quantities and can be easily solved by the Fourier-Laplace method when the longitudinal velocities have a Maxwell distribution initially. From (19) and (20) one can also see that the average random thermal energy will

not increase during the wave damping, i.e., the field energy will appear in the kinetic energy of the ordered motion of the plasma electrons. This is because the electrons which receive energy from the high-frequency field are always in phase with the field and, therefore, with one another. As a result, the thermal distribution of electron velocities associated with the distribution of the phases of their Larmor rotation will remain unchanged. To transform the energy of regular electron rotation into energy of random motion, it is necessary that the phases be "mixed". In addition to phase mixing due to the extremely rare collisions in a high-temperature plasma, randomization of phases can be caused by the nonuniformity of the magnetic field along a line of force. This can occur because the electrons rotating at different places and at different frequencies, ω_{ce} , have their phases mixed due to the thermal motion along the line of force.

In the above case of cyclotron resonance the energy is absorbed by charged particles moving along a constant field with a velocity $v_z = (\omega_c - \omega)/k_z$. Here, the energy in the transverse motion increases and, therefore, a change takes place mainly in the distribution of transverse velocities. Since the number of particles which are in resonance is determined by the longitudinal velocity distribution (which is almost unchanged), the absorption takes place continuously.

The situation is different with Cherenkov absorption since during the process of absorbing energy there is a change in the longitudinal velocity distribution which determines the resonance of the charged particles. Here the particles are brought out of resonance and a balance may occur between the absorption and emission of radiation for certain particles. Mathematically, this is expressed by the vanishing of the derivative $\partial f_0 / \partial v_z$ in the velocity range for which the resonance $v_z = \omega/k_z$ takes place.

The problem of noncollisional absorption influenced by nonlinear effects is a subject under further investigation.

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Generation and Thermalization of Plasma Waves

By T. H. Stix *

A fully ionized gas immersed in an axial magnetic field may exhibit low-frequency oscillations where the electric field and macroscopic velocity perturbations are perpendicular to the axis. The ordinary and extraordinary hydromagnetic waves of Alfvén¹ and Åström² are examples of these transverse oscillations. Ion cyclotron waves³ are another example. The physical environment appropriate to these low-frequency transverse waves is present in several types of proposed thermonuclear reactors, including the stellarator⁴ and the pyrotron.⁵

For several reasons, the thermonuclear experimenter may be interested in generating these waves. First, they are a hydromagnetic phenomenon that can be produced in the laboratory and experimental data can be checked against hydromagnetic theory. Second, generating and observing these waves at low power may be a useful diagnostic technique for determining plasma characteristics which are important parameters of these waves such as magnetic field strength, plasma density and density distribution, plasma temperature and composition, and the state of ionization of the plasma. Finally, generating these waves at high power is a possible scheme for rapid heating of the plasma. Calculations on ion cyclotron wave generation and thermalization indicate an efficiency, in a typical example, of more than 65% for the transfer of power from an rf power source into effective random ion motion.

An effect which is termed "cyclotron damping" is found to be extremely important in the theory of ion cyclotron waves. Cyclotron damping causes free ion cyclotron waves to decay exponentially with time, or propagated ion cyclotron waves to decay exponentially with axial distance. Furthermore, cyclotron damping converts the energy of the wave motion into effectively random transverse ion motion, which makes ion cyclotron heating especially useful for a pyrotron-type device.

Cyclotron damping occurs for transverse oscillations which are periodic both in time and in axial distance. Ions moving along lines of force will "feel" the oscillations of the transverse electric field at a frequency which differs from the plasma rest frame

frequency by the Doppler shift. Some ions will "feel" the oscillations at their own cyclotron frequency, and Lenard⁶ has pointed out that these ions will absorb energy at a constant rate from the electric fields. Dawson⁶ has shown that the presence of classes of such ions leads to the damping with distance of propagated oscillations. In the excellent paper of Gershman⁷ the effects of cyclotron damping on ordinary and extraordinary hydromagnetic waves have been computed.

The kinematic effects of cyclotron damping appear in the off-diagonal components,

$$n_i m \langle w_x w_z \rangle_{Av} \text{ and } n_i m \langle w_y w_z \rangle_{Av},$$

of the stress tensor. The axial ion velocity w_z is a constant of the motion through first order in these transverse oscillations, and is a zero-order quantity. The off-diagonal stress tensor terms are then first order terms, and must be included in a theory of small-amplitude oscillations. Although the stress tensor does not appear explicitly, it is the calculation of these momentum transport quantities which occupies the first section of this paper. A dispersion relation is obtained at the end of this section.

Then the physical effects of cyclotron damping are discussed in terms of phase relations, power absorption and damping rates. The reduction of the dispersion relation and the phase relations to the undamped results previously obtained (Ref. 3) is shown for frequencies slightly removed from the ion cyclotron frequency.

After that we calculate the energy input from an inducing coil to a cylindrical plasma. The circumstances include excitation of the plasma in the immediate neighborhood of the ion cyclotron frequency and excitation of ion cyclotron waves and hydromagnetic waves. Estimates are made for the width of the power absorption resonances, and the power transferred to the plasma is compared to the ohmic losses in the induction coil. For a plasma of appreciable density, the power transfer is efficient only in the cases of wave generation. A plasma heating scheme proposed in the last section is to generate ion cyclotron waves at a long wave-length, in order to achieve efficient transfer from the rf power source, and allow the waves to propagate through a region of magnetic field which decreases with axial distance

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from the coil. The wavelength of the ion cyclotron waves becomes shorter and shorter, and finally local cyclotron damping starts to occur. In this local region, the waves damp out and the wave energy is converted into energy of effectively random transverse ion motion.

DISPERSION RELATION FOR CYCLOTRON DAMPING

Small amplitude oscillations perpendicular to the lines of force are considered for a fully ionized gas immersed in an axial magnetic field. Particle collisions are infrequent and the temperature of the gas corresponding to ion motions perpendicular to the lines of force is assumed to be zero. The temperature of ions corresponding to motions parallel to the lines of force is taken to be finite. We consider oscillations which are periodic in axial distance and in time, and neglect effects of resistivity, gravity, viscosity and electron inertia.

For transverse oscillations the velocity of an individual ion along a line of force is approximately constant. We divide up the plasma into a massless electron fluid and into constituent streams of ions, where each stream is populated by ions with the same zero-order axial velocity. We solve the equation of motion for each constituent stream separately in terms of the transverse electric field along a line of force. Adding the streams together gives a charge and current distribution due to the ion motions in terms of the electric field. Inserting this ion charge and current distribution into Ohm's Law and Maxwell's equations yields homogeneous equations in the transverse electric fields. The solution of these equations determines the x and y dependence of the electric fields for a given z and t dependence, and the elimination of the electric field variables from the equation yields a dispersion relation for the system.

The coefficients in the dispersion relation are integrals over the distribution of zero-order parallel ion velocities. Following a suggestion made by Ira Bernstein⁸ we can avoid divergences in these integrals at resonance by considering an initial value problem in the ion motion.

A list of symbols used in this paper is given in Table 1, together with their definitions. Dimensionless quantities will be used extensively.

The first-order linearized equation for the macroscopic motion of ions in one of the constituent streams is

$$\frac{\partial \mathbf{u}}{\partial t} + w_z \frac{\partial \mathbf{u}}{\partial z} = \frac{Z_1 e}{m} [\mathbf{E} + \mathbf{u} \times \mathbf{e}_z B_0 + w_z \mathbf{e}_z \times \mathbf{B}] \quad (1)$$

We assume that the first-order electric and magnetic fields vary as the real parts of $\tilde{\mathbf{E}}(x, y) \exp(ikz + i\omega t)$ and $\tilde{\mathbf{B}}(x, y) \exp(ikz + i\omega t)$, where the components of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ may be complex. A consequence of the massless electron motion is that $\tilde{E}_z$ is zero. Using this fact and Maxwell's equations, $\tilde{\mathbf{B}}$ in equation (1) can be expressed in terms of $\tilde{E}_x$ and $\tilde{E}_y$.

A solution to (1) will then express the transverse velocity in terms of the transverse electric fields. We

choose that particular solution $\mathbf{u}_\perp$ which fits the initial values $u_x = u_y = 0$ at $T = T_0$ and sum $\mathbf{u}_\perp$ over all the constituent streams, to obtain $\mathbf{v}_\perp$. In dimensionless form, we have

$$\mathbf{v}_\perp = \text{Re}\{ (1/\Omega) [-A_+ + A_- + iD_+ + iD_-] \times \tilde{\mathbf{E}}_\perp e^{i\kappa z + i\Omega T} + (i/\Omega) [-A_+ - A_- + iD_+ - iD_-] \times \tilde{\mathbf{E}} \times \mathbf{e}_z e^{i\kappa z + i\Omega T} \} \quad (2)$$

where

$$A_\pm = \mp \frac{1}{2|\kappa|} \int_{-\infty}^{\infty} g\left(\frac{\Lambda - \Omega}{\kappa}\right) \frac{\Lambda}{1 \pm \Lambda} \times \sin[(T - T_0)(1 \pm \Lambda)] d\Lambda, \quad (3)$$

$$D_\pm = \mp \frac{1}{2|\kappa|} \int_{-\infty}^{\infty} g\left(\frac{\Lambda - \Omega}{\kappa}\right) \frac{\Lambda}{1 \pm \Lambda} \times \{1 - \cos[(T - T_0)(1 \pm \Lambda)]\} d\Lambda \quad (4)$$

The integrals in (3) and (4) approach long-time asymptotic values. For values of $T - T_0 \gg |\kappa s_0|^{-1}$, which means a time long compared with the time it takes ions with the rms velocity to travel an axial distance equal to $\lambda/9$, the integrals may be reasonably approximated by their asymptotic values.

For a Maxwellian distribution of axial ion velocities characterized by a mean axial ion energy of $kT_1/2$, $A_\pm$ has the long-time asymptotic form

$$A_\pm = \frac{\pi}{2|\kappa|s_0} e^{-p_\pm^2} = \frac{\lambda B_0 Z_1}{9.47 A^{1/2} T_1^{1/2}} e^{-p_\pm^2}, \quad (5)$$

where $p_\pm \equiv (1 \pm \Omega)(\kappa s_0)^{-1}$.

For large values of $T - T_0$ we may approximate $D_\pm$ by dropping the cosine term and taking the principal part of the remaining integral. For a Maxwellian distribution, the long-time asymptotic form is conveniently given by $D_\pm \rightarrow I_\pm - \frac{1}{2}$. A graph of $\kappa s_0 I_\pm$ versus $p_\pm$ is shown in Fig. 1. Also shown on the graph is $|\kappa|s_0 A_\pm$. $I_\pm$ reaches peak values of $0.54|\kappa s_0|^{-1}$ at $|p_\pm| \approx 1.0$. Useful approximations to $I_\pm$ are $I_\pm \approx (2\kappa s_0 p_\pm)^{-1}$ for $|p_\pm| \gg 1$ and $I_\pm \approx p_\pm (\kappa s_0)^{-1}$ for $|p_\pm| \ll 1$.

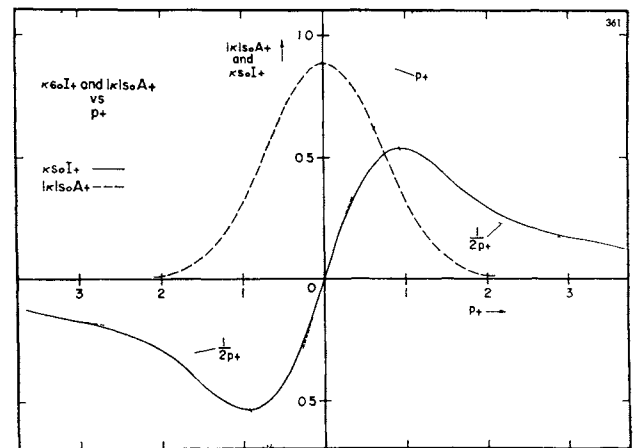


Figure 1 Graphs of $\kappa s_0 I_\pm$ and $|\kappa|s_0 A_\pm$

Table 1

<i>Integers and Operators</i>			
A	atomic weight	$\mathbf{Q}$	$\epsilon_x Q_x + \epsilon_y Q_y + \epsilon_z Q_z$
Z_i	ionic atomic number	$\mathbf{Q}_\perp$	$\epsilon_x Q_x + \epsilon_y Q_y$
∇	$\epsilon_x \frac{\partial}{\partial X} + \epsilon_y \frac{\partial}{\partial Y} + \epsilon_z \frac{\partial}{\partial Z}$	$\tilde{Q}$	$Q = \text{Re} \{ \tilde{Q} e^{i\omega Z + i\Omega T} \}$
Re	real part of	$\epsilon_x, \epsilon_y, \epsilon_z, \epsilon_r, \epsilon_\theta$	unit vectors
<i>cgs-Gaussian Units</i>			
$\mathbf{B}$	first-order magnetic field		
B_0	zero-order axial magnetic field strength		
c	velocity of light		
$\mathbf{E}$	first-order electric field		
e	electronic charge		
$\mathbf{j}^*$	sheet current density in "ideal coil"		
k	axial wave-number, $\pm 2\pi/\lambda$		
k	Boltzmann's constant		
l_{coil}	over-all length of induction coil		
m	ion mass		
n_e, n_i	number of electrons, ions per unit volume		
$n(w_z)dw_z$	number of ions per unit volume with w_z in range dw_z		
r_c	radius of induction coil		
r_0	radius of plasma boundary		
$r, \theta, z; x, y, z; t$	cylindrical and Cartesian coordinates; time		
T_i	ion temperature in $^\circ\text{K}$		
$\mathbf{u}$	first order macroscopic velocity for a single constituent stream		
$\mathbf{u}$	macroscopic plasma velocity		
$\mathbf{w}$	individual ion velocity		
$\lambda = \lambda $	axial wave-length		
ω	angular frequency		
ω_{ci}	angular ion cyclotron frequency, $ZieB_0/mc$		
<i>Dimensionless Units</i>			
$A_\pm$	Eq. (3), (5)	R_0	$r_0 \omega_{ci}/c$
B	$\mathbf{B}/B_0$	S	Eq. (23)
$D_\pm$	Eq. (4)	s_0	$[2T_i/mc^2]^{1/2}$
$\mathbf{E}$	$\mathbf{E}/B_0$	T	ω_{ci}/t
$F_\pm$	Eq. (6)	$\mathbf{v}$	$\mathbf{u}/c$
G	Eq. (21)	X, Y, Z	$\frac{\omega_{ci}}{c} [x, y, z]$
$g \left(\frac{w_z}{c} \right) d \left(\frac{w_z}{c} \right)$	$\frac{n(w_z)dw_z}{n_i}$	α	$4\pi n_i mc^2 B_0^{-2}$
$I_\pm, J_\pm, K_1$	Ref. 9	β_m	Eq. (20)
$I_\pm$	Fig. 1	γ	Eq. (9)
$\mathbf{j}^*$	$4\pi \mathbf{j}^*/B_0$	η	Eq. (18)
L	$\omega_{ci} l_{\text{coil}}/c$	κ	kc/ω_{ci}
M, N	Eq. (14)	Λ	$\Omega + \kappa \omega_{ci} c^{-1}$
P_m	Eq. (22)	ν	Eq. (8)
$p_\pm$	$(1 \pm \Omega)(\kappa s_0)^{-1}$	ξ	Eq. (23)
Q, U, V, W	Section 3	σ_1, σ_2	cf Eq. (7)
R	$r \omega_{ci}/c$	Ω	ω/ω_{ci}

We look now at particle energies rather than velocities. Let $m\langle u_x^2 + u_y^2 \rangle/2$ denote the average over an oscillatory cycle of the transverse energy of an ion in a single constituent stream. Summing this average energy over all the constituent streams, we obtain

$$\frac{1}{2} m \langle u_x^2 + u_y^2 \rangle = mc^2 \{ F_+ |\tilde{E}_x + i\tilde{E}_y|^2 + F_- |E_x - iE_y|^2 \} \quad (6)$$

where F_+ has the long-time asymptotic form for a Maxwellian distribution, $F_\pm \rightarrow F_0 + (T - T_0)A_\pm/2$. F_0 is the contribution to the sum from ions which do

not "feel" their ion cyclotron frequency, and represents the kinetic energy associated with the non-resonant part of the forced ion oscillations. The second term is the contribution to the sum from ions which do "feel" their ion cyclotron frequency. The energy associated with a single-constituent stream at exact resonance increases as the square of the elapsed time. However, because of the spread in axial velocities, fewer and fewer streams remain in resonance as time goes on, so that the total energy associated with all the streams increases only linearly with time. Similarly, the velocity $\mathbf{u}_\perp$ of a single-constituent stream at exact resonance increases linearly with elapsed time, but the summed velocity $\mathbf{v}_\perp$ asymptotically approaches a constant magnitude.

The physical situation we wish to approximate in a self-consistent manner is a plasma with steady-state oscillations. A low but finite collision rate in the plasma creates a quasi-equilibrium situation. (There may be a heating of the plasma due to the oscillations, which we can make arbitrarily slow by reducing the amplitude of oscillation.) To help understand this quasi-equilibrium, we consider, for simplicity, those collisions which interchange the axial velocities of the colliding ions but leave the transverse velocities of each ion unchanged. Each colliding ion will be removed from one constituent stream and put into another. The transverse velocity an ion carries with it from the first stream may be different from the transverse velocity at that point associated with the second stream. If so, this difference will be random. In particular, an ion moving from a resonant stream to a non-resonant stream will bring with it a large transverse velocity, almost all of which will be random with respect to the non-resonant stream. A large number of such transitions will increase the temperature of the transverse motion of the non-resonant ions. Conversely, an ion moving from a non-resonant stream to a resonant stream will bring with it almost no transverse velocity. It will then be accelerated by the transverse electric fields from an initial state of almost zero transverse velocity. A large number of such transitions will cause energy to be absorbed by the resonant ions from the electric fields in the plasma. These two processes are the thermalization and energy absorption mechanisms of cyclotron damping.

With these processes as guides, we can construct a simplified model of a quasi-equilibrium plasma to which our ion orbit calculations would accurately apply. We consider a plasma in which each ion undergoes a catastrophe at a time τ after its previous catastrophe. Each catastrophe causes the ion's transverse velocity to be reduced to zero. The macroscopic velocity $\langle \mathbf{v}_\perp \rangle$ for this plasma is obtained from (2) by averaging over catastrophe times,

$$\langle \mathbf{v}_\perp \rangle = \tau^{-1} \int_{-\tau}^0 \mathbf{v}_\perp dT_0.$$

This average has the same asymptotic form for large τ that $\mathbf{v}_\perp$ has for large $T - T_0$. $\langle \mathbf{v}_\perp \rangle$ gives us

the plasma ion current density, which is all we require. Plasma electron current is obtained from Ohm's Law (Eq. 5a, Ref. 3) which states that electrons move freely along magnetic lines of force and with a velocity $c\mathbf{E} \times \mathbf{e}_z$ across magnetic lines of force. In Maxwell's equations (Eq. 6a-9a, Ref. 3) we can neglect displacement current in both the plasma and vacuum for the wave-lengths and frequencies of interest to us. We obtain the differential equation for the transverse electric field in the plasma

$$[\nabla \times (\nabla \times \tilde{\mathbf{E}})]_{\perp} = \sigma_1 \tilde{\mathbf{E}} + i\sigma_2 \tilde{\mathbf{E}} \times \mathbf{e}_z \quad (7)$$

where $\sigma_1 \equiv \alpha(iA_+ - iA_- - 1 + I_+ + I_-)$, and $\sigma_2 \equiv \alpha(iA_+ + iA_- + \Omega + I_+ - I_-)$, and the long-time asymptotic expressions are to be used for $A_{\pm}$.

In cylindrical coordinates, with no dependence on θ , and when σ_1 and σ_2 are independent of position, the r and θ components of Eq. (7) may be combined to give Bessel's equation. The solution which is finite at the origin⁹ is $\tilde{E}_r \sim \tilde{E}_{\theta} \sim J_1(\nu R)$, where

$$\nu^2 = \frac{\sigma_1^2 - 2\sigma_1\kappa^2 + \kappa^4 - \sigma_2^2}{\sigma_1 - \kappa^2}. \quad (8)$$

ν is the radial wave-number. By going to the limit of large radii, Eq. (8) may be thought of as the dispersion relation for a Cartesian coordinate system, where ν and κ are wave-numbers in the appropriate Cartesian directions.

PHASE RELATIONS AND CYCLOTRON DAMPING RATES

We distinguish four regions, determined by frequency, wavelength, density, and temperature,

Table 2. Approximate Dispersion Relations, σ_1 and σ_2

Dispersion relation	σ_1	σ_2
I. $\nu^2 = -\kappa^2 \left[\frac{1-2i\gamma}{1-i\gamma} \right]$	$\pm \sigma_2$	$i\alpha A_{\pm}$
II. $\frac{1}{\sigma_1} = \frac{1}{\kappa^2} + \frac{1}{\kappa^2 + \nu^2}$	$\pm \sigma_2 \alpha \left(iA_{\pm} \pm \frac{1}{2(1 \pm \Omega)} \right)$	
III. $\Omega^2 = \frac{\kappa^2}{\alpha}$	$-\frac{\sigma_2}{\Omega}$	$-\frac{\alpha\Omega^3}{1-\Omega^2}$
IV. $\Omega^2 = \frac{\kappa^2 + \nu^2}{\alpha}$	$-\frac{\sigma_2}{\Omega}$	$-\frac{\alpha\Omega^3}{1-\Omega^2}$

where the oscillations described by the dispersion relation (8) may be rather easily characterized. The spectrum also contains other regions, where the plasma behavior is not so easily characterized. The kinematics in the other regions can, however, also be obtained from the equations of the preceding section. The four regions are labelled:

- Case I: Neighborhood of ion cyclotron frequency
- Case II: Ion cyclotron waves
- Case III: Torsional hydromagnetic waves
- Case IV: Compressional hydromagnetic waves.

Four parameters are particularly important in characterizing these regions, $|\kappa|s_0$, $p_{\pm}$, γ , and $^{10} \kappa^2/\alpha$, where

$$\gamma \equiv \left(\frac{\alpha A_{\pm}}{\kappa^2} \right)_{\Omega=\mp 1} = 5.20 \times 10^{-18} \frac{Z_i^3 B_0 n_i \lambda^3}{A^{3/2} T_i^{1/2}}, \quad (9)$$

$$\frac{\kappa^2}{\alpha} = \frac{mk^2 c^2}{4\pi n_i Z_i^2 e^2} = 2.03 \times 10^{16} \frac{A}{Z_i^2 n_i \lambda^2}. \quad (10)$$

$|\kappa|s_0$ is the ratio of the velocity of an ion of energy $\frac{1}{2}T_i$ to the phase velocity of the oscillation, and is small in cases of practical interest. It may be evaluated numerically by inspection of equation (5). $p_{\pm}$ measures the proximity of the oscillation frequency to the ion cyclotron frequency. γ large or small will determine whether or not the electric field in the plasma is appreciably affected by the induced field due to plasma currents. κ^2/α large or small was the principal criterion found in Ref. (3) separating ion cyclotron waves from hydromagnetic waves. In detail, the conditions in the four regions are:

- Case I: $|p_{+}| \ll 1$ or $|p_{-}| \ll 1$, $|\kappa|s_0 \ll 1$, $\kappa^2/\alpha \gg 1$
- Case II: $|p_{+}| \gg 1$, $|p_{-}| \gg 1$, $\kappa^2/\alpha \gg 1$, $\Omega \approx \pm 1$
- Case III: $|p_{+}| \gg 1$, $|p_{-}| \gg 1$, $\kappa^2/\alpha \ll 1$
- Case IV: $|p_{+}| \gg 1$, $|p_{-}| \gg 1$, $\kappa^2/\alpha \ll 1$. (11)

Using (2), the radial component of (7), and approximations based on (11), we obtain the approximate dispersion relations and phase relations for each of the four cases given in Tables 2 and 3. Some additional comments may be made on the separate cases. References will be made to (11), Table 2 and Table 3 without specific mention.

Case I

The ions rotate in circular orbits with the same sense of rotation as that for a free ion in a magnetic field. The ion velocity vector is thus circularly polarized. An electric field with the same circular polarization and phase exists in the plasma, so that energy is being put into the ions. Taking the time average of $\mathbf{v} \cdot \mathbf{E}$, and using the phase relations of Table 3 to obtain $\mathbf{v}$ in terms of $\mathbf{E}$, we obtain a power

Table 3. Phase Relations

	$\frac{\tilde{E}_r}{i\tilde{E}_{\theta}}$	$\frac{\tilde{v}_r}{i\tilde{v}_{\theta}}$	$\frac{\tilde{v}_r \pm i\tilde{v}_{\theta}}{E_r \pm i\tilde{E}_{\theta}}$
I.	$\frac{i\gamma}{1 \mp i\gamma}$	$\frac{\Omega}{ \Omega }$	$2A_{\pm}$
II.	$-\left(1 + \frac{\nu^2}{\kappa^2}\right) \frac{\Omega}{ \Omega }$	$-\Omega$	$\frac{2i\sigma_1}{\alpha\Omega}$
III. ^a	$\frac{-\nu^2}{\Omega\kappa^2}$	$-\Omega \left(1 + \frac{\kappa^2}{\nu^2}\right)$	$\mp i$
IV. ^a	$\Omega \left(1 + \frac{\kappa^2}{\nu^2}\right)$	$\frac{\nu^2}{\Omega\kappa^2}$	$\pm i$

^a Valid for $\alpha\nu^4 \gg \kappa^4$.

input for Cases I and II which is the same as given by (6), where the energy in each constituent stream was summed over all the streams. The same power input for Case I was also obtained by Kulsrud and Lenard⁶ through quite a different calculation.

For Case I, E_θ varies as I_1 (constant κR), where the absolute value of the constant lies between 1 and $\sqrt{2}$. For $|\kappa|R_0 \ll 1$, $E_\theta \sim R$ for $R \leq R_0$, and is equal to the E_θ that would exist in the region in the absence of plasma due to the induction coil, $E_{\theta\text{vacuum}}$. This E_θ cannot match the boundary condition for a free oscillation (see Ref. 3), so the oscillations can only be forced or driven.

For sufficiently low density or short wavelength, $\gamma \ll 1$ and E_r is very small compared to E_θ . The total vector electric field in the plasma is approximately equal to $\epsilon_\theta E_{\theta\text{vacuum}}$. E_θ is in phase with v_θ , and in this circumstance cyclotron damping is a strong effect. For $\gamma \gg 1$, corresponding to higher density or longer wavelength, E_r is approximately equal in magnitude to E_θ . The physical explanation for this set of circumstances was offered first by Kulsrud.⁶ The currents due to the radial ion motion and the associated longitudinal electron motion induce a fluctuating magnetic field and in turn a radial electric field, E_r . It turns out that this radial electric field is phased so that the resultant E_r and E_θ combination have principally a circular polarization which is opposite to that of the ion motion, v_r and v_θ . The wrongly polarized electric field does not put energy into the ions. There is a small electric field of the correct polarization which supplies enough energy to the ions to overcome the now much reduced cyclotron damping effect. It will turn out that at appreciable densities this inductive effect greatly reduces the efficiency of ion cyclotron heating in the Case I region.

Cases II, III and IV

These are possible oscillatory modes of the plasma. The simple criterion, $|p_+| \gg 1$ and $|p_-| \gg 1$ is sufficient to reduce the formalism in this paper to the simpler approach of Ref. (3), where the stress tensor was neglected in the equation of motion.

In Cases III and IV and in the undamped form of Case II, the electric field in the plasma is in quadrature with the ion velocity. No energy is transferred to or from the plasma wave in the steady state. However, when these waves are externally generated by an induction coil, the electric field due to the coil will appear in the plasma in phase with the ion velocity and in quadrature with the coil current. The electric field due to plasma currents associated with the generated wave will be in quadrature with the ion velocity but in phase with the coil current. This field is the "back emf". Power flows from the coil to the plasma wave.

Of particular interest is the case for ion cyclotron waves which are slightly damped by a small amount of cyclotron damping. This situation occurs for values of $p_\pm$ such that $A_\pm$ is small but not negligibly small compared to $|I_\pm|$. (See Fig. 1.) In the

constituent stream derivation, κ and Ω were assumed real. The dispersion relation is satisfied by complex values of ν . Physically, the cyclotron damping power is furnished by a divergence in the radial (or κ -direction) flow of energy. Writing $\nu = \nu_0 + i\nu_1$ where ν_0 and ν_1 are real, $(\nu_1)^{-1}$ is the κ -direction e -folding distance for the damping of velocity and field amplitudes. We are also interested in two other cases.

When Ω and ν are real, but κ is complex, the cyclotron damping energy is supplied by a divergence in the axial flow of power. Writing $\kappa = \kappa_0 + i\kappa_1$, $(\kappa_1)^{-1}$ is the axial damping distance for the wave amplitudes. Finally, when ν and κ are real but $\Omega = \Omega_0 + i\Omega_1$ is complex, the cyclotron damping energy is supplied by the wave itself, causing it to damp out with time at a rate Ω_1 .

For slightly damped waves, the damping rate and distances are linearly related to one another with coefficients determined by the undamped waves. Using such linearized perturbations of the dispersion relation, we obtain the damping rate and reciprocal damping distance for slightly damped ion cyclotron waves,

$$\Omega_1 = 2A_\pm(1 \pm \Omega)^2 = \frac{A_\pm}{2} \left(\frac{\alpha}{\kappa^2} + \frac{\alpha}{\kappa^2 + \nu^2} \right)^2, \quad (12)$$

$$\kappa_1 = \pm \frac{\alpha A_\pm}{\kappa} \frac{(\nu^4 + 4\nu^2\kappa^2 + 4\kappa^4)}{(2\nu^4 + 4\nu^2\kappa^2 + 4\kappa^4)}. \quad (13)$$

Equation (13), for the case $\nu = 0$, has also been obtained by Dawson.⁶

ENERGY INPUT TO A PLASMA FROM AN INDUCTION COIL OF FINITE LENGTH

The relations given in the preceding sections apply to steady-state oscillations in a cylinder of plasma of infinite length. In practice, one has a plasma of finite length and an induction coil considerably shorter than the plasma. The true resonant modes of such a plasma are periodic in the axial length. However, we shall assume that wave energy which is propagated axially away from the coil is thermalized and absorbed without reflection. Assuming rapid thermalization, the rate of temperature rise of a plasma in a finite size reactor (e.g., stellarator or pyrotron) may be found by dividing the net power input to the plasma by the heat capacity of the plasma. For practical cases, it is useful to have a ratio of power inputs rather than an absolute level. We choose to compare the power input to the plasma with the rf power which is wasted, in a practical system, in ohmic losses in the induction coil. This ohmic loss is the principal source of inefficiency in a practical plasma heating system. And in a diagnostic experiment, power absorption by the plasma must be measured above the background of this ohmic loss. Fortunately, in both plasma heating and plasma diagnostic experiments, this power ratio is convenient to measure.

We call this power ratio W , and break W up into a product of three more ratios, such that $W = QVU$. We discuss these ratios in turn.

W

W is the average rf power input to the plasma divided by the average rf ohmic power loss in the coil system. The coil system includes not only the exciting coil, but also the connecting leads and the resonating elements, such as capacitors.

Q

Q is the figure of merit for the coil system. It is defined as the peak rf magnetic energy stored in the coil system multiplied by the angular frequency, ω , and divided by the average rf ohmic power loss in the coil system. Typical Q values in the megacycle range would run from 200 to 400.

V

A practical coil system stores magnetic energy in regions other than inside the volume normally occupied by plasma. To allow room for the vacuum chamber wall and for electrical insulation, the radius of the plasma boundary may be only one half the coil radius. Magnetic energy will also be stored outside the coil volume in the stray inductances of the resonating elements and connecting leads. We therefore introduce a storage ratio, V , defined under no-plasma conditions. V is the peak rf magnetic energy stored in an "ideal coil" divided by the peak rf magnetic energy stored in the coil system. The "ideal coil" is defined to have the following characteristics: (a) its radius is the same as the radius of the plasma boundary, (b) it carries a sheet current proportional to $\cos(kz + \omega t)$, (c) it has zero stray inductance, and (d) internal to the ideal coil, the electromagnetic field produced by the sheet current of the ideal coil is the same as the electromagnetic field produced by the actual coil system.

If the current distribution in a physical coil system is best approximated by standing waves, we would superimpose two running-wave ideal coils. The numerator of V should be the sum of the magnetic energies for both ideal coils. For a practical case, we consider an induction coil constructed of individual sections which are spaced axially every half wavelength and which are electrically connected in series. The azimuthal direction of current flow alternates from section to section. We denote by $\mathcal{L}_i$ that part of the total coil system inductance, $\mathcal{L}_{cs}$, which is associated with the fundamental $(\cos kz \cos \omega t)$ component of the induction-coil current. If each section contains N turns spread evenly over a $1/M$ th part of a wavelength, we obtain, neglecting end effects, the end-to-end $\mathcal{L}_i$,

$$\mathcal{L}_i = \frac{16}{\pi} \times 10^{-9} \left[MNkr_c \sin\left(\frac{\pi}{M}\right) \right]^2 l_{\text{coil}} \times [2I_1(kr_c)K_1(kr_c)] \quad (14)$$

and the storage ratio is

$$V = \frac{I_1(kr_0)K_1(kr_c)\mathcal{L}_i}{I_1(kr_c)K_1(kr_0)\mathcal{L}_{cs}} \quad (15)$$

where $\mathcal{L}_i$ and $\mathcal{L}_{cs}$ are in henries, and l_{coil} is in cm. Evaluating V in (14) for typical cases will reveal the high penalty for short wavelengths and small r_0 to r_c ratios. In practice, values of V may run from 0.05 to 0.2.

U

The numerator of U is the average power input to the plasma for a given ideal-coil current. The denominator of U is the product of the angular frequency, ω , and the peak rf magnetic energy stored in the ideal coil under a no-plasma condition with the same ideal-coil current.

Long-coil Approximation

We consider now an induction coil of finite length. We divide the plasma into three sections: to the left of, underneath, and to the right of the coil. We consider the plasma section underneath the coil as though it were a plasma section underneath an induction coil of infinite length. End effects are neglected.

A coil of finite length will excite a band of wavelengths. If the band is too broad, it may be broken into groups of narrower band-widths. A calculation may be made for each group, and the results superimposed. In treating Case I, we shall assume that the band of excited wavelengths lies entirely within the range allowed by the criteria in (11), and we shall neglect the change of plasma kinematics within the band. In Cases II, III and IV, the effect of the wide band-width for a short coil will appear in the shape and amplitude of the power absorption curve.

In Case I, the power absorption calculation will be based on the cyclotron damping integrals (4). In discussing these integrals, it was mentioned that ions would reach their asymptotic velocity after being subjected to the electric fields for times long compared with the time it would take an ion with energy $\frac{1}{2}T_1/2$ to travel the distance $\lambda/9$. For these integrals to apply to ions drifting through the coil section, the coil length must be long compared to $\lambda/9$.

Case I

We consider the case $|\kappa|R_0 \ll 1$ so that $E_\theta \approx E_\theta^{\text{vacuum}}$. Underneath the induction coil, the power input per unit volume may be obtained from (6). We use the phase relation in Table 3 to obtain $\tilde{E}_x \pm i\tilde{E}_y$ in terms of $\tilde{E}_\theta^{\text{vacuum}}$. To obtain the power ratio U_I , we integrate the power input per unit volume over the plasma volume, and divide by the reactive power stored magnetically in the ideal coil. For $|\kappa|R_0 \ll 1$, we may use a solenoid approximation. We obtain, at $\Omega = -1$ or 1 ,

$$U_I = \frac{\gamma}{1 + \gamma^2} \cdot \frac{\pi^2 r_0^2}{2\lambda^2} \quad (16)$$

U_I is a maximum for $\gamma = 1$. At higher values of γ (higher density, longer wavelength), the plasma-current inductive effect reduces the efficiency. The shape and width of the resonance depend on the magnitude of γ . We derive an improved version of $\tilde{E}_r/i\tilde{E}_\theta$ in Table 3, Case I, using $\sigma_1 = \sigma_2 = \alpha(iA_+ + I_+)$, and obtain by numerical calculation the following half-power frequencies.

$$\begin{aligned} |\Omega_{\pm}| &= 1 \pm 7.0 \eta & \text{for } \gamma \ll 1, \\ |\Omega_{\pm}| &= 1 \pm 11.5 \eta & \text{for } \gamma \gg 1, \\ |\Omega_{\pm}| &= \begin{cases} 1 - 1.7 \eta \\ 1 - 11.4 \eta \end{cases} & \text{for } \gamma = 1, \end{aligned} \quad (17)$$

where

$$\eta \equiv \frac{A^{\frac{1}{2}} T_i^{\frac{1}{2}}}{\lambda B_0 Z_i}. \quad (18)$$

The resonant peak in the first two cases occurs at $|\Omega| = 1$. In the $\gamma = 1$ case, the peak occurs at $|\Omega| = 1 - 8.0 \eta$, and the calculated power absorption at the peak is 2.6 times the absorption at $|\Omega| = 1$ (Eq. 16). For larger values of γ , this peak moves into the region of ion cyclotron waves. If one evaluates $p_{\pm}$, which measures when damping can occur (see Fig. 1), at the resonant frequency for undamped ion cyclotron waves, its numerical value turns out to be approximately equal to γ . For $\gamma \ll 1$ then, ion cyclotron waves cannot exist because they are overdamped. $\gamma = 1$ is an intermediate case. For $\gamma \gg 1$, cyclotron damping will be small, and there is a weak power absorption peak at $|\Omega| = 1$ and a strong peak associated with wave generation. The calculation of power absorption at the strong peak is done in the following section.

Cases II, III and IV

We consider now the cases where we have a resonance of the total plasma. The power which the exciting coil puts into the plasma in the coil section is propagated or radiated, axially out each end of the section as a hydromagnetic or ion cyclotron resonance wave. (For Case II, a small amount of power will also go into cyclotron damping inside the coil section.)

We expect running waves to squirt out each end of the coil, and to run axially down the plasma to infinity in each direction. We divide the plasma into three axial sections, and join the solutions for each section to one another by making the wave amplitudes continuous at the joint. Further details of end effects at the joints are not considered. For a left-moving running wave, for instance, we assume that everywhere to the left of the left end of the exciting coil the wave motion is a natural mode of oscillation of the system, and that the wave amplitude is constant. To the right of the right end of the exciting coil, the wave amplitude for this left moving wave is assumed to be zero. And underneath the exciting coil, we assume that the plasma behaves as though it were underneath an exciting coil of infinite length.

The solution underneath the coil must satisfy the boundary conditions at the plasma-vacuum interface $R = R_0$ and at the induction coil radius $R = R_c$. Without loss of generality, we can replace the induction coil by an ideal coil at $R = R_0$ carrying a current $\mathbf{j}^* = \mathbf{e}_\theta \text{Re}[j^* \exp(i\kappa Z + i\Omega T)]$. We use the boundary conditions in Ref. (3), Eqs. (4b)–(9b), and expand $J_1(\nu R)$, where it occurs, in an infinite series of $J_1(\nu_m R)$, where ν_m is the radial wave number for the m th radial mode of oscillation of the free cylindrical plasma. The various ν_m are the various solutions of Eq. (12), Ref. (3).

For each mode m we choose particular solutions E_θ^m for the left moving wave which are zero at the right end of the coil, $Z = Z_0$. We obtain, underneath the coil and for $R \leq R_0$,

$$\begin{aligned} \tilde{E}_\theta^m &= \frac{\beta_m J_1(\nu_m R)}{\nu^2 - \nu_m^2} [1 - \cos\{(\kappa - \kappa_m)(Z - Z_0)\} \\ &\quad + i \sin\{(\kappa - \kappa_m)(Z - Z_0)\}] \end{aligned} \quad (19)$$

where

$$\beta_m = \frac{2i\Omega \tilde{j}^*}{R_0[1 + (G^2 - 1)(\nu_m R_0)^{-2}]J_1(\nu_m R_0)}, \quad (20)$$

and

$$G = -\frac{\kappa R_0 K_1'(\kappa R_0)}{K_1(\kappa R_0)}. \quad (21)$$

κ_m is the axial wave-number for the m th radial mode of free oscillation at the frequency Ω . Other amplitudes in the plasma beside $\tilde{E}_\theta$ may be obtained from $\tilde{E}_\theta$ via the phase relations of Table 3.

The power input to the plasma from the ideal coil is found by integrating over the surface of the ideal coil the product of the coil current density with the electric field due to plasma currents. The average power going into the m th radial mode is

$$P_m = \left| \left(\frac{B_0^2 c^3}{8\omega_{ci}^2} \right) \frac{\beta_m \tilde{j}^* L^2 R_0 (\kappa - \kappa_m) J_1(\nu_m R_0)}{\nu^2 - \nu_m^2} \right| S(\xi), \quad (22)$$

where

$$S(\xi) \equiv \frac{1 - \cos 2\xi + i(2\xi - \sin 2\xi)}{2\xi^2} \quad (23)$$

and $\xi \equiv (\kappa - \kappa_m)L/2$, and where we use the real term in $S(\xi)$ to represent real power and the imaginary term to represent reactive power. $S(\xi)$ approaches 1 as $\xi \rightarrow 0$ so that P_m is real at resonance.

The power ratio U_{wave} is the ratio of the real part of P_m to the reactive power in an ideal coil. Evaluating this ratio at resonance, $\nu = \nu_m$ and $\kappa = \kappa_m$, we obtain

$$\begin{aligned} U_{\text{wave}} &= \left[(2I_1(\kappa R_0) K_1(\kappa R_0)) \right. \\ &\quad \times \left. \left\{ 1 + \frac{G^2 - 1}{(\nu R_0)^2} \right\} \frac{\kappa R_0^2}{L} \frac{\partial \nu^2}{\partial \kappa^2} \right]^{-1}. \end{aligned} \quad (24)$$

A useful approximation to (24) is the $|\kappa|R_0 \ll 1$ limit. Then

$$U_{\text{wave}} \approx \left[\frac{\kappa R_0^2}{L} \frac{\partial v^2}{\partial \kappa^2} \right]^{-1}. \quad (25)$$

The quantity $\partial v^2 / \partial \kappa^2$ is obtained by differentiating the relevant dispersion relation, (8) or Table 2. In the low κR_0 limit, $v_m R_0$ are the roots of $J_0(v_m R_0)$, and are given approximately by $v_m R_0 \approx \pi(m - \frac{1}{4})$ for $m \geq 1$. (Compare Ref. 3.) In this limit, $v^2 \gg \kappa^2$. It is frequently useful to drop the $\kappa^2/\alpha \gg 1$ criterion on ion cyclotron waves. For the undamped waves, it is sufficient to require only that $v^2 \gg \kappa^2$ and $v^2/\alpha \gg \Omega^2$ to obtain the approximate dispersion relation

$$\Omega^2 = [1 + \alpha/\kappa^2]^{-1}. \quad (26)$$

Using the $v^2/\alpha \gg \Omega^2$ and $v^2 \gg \kappa^2$ approximation for ion cyclotron waves, we obtain the $|\kappa|R_0 \ll 1$ approximation to (24) for cases II, III, IV:

$$U_{\text{II}} = \frac{8l_{\text{coil}} r_0^2}{\pi \lambda^2 (m - \frac{1}{4})^4} (1 + \alpha/\kappa^2)^{-1} \quad (27)$$

$$U_{\text{III}} = \frac{8l_{\text{coil}} r_0^2}{\pi \lambda^2 (m - \frac{1}{4})^4} (\kappa^2/\alpha) \quad (28)$$

$$U_{\text{IV}} = \frac{l_{\text{coil}} \lambda}{2\pi r_0^2}. \quad (29)$$

The resonance curve of plasma load versus frequency is given by (22). Since the quantity in absolute value signs in Eq. (22) varies only slowly in the vicinity of a resonance, the shape of the resonance curve is essentially determined by $S(\xi)$, Eq. (23). One may substitute for ξ , $\xi = (L/2)(\Omega - \Omega_m)(\partial\Omega/\partial\kappa)^{-1}$. The half-power points occur at $\xi = \pm 1.4$. $\partial\Omega/\partial\kappa$ is the group velocity of the wave, and may be determined from the dispersion relation. For our various cases, we obtain the half-power frequencies

$$\frac{\Omega_{\frac{1}{2}} - \Omega_m}{\Omega_m} = \begin{cases} \pm \frac{0.45\lambda}{l_{\text{coil}}} [1 + \kappa^2/\alpha]^{-1} & \text{Case II} \\ \pm \frac{0.45\lambda}{l_{\text{coil}}} & \text{Case III} \\ \pm \frac{1.8r_0^2}{l_{\text{coil}} \lambda (m - \frac{1}{4})^2} & \text{Case IV.} \end{cases} \quad (30)$$

THERMALIZATION OF PLASMA WAVES

We can list some of the processes which will act to thermalize the energy of plasma wave motion and which are amenable to calculation. In addition to these known processes, there may occur in an actual plasma a rapid thermalization of wave energy through not yet understood processes similar to those observed by Langmuir¹¹ and Gabor.¹²

In all three types of plasma waves, ion-electron collisions will transform the wave energy slowly into ohmic heat. A faster thermalization will take place in hydromagnetic compressional waves through the heating processes of magnetic pumping,⁶ which will act on both ions and electrons.

An interesting situation occurs for ion cyclotron waves in a gas containing ions with different charge to mass ratios. Only the resonant ions move with the wave. One might consider heating a deuterium-tritium mixture in this manner. Even if the relative average velocities of deuterium and tritium were insufficient for nuclear reaction without wave thermalization, the ion-ion collisions will cause the wave to be thermalized fairly rapidly.

Cyclotron damping offers a possibility for thermalizing ion cyclotron waves extremely rapidly. It will generally be highly inefficient to put power into a plasma of appreciable density at the short wavelengths required for strong cyclotron damping (Case I). At longer wavelengths, however, one can couple energy into ion cyclotron waves, and the power transfer can be extremely efficient (Case II). For a plasma confinement device, such as a stellarator or a pyrotron, one could cause the steady-state confining field to decrease with the axial distance from each end of the coil section. If the rate of change with distance of the confining field is sufficiently small, the ion cyclotron waves which squirt out each end of the coil section will change in an adiabatic fashion. As they move through the decreasing B_0 , their wavelength will gradually decrease, and when it becomes sufficiently short, cyclotron damping will occur locally. The ion cyclotron waves will damp out (Eq. (13)), and the wave energy will be transformed into energy of individual ion motions which are still perpendicular to the lines of force, but with effectively random amplitudes and phases. One might say that the actual heating of the plasma occurs in this region of cyclotron damping.

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Addendum

The author also wishes to thank Dr. J. Neufeld, who has recently brought to his attention the paper of Gershman.⁷ It may be noted that the damping rate computed from the dispersion relation (8) in the present paper will be equal to Ω^{-1} times the damping rate computed from the dispersion relation (31) in Gershman's work. This difference has its origin in the assumption here that the temperature of transverse ion motion was zero, while Gershman assumed an isotropic distribution.

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Plasma Oscillations in a Stationary External Magnetic Field

By S. Ciulli * and M. Micu †

It has been shown by Kruskal and Schwarzschild,¹ and also by Tayler² that a stationary plasma contracted by its own magnetic field is unstable for small perturbations when the computations are based on the magnetohydrodynamic equations. However, a homogeneous axial magnetic field can be employed to stabilize the plasma.

We shall study the stationary distribution function as well as the small deviations from the stationary state of a gas discharge using the Boltzmann equation as a basis for our calculations.

An external magnetic field which is applied after or during the time of ionization causes the plasma to rotate and thereby produces an axial magnetic field within the plasma whose direction is opposite to the externally generated field.

The stationary problem which we are discussing here corresponds to a plasma with cylindrical symmetry at a temperature T . The ions and electrons have axial velocity components v_i and v_e and azimuthal components Ω_i and Ω_e . The azimuthal components correspond to the Foucault rotations caused by the action of the external magnetic field. For the sake of simplicity, we have taken these rotation frequencies to be equal to the cyclotron frequency.

The stationary problem admits separation of variables which leads to a Maxwell velocity distribution and to a spatial distribution function which has the asymptotic form $1/r^4$.

In all our computations we have supposed that the condition of electrical neutrality for the plasma has been fulfilled. The tendency towards neutrality is a basic property of the plasma, unless we are considering distances less than the Debye shielding distance.³ Furthermore, for the study of the perturbations which affect the stability of our plasma, such an assumption is not an embarrassing one because of the low frequencies which are involved.

The plasma perturbations are estimated by the deviations of the first order moments from their Maxwell values. To avoid an excessive amount of algebra only the perturbations of one of the components are analyzed here. Since the perturbations have been assumed to be very small one may assume that the higher moments can be found by using the

Maxwell distribution. In this way one obtains linear equations when terms quadratic in the perturbed quantities are neglected.

The linearization of the equations allows us to Fourier analyze the perturbed quantities. We then obtain a homogeneous system of differential equations for the Fourier amplitudes of the magnetic field and we can study their asymptotic form.

The dispersion relation is obtained by joining smoothly a solution regular at the origin with the regular asymptotic one.

STATIONARY SOLUTIONS

The Boltzmann equation for electrons in the presence of internal and external forces is

$$\partial F / \partial t + \mathbf{v} \cdot \partial F / \partial \mathbf{v} + \mathbf{a} \cdot \partial F / \partial \mathbf{v} = \delta F / \delta t|_c \quad (1)$$

where F is the distribution function, $\mathbf{r}$ is the position vector, $\mathbf{v}$ the particle velocity, and $\mathbf{a} = (e/mc)\mathbf{v} \times \mathbf{H}$ is the acceleration of an electron moving with a velocity $\mathbf{v}$ in a magnetic field $\mathbf{H}$. $\delta F / \delta t|_c$ is the collision term which we assume can be neglected.

For problems with cylindrical symmetry and which are time independent Eq. (1) takes the form

$$v_r \frac{\partial F}{\partial r} + \frac{v_\phi}{r} \left(v_\phi \frac{\partial F}{\partial v_r} - v_r \frac{\partial F}{\partial v_\phi} \right) + a_r \frac{\partial F}{\partial v_r} + a_\phi \frac{\partial F}{\partial v_\phi} + a_z \frac{\partial F}{\partial v_z} = 0. \quad (2)$$

The letters r , ϕ and z denote the components of vectors in the cylindrical coordinate system (r, ϕ, z) . The direction of the z axis has been chosen so that the external field has the value $-H_0$. The total field has the components

$$\mathbf{H}^{(0)} = (0, H_\phi^{(0)}, H_z^{(0)} - H_0). \quad (3)$$

$H_\phi^{(0)}$ and $H_z^{(0)}$ are the components of the magnetic field generated by the ordered motion of the electrons and the positive ions in the plasma. We can now write Eq. (2) as follows

$$v_r \frac{\partial F}{\partial r} + \left[\left(\frac{e}{mc} \right) (H_z^{(0)} - H_0) + \frac{v_\phi}{r} \right] \left(v_\phi \frac{\partial F}{\partial v_r} - v_r \frac{\partial F}{\partial v_\phi} \right) - \left(\frac{e H_\phi^{(0)}}{mc} \right) \left(v_z \frac{\partial F}{\partial v_r} - v_r \frac{\partial F}{\partial v_z} \right) = 0. \quad (4)$$

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Equation (4), after separation of variables, has a solution of the form

$$F = f(r) (m/2\pi kT)^{1/2} \exp - [v_r^2 + (v_\phi - \Omega r)^2 + (v_z - v_T)^2], \quad (5)$$

where $f(r)$ satisfies the equation

$$(1/f)df/dr - (e/c kT) (\Omega r H_z^{(0)} + v_r H_\phi^{(0)}) = 0, \quad (6)$$

and the field components are given by

$$H_z^{(0)} = [4\pi e (\Omega - \bar{\Omega})/c] \int_r^\infty f(r') r' dr',$$

$$H_\phi^{(0)} = [e(v_T - \bar{v}_T)/c] \int_0^r f(r') r' dr', \quad (7)$$

where v_T and Ω denote the axial and angular velocities of the other species. It is easily verified that the solution of Eq. (6) is

$$f = \frac{2c^2 kT / \pi e^2}{[v_T(v_T - \bar{v}_T) - \alpha^{-2} \Omega (\Omega - \bar{\Omega})]^{-1} (\alpha r^2 - \alpha^{-1})} \quad (8)$$

where α is an integration constant which is determined from the normalisation condition.

THE PERTURBED SOLUTION

We assume that the deviations from the equilibrium position are small so that, up to first order, the moments are well approximated by those computed with a Maxwell distribution.

If we multiply Eq. (1) by $\mathbf{v}$ and integrate over all velocities, we obtain

$$\partial G_i / \partial t + \partial G_{ij} / \partial x_j + (e/mc) [\mathbf{H} \times \mathbf{G}]_i = 0 \quad (9)$$

where G_i and G_{ij} are defined by the following integrals

$$G_i = \int v_i F d^3\mathbf{v}, \quad G_{ij} = \int v_i v_j F d^3\mathbf{v}. \quad (10)$$

We now separate the velocity moments and fields into two parts, one of which corresponds to the equilibrium solution (labelled by the superscript (0)) and the other to deviations from the equilibrium position (labelled by the superscript (1)).

$$G_i = G_i^{(0)} + G_i^{(1)}, \quad G_{ij} = G_{ij}^{(0)} + G_{ij}^{(1)} \approx G_{ij}^{(0)};$$

$$\mathbf{H} = \mathbf{H}^{(0)} + \mathbf{H}^{(1)}. \quad (11)$$

From Eq. (9) we then obtain

$$\partial G_{ij} / \partial x_j = (e/mc) [\mathbf{H}^{(0)} \times \mathbf{G}^{(1)}]_i = 0 \quad (12)$$

for the equilibrium quantities and

$$\partial \mathbf{G} / \partial t + (e/mc) [\mathbf{H}^{(0)} \times \mathbf{G}^{(1)} + \mathbf{H}^{(1)} \times \mathbf{G}^{(0)}] = 0 \quad (13)$$

for the perturbations $\mathbf{G}^{(1)}$ by neglecting the quadratic terms.

The value quantity of $\mathbf{H}^{(0)}$ is determined by (3), (7) and (8), and $\mathbf{G}^{(0)}$ can be calculated with F from Eq. (5)

$$\mathbf{G}^{(0)} = \int \mathbf{v} F d^3\mathbf{v} = f(r) \mathbf{v}_T. \quad (14)$$

$\mathbf{H}$ is related to $\mathbf{G}$ by the Maxwell equation

$$\nabla \times \mathbf{H} = (4\pi e/c) \mathbf{G}. \quad (15)$$

Substituting the first two relations of Eq. (11) into Eq. (15) and identifying terms yields

$$\nabla \times \mathbf{H}^{(0)} = (4\pi e/c) \mathbf{G}^{(0)}, \quad (16)$$

$$\nabla \times \mathbf{H}^{(1)} = (4\pi e/c) \mathbf{G}^{(1)}. \quad (17)$$

We now substitute (14) and (17) in (13) and obtain a system of differential equations which define the quantity $\mathbf{H}^{(1)}$

$$\partial [\nabla \times \mathbf{H}^{(1)}] / \partial t + (e/mc) \mathbf{H}^{(0)} \times \nabla \times \mathbf{H}^{(1)} + (4\pi e^2/mc^2) f(r) \mathbf{H}^{(1)} \times \mathbf{v}_T = 0. \quad (18)$$

The three components of the vector $\mathbf{H}^{(1)}$ are not independent. This can be shown by taking the inner product of Eq. (18) with $\mathbf{v}_T$. We then have a linear relation between the three components of $\nabla \times \mathbf{H}^{(1)}$.

We shall take the perturbation $\mathbf{H}^{(1)}$ to be of the form

$$\mathbf{H}^{(1)} = \mathbf{h}(r, \omega, k, n) \exp i(kz + n\phi - \omega t). \quad (19)$$

Substituting this expression into Eq. (18) yields

$$i\omega [in/rh_z - ikh_\phi] + (e/mc) [H_z^{(0)} (ikh_r - h_z') - H_\phi^{(0)} (rh_\phi)' / r - inh_r / r] + (4\pi e^2/mc^2) f(r) (\Omega rh_z - v_T h_\phi) = 0, \quad (20a)$$

$$i\omega [ikr - h_z'] - (e/mc) H_z^{(0)} (inh_z / r - ikh_\phi) + (4\pi e^2/mc^2) f(r) v_T h_r = 0, \quad (20b)$$

$$i\omega [(rh_\phi)' / r - inh_r / r] + (e/mc) H_\phi^{(0)} (imh_z / r - ikh_\phi) - (4\pi e^2/mc^2) f(r) \Omega rh_r = 0. \quad (20c)$$

The derivative of h_r does not occur here so that this component can be found directly from Eq. (20b)

$$h_r = \frac{i[\omega h_z' + (eH_z^{(0)}/mc) (nh_z / r - kh_\phi)]}{-\omega k + (4\pi e^2/mc^2) f(r) v_T}. \quad (21)$$

and substituted into (20a) and (20b). This yields a first order homogeneous set of equations for h_ϕ and h_z

$$\frac{e H_\phi^{(0)}}{mc r} (rh_\phi)' + \left[\frac{e H_z^{(0)}}{mc} + \frac{(\omega e/mc) (kH_z^{(0)} + nH_\phi^{(0)}/r)}{-\omega k + (4\pi e^2/mc^2) f v_T} \right] h_z' = \left[\omega k + \left(\frac{4\pi e^2}{mc^2} \right) f v_T + \frac{(e/mc)^2 H_z^{(0)} k (kH_z^{(0)} + nH_\phi^{(0)}/r)}{-\omega k + (4\pi e^2/mc^2) f v_T} \right] h_\phi + \left[-n\omega/r + \left(\frac{4\pi e^2}{mc^2} \right) f \Omega r - \frac{[(e/mc)^2 nH_z^{(0)}/r] (kH_z^{(0)} + nH_\phi^{(0)}/r)}{-\omega k + (4\pi e^2/mc^2) f v_T} \right] h_z, \quad (22a)$$

$$\begin{aligned}
& \frac{\omega}{r} (r h_\phi)' + \left[\omega \frac{(n\omega/r) - (4\pi e^2/mc^2) \Omega r}{-\omega k + (4\pi e^2/mc^2) f v_T} \right] h_z' \\
&= \left[\frac{ke H_\phi^{(0)}}{mc} \right. \\
&+ \frac{k (e H_z^{(0)}/mc) [n\omega/r - (4\pi e^2/mc^2) \Omega r]}{-\omega k + (8\pi e^2/mc^2) f v_T} \left. \right] h_\phi \\
&- \left[\frac{ne H_\phi^{(0)}}{mc r} \right. \\
&+ \frac{(e H_\phi^{(0)}/mc) (n/r) [n\omega/r - (4\pi e^2/mc^2) f \Omega r]}{-\omega k + (4\pi e^2/mc^2) f v_T} \left. \right] h_z. \quad (22b)
\end{aligned}$$

THE DISPERSION RELATION

The behavior of the solution at the origin is obtained from (22) by retaining the smallest powers of r . This gives

$$\begin{aligned}
h_\phi'' + (3/r) h_\phi' + [(1 - n^2)/r^2] h_\phi &= 0, \\
h_\phi'' - (1/r) h_\phi' - (3/r^2) h_\phi &= 0. \quad (23)
\end{aligned}$$

Equation (23) has solutions of the form r^β , where β takes the values

$$\begin{aligned}
\beta &= -1 \pm n \quad \text{for } n \neq 0, \\
\beta &= 3, 1 \quad \text{for } n = 0. \quad (24)
\end{aligned}$$

There is a single finite solution at the origin. We obtain the behavior of the solutions at infinity from (22) by retaining the largest powers of r . In this manner we obtain for h_z

$$\begin{aligned}
h_z'' + k(1 + \omega/\Omega - \Omega/\omega) h_z' \\
- k(3 + n - na^2/\Omega b) h_z/r = 0, \quad (25)
\end{aligned}$$

where a and b are defined by

$$\begin{aligned}
a &= \lim (re H_\phi^{(0)}/mc) \\
&= 4(v_T - \bar{v}_T) (kT/m) / [v_T(v_T - \bar{v}_T) \\
&\quad - \Omega(\Omega - \bar{\Omega})/\alpha^2], \\
b &= \lim (4\pi e^2/mc^2) f \Omega r^4 \\
&= (8\Omega kT/\alpha^2 m) / [v_T(v_T - \bar{v}_T) \\
&\quad - \Omega(\Omega - \bar{\Omega})/\alpha^2]. \quad (26)
\end{aligned}$$

Equation (25) has two linearly independent solutions whose asymptotic forms are

$$h_z^{(1)} = \exp -kr(1 + \omega/\Omega - \Omega/\omega), \quad (27)$$

$$h_z^{(2)} = (3 + n - na^2/\Omega b)/r^{(1 + \omega/\Omega - \Omega/\omega)}. \quad (28)$$

We always have solutions regular at infinity if we take

$$s = (3 + n - na^2/\Omega b)/(1 + \omega/\Omega - \Omega/\omega), \quad (29)$$

where $\text{Re}(s) \geq 0$ and s is determined from the condition that the solution must be regular at the origin.

For non-zero n , the dispersion relation is found by imposing the asymptotic form (28) on the solution $h_z^{reg}(r)$ regular at the origin:

$$\lim_{r \rightarrow \infty} (h_z^{reg}(r)/r h_z^{reg'}(r)) = -s. \quad (30)$$

This gives a numerical relationship between ω , k , n , H_0 and T . Stability is obtained for those values of the experimental parameters H_0 and T which lead to $\text{Im}\omega \leq 0$ in the dispersion relation.

Numerical solutions must be used in Eq. (30) because Eq. (22), which defines h_z^{reg} , is too complicated to admit an analytical solution. The numerical solutions are now being carried out by the computing group at the Joint Institute for Nuclear Research in Dubna.

A number of idealizations and approximations have been made in the present analysis which limit the utility of the formulae obtained. For example, if we include the collision term in the Boltzmann equation additional quantities appear in the final results, i.e., viscosity, thermal and electrical conductivity, etc. Furthermore, the finite dimensions of linear discharge tubes or the toroidal form of some discharges strongly modify the present results. Taking into account these additional effects, exact solutions will be obtained numerically for different values of the experimental parameters.

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Stability of Plasma in Static Equilibrium

By M. D. Kruskal and C. R. Oberman *

Our purpose is to derive from the Boltzmann equation in the small m/e limit¹, criteria useful in the discussion of stability of plasmas in static equilibrium. At first we ignore collisions but later show their effects may be taken into account. Our approach yields a generalization of the usual energy principles^{3,4,5} for investigating the stability of hydromagnetic systems to situations where the effect of heat flow along magnetic lines is not negligible, and hence to situations where the strictly hydrodynamic approach is inapplicable.

In the first two sections we characterize our general method of approach and delineate the properties of the small m/e limit which we use to determine the constants of the motion and the condition for static equilibrium. In the next two sections we calculate the first and second variations of the energy and conclude with a statement of the general stability criterion. In the final three sections we state several theorems which relate our stability criterion to those of ordinary hydromagnetic theory,⁵ we show how to take into account the effect of collisions, and briefly discuss the remaining problem of incorporating the charge neutrality condition into the present stability theory.

GENERAL METHOD

Our method consists of writing down the energy of the system to second order in the perturbation fields of f and $\mathbf{B}$, where f is the distribution function in $\mathbf{x}, \mathbf{v}$ space and $\mathbf{B}$ is the magnetic field intensity. We eliminate the terms involving the second order perturbation $\hat{f}$ by employing certain constraints, namely, that certain constants of the motion have their equilibrium values. The constants of the motion we employ are time-independent and are functionals of f and $\mathbf{B}$ which are regular at, and permit expansion about, their equilibrium values.

The resulting expression for the energy is a quadratic form in $\hat{f}$ and $\hat{\mathbf{B}}$ jointly whose positive-definiteness provides a sufficient condition for stability. (More about $\hat{\mathbf{B}}$ later, let it suffice for now to say $\hat{\mathbf{B}}$ describes

the displacement of magnetic lines of force away from their equilibrium positions.) We rid ourselves of the dependence on $\hat{f}$ by minimizing the energy with respect to it, subject to the constraint that *all* constants of the forementioned type have their equilibrium values. We then have a sufficient condition for stability involving $\hat{\mathbf{B}}$ alone.

Generally, the constants of the motion of the type we employ do not specify the motion completely so that there exist many motions evolving from the same equilibrium (at $t = -\infty$). By restricting the constants of motion to their equilibrium values, the only possible motions other than the equilibrium behavior are instabilities (the pure modes of which have an exponential time behavior and hence vanish at $t = -\infty$).

To illustrate the method we consider the simplest of examples, the equation of motion

$$\ddot{x} = \lambda x. \quad (1)$$

This system has one time-independent constant, the energy

$$\mathcal{E} = \frac{1}{2} (\dot{x}^2 + \lambda x^2). \quad (2)$$

There exists another constant (the initial phase) of more complicated behavior and involving the time explicitly. We do not employ this latter constant.

The first order variation of the energy vanishes since in static equilibrium $x(t)$ vanishes. The second order variation leads to the form

$$0 = \delta^2 \mathcal{E} = \frac{1}{2} (\dot{x}^2 + \lambda x^2) \quad (3)$$

for the perturbation. Clearly if $\lambda > 0$ then the energy is a positive-definite form and the system is stable, for there exist no motions away from equilibrium. The stable oscillatory motions must necessarily increase the energy from its equilibrium value and hence are disregarded. If $\lambda < 0$, however, the form is indefinite, (3) can be satisfied nontrivially, and there exist (exponential) motions away from equilibrium.

This method was suggested by a technique used by W. Newcomb² to show stability in the much simpler case of a plasma with a Maxwellian equilibrium distribution.

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SMALL m/e LIMIT

In the present investigation we obtain these criteria by examining the second order variation of the energy

$$\mathcal{E} = \int d^3x \frac{1}{2} (B^2 + E^2) + \sum \int \int \int (B/q) dv d\epsilon d^3x [mf(\frac{1}{2}q^2 + \frac{1}{2}\alpha^2 + \nu B)] \quad (4)$$

from its equilibrium value. Here $\mathbf{E}$ and $\mathbf{B}$ are the electromagnetic field intensities, f is the distribution function in $\mathbf{x}, \mathbf{v}$ space of a particular species of charged particles, the summation is over all species, and

$$\mathbf{v} = \alpha + \mathbf{v}_\perp + q\mathbf{n}, \quad (5)$$

$$\mathbf{B}(\mathbf{x}, t) = B(\mathbf{x}, t)\mathbf{n}(\mathbf{x}, t) \quad (6)$$

$$\alpha = \mathbf{E} \times \mathbf{B} / B^2, \quad (7)$$

$$q = \mathbf{v} \cdot \mathbf{n}, \quad (8)$$

$$\nu = v_\perp^2 / 2B, \quad (9)$$

$$\epsilon = q^2 / 2 + \nu B. \quad (10)$$

The quantity $(B/q) dv d\epsilon$ represents the volume element in velocity space. We assume for simplicity that any boundaries present are such as to present no complications, e.g., rigid and perfectly conducting walls with $\mathbf{B}$ entirely tangential. The properties of the small m/e limit we employ are:

- (a) ν is constant following a particle motion.
- (b) f is rotationally symmetric in velocity space about a line parallel to $\mathbf{B}$ and passing through the point α .
- (c) α is the common drift velocity of all particles.

This last fact, as is well known,^{1, 3, 4, 5} permits the introduction into the formalism of a displacement vector $\xi(\mathbf{x}, t)$ which governs not only the development of the field quantities but also the transverse motion of the particles. We defer consideration of the additional property of charge neutrality until later, but we remark at this time that in contradistinction to the Chew-Goldberger-Low¹ (CGL) theory, where one particle species is taken to have a much smaller mass than another in order to satisfy the condition that $\mathbf{E} \cdot \mathbf{n}$ vanishes, we treat all particle species on an equal footing, regard $\mathbf{E}$ and $\mathbf{B}$ as participants in the m/e expansion and find that $\mathbf{E} \cdot \mathbf{n}$ is indeed zero to lowest order in m/e , which is all that is necessary for the evolution of the expansion.

The equilibrium distribution function we denote by $g(\nu, \epsilon, L)$ where L labels the line of force passing through a point in space. We take g to be monotonic in ϵ with

$$g\epsilon < 0 \quad (11)$$

for reasons we shall see later. The equilibrium condition is

$$0 = -\nabla \cdot \mathbf{P} + (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \cdot (p_+ \mathbf{I} + p_- \mathbf{n}\mathbf{n}) + \frac{1}{2} \nabla B^2 - \mathbf{B} \cdot \nabla \mathbf{B} \quad (12)$$

where $\mathbf{P}$ is the stress dyadic, $\mathbf{I}$ is the unit dyadic,

$$p_+ = p_\perp, \quad p_- = p_\parallel - p_\perp, \quad (13)$$

with $p_\perp$ and $p_\parallel$ given by

$$p_\parallel = m \int \int (B/q) dv d\epsilon q^2 g, \quad (14)$$

and

$$p_\perp = m \int \int (B/q) dv d\epsilon \nu B g. \quad (15)$$

FIRST ORDER VARIATION OF ENERGY

The first order change in $\mathcal{E}$ is given by

$$\begin{aligned} \mathcal{E} = \int d^3x (\frac{1}{2} B^2 \nabla \cdot \xi + \mathbf{B} \cdot \hat{\mathbf{B}}) - m \int \int \int dv d\epsilon d^3x \\ \times [(g + \epsilon g_\epsilon)(Bq \nabla \cdot \xi + (Bq)^\wedge) + Bq(\hat{f} + \epsilon \hat{f}_\epsilon)]. \end{aligned} \quad (16)$$

Here we take for convenience the change following the displacement ξ rather than at a fixed point, and we now take equilibrium quantities without designation and denote perturbation quantities with circumflexes. (We shall frequently perform an integration by parts with respect to ϵ , as we have in arriving at (16), by making use of the fact that $1/q$ equals q_ϵ , which follows immediately from (7). We do this in order to avoid the appearance of non-integrable integrands like $1/q^3$.) The perturbed magnetic field intensity is given up to second order in ξ by⁴

$$\begin{aligned} \mathbf{B}(\mathbf{x} + \xi) &= \mathbf{B}(\mathbf{x}) + \hat{\mathbf{B}}(\mathbf{x}) + \hat{\hat{\mathbf{B}}}(\mathbf{x}) \\ &= \mathbf{B} + [\mathbf{B} \cdot \nabla \xi - \mathbf{B} \nabla \cdot \xi] \\ &\quad + \frac{1}{2} \{ \mathbf{B} [(\nabla \cdot \xi)^2 + \nabla \xi : \nabla \xi] - 2(\nabla \cdot \xi) \mathbf{B} \cdot \nabla \xi \}, \end{aligned} \quad (17)$$

and the volume element at the displaced point is given by

$$d^3x(\mathbf{x} + \xi) = d^3x \{ 1 + \nabla \cdot \xi + \frac{1}{2} [(\nabla \cdot \xi)^2 - \nabla \xi : \nabla \xi] \}. \quad (18)$$

However, we find $\mathcal{E}$ vanishes trivially when we make use of the general constraint condition that all constants of the motion have their equilibrium values. Indeed, to first order, we have

$$\left\{ \int \int \int (B/q) dv d\epsilon d^3x G(f, \nu, L) \right\}^\wedge = 0, \quad (19)$$

where $G(f, \nu, L)$ is an arbitrary function of the distribution function f (remember $\dot{f} = 0$), ν , and L where again L labels a line of force passing through a point in space. That magnetic lines of force maintain their identity during a displacement⁶ is a consequence of the fact that $\mathbf{E} \cdot \mathbf{n}$ is zero (to lowest order in m/e); that particles stick to magnetic lines of force is a consequence of (c). We may write condition (19) as

$$\begin{aligned} 0 = - \int \int \int dv d\epsilon d^3x G \hat{f}(g(\nu, \epsilon, L), \nu, L) \\ \times [Bq(\nabla \cdot \xi) g_\epsilon + (Bq)^\wedge g_\epsilon - (B/q) \hat{f}], \end{aligned} \quad (20)$$

and now regard G_f as an arbitrary function of ε, ν, L because, by (11), g is monotonic in ε . Accordingly we may strip (20) to the basic constraint condition

$$0 = \int_T d^3x [Bq(\nabla \cdot \xi)g_e + (B/q) \times (-\nabla \cdot \xi + \mathbf{nn} : \nabla \xi)(q^2 - \nu B)g_e - B\hat{f}/q], \quad (21)$$

where the integration is over a thin tube of force T . (We may, in general, transform integrals over thin tubes of force of flux $d\psi$ to integrals along lines of force according to the prescription

$$\int_T d^3x BA(\mathbf{x}) = d\psi \int_L dl A(\mathbf{x}), \quad (22)$$

for arbitrary $A(\mathbf{x})$.

We now make the particular choice

$$G_f(g(\nu, \varepsilon, L), \nu, L) = -m\varepsilon \quad (23)$$

for G_f in (20), add the resulting expression to (16) in order to eliminate $\hat{f}$, and obtain

$$\hat{\mathcal{E}} = \int d^3x \left\{ \frac{1}{2} B^2 \nabla \cdot \xi + \mathbf{B} \cdot \hat{\mathbf{B}} \right\} - m \int \int \int d\nu d\varepsilon d^3x g [Bq \nabla \cdot \xi + (Bq)^\wedge]. \quad (24)$$

The right-hand side of (24) now vanishes identically as stated when use is made of (10), (17), (12), (14), and (15).

SECOND ORDER VARIATION OF ENERGY

The second order change in energy is given by

$$\begin{aligned} \hat{\mathcal{E}} = \int d^3x \{ & \frac{1}{2} [(\nabla \cdot \xi)^2 - \nabla \xi : \nabla \xi] B^2 + (\nabla \cdot \xi) \mathbf{B} \cdot \hat{\mathbf{B}} + \frac{1}{2} \hat{B}^2 + \mathbf{B} \cdot \hat{\mathbf{B}} + \frac{1}{2} \hat{E}^2 + \frac{1}{2} \rho \alpha^2 \} \\ & + \frac{1}{3} m \int \int \int d\nu d\varepsilon d^3x \{ \frac{1}{2} [(\nabla \cdot \xi)^2 - \nabla \xi : \nabla \xi] Bq^3 + \nabla \cdot \xi (Bq^3)^\wedge + (Bq^3)^\wedge (\varepsilon g_{ee} + 2g_e) \\ & + ((Bq^3)^\wedge + (\nabla \cdot \xi) Bq^3) (\varepsilon \hat{f}_{ee} + 2\hat{f}_e) + Bq^3 (\varepsilon \hat{f}_{ee} + 2\hat{f}_e) \}. \end{aligned} \quad (25)$$

Here ρ is the mass density. If we write the constraint condition (19) to second order,

$$\begin{aligned} 0 = \{ & \frac{1}{3} \int \int \int d\nu d\varepsilon d^3x (G''f_e^2 + G'f_{ee}) \}^\wedge \\ = & \frac{1}{3} \int \int \int d\nu d\varepsilon d^3x \{ G'f^\wedge (Bq^3)_{ee} + \frac{1}{2} (G''\hat{f}^2)_{ee} (Bq^3)_{ee} (Bq^3) \\ & + G'f^\wedge \nabla \cdot \xi (Bq^3)_{ee} + (\frac{1}{2} [(\nabla \cdot \xi)^2 - \nabla \xi : \nabla \xi] (Bq^3) \\ & + \nabla \cdot \xi (Bq^3)^\wedge + (Bq^3)^\wedge (G''g_e^2 + G'g_{ee}) \} \end{aligned} \quad (26)$$

and make the same choice (23) for G_f , we can eliminate $\hat{f}$ in the same way $\hat{f}$ was eliminated in first order, and $\hat{\mathcal{E}}$ then becomes a quadratic form in ξ and $\hat{f}$ jointly. (We have not explicitly introduced the next order correction to the displacement ξ since its contribution to $\hat{\mathcal{E}}$ vanishes in second order, as the contribution of ξ to $\hat{\mathcal{E}}$ vanished in the first order.) We now have

$$\hat{\mathcal{E}} = \frac{1}{2} \int d^3x (\hat{\mathbf{E}}^2 + \rho \alpha^2) + \delta W \quad (27)$$

with δW defined by

$$\begin{aligned} \delta W = & \frac{1}{2} \int d^3x \{ \mathbf{Q}^2 + (\nabla \times \mathbf{B}) \cdot \xi \times \mathbf{Q} \\ & + \xi \cdot \nabla \xi \cdot (\nabla \cdot \mathbf{P}) \} - \frac{1}{2} m \int \int \int d\nu d\varepsilon d^3x (B/q) (f^2/g_e) \\ & + \frac{1}{3} m \int \int \int d\nu d\varepsilon d^3x g_e \{ \frac{1}{2} [(\nabla \cdot \xi)^2 \\ & - \nabla \xi : \nabla \xi] Bq^3 + \nabla \cdot \xi (Bq^3)^\wedge + (Bq^3)^\wedge \}, \end{aligned} \quad (28)$$

and

$$\mathbf{Q} = \nabla \times (\xi \times \mathbf{B}). \quad (29)$$

We find after using (10), (17), (14), and (15) that (28) becomes

$$\begin{aligned} \delta W = & \frac{1}{2} \int d^3x \{ \mathbf{Q}^2 + (\nabla \times \mathbf{B}) \cdot (\xi \times \mathbf{Q}) \\ & + (\nabla \cdot \xi) \xi \cdot \nabla p_+ + 2p_+ [\nabla \cdot \xi - \mathbf{nn} : \nabla \xi]^2 \\ & + p_- [-\mathbf{n} \cdot \nabla \xi \cdot \nabla \xi \cdot \mathbf{n} - \mathbf{nn} : (\xi \cdot \nabla \nabla \xi) \\ & - (\mathbf{n} \cdot \nabla \xi)^2 + (\mathbf{nn} : \nabla \xi)^2] \} \frac{1}{2} m \int \int \int d\nu d\varepsilon d^3x \\ & \times [f^2/g_e - g_e \nu^2 B^2 (-\nabla \cdot \xi + \mathbf{nn} : \nabla \xi)^2]. \end{aligned} \quad (30)$$

We are now prepared to state our stability criterion: δW is a quadratic form in ξ and $\hat{f}$ jointly, otherwise depending only on equilibrium quantities. If this form is positive definite (i. e. positive for all nontrivial permissible ξ and $\hat{f}$), then our system is stable. Indeed, the only ξ and $\hat{f}$ for which $\hat{\mathcal{E}}$ can vanish (as it must since $\hat{\mathcal{E}}$ is a constant of the motion) are the trivial ones and hence no instability can develop. We hope to show this condition is necessary as well as sufficient for stability.

Let us now minimize this expression with respect to $\hat{f}$ (find the worst $\hat{f}$ from the point of view of stability) subject to the general constraint condition (21). To do this we multiply (21) by the Lagrange multiplier $\lambda(\varepsilon, \nu, L)$, integrate over ν and ε and then integrate (sum) over tubes of force to obtain

$$0 = \int \int \int d\nu d\varepsilon d^3x \lambda(\nu, \varepsilon, L) [Bq g_e \nabla \cdot \xi + (B/q) g_e (-\nabla \cdot \xi + \mathbf{nn} : \nabla \xi)(q^2 - \nu B) - B\hat{f}/q]. \quad (31)$$

We now add this expression to (30) and then vary with respect to $\hat{f}$, obtaining the Euler equation

$$-\hat{f}/g_e + \lambda = 0. \quad (32)$$

If we now use this to eliminate $\hat{f}$ in (16), we find

$$\lambda = \int_T d^3x (B/q) [q^2 \nabla \cdot \xi + (-\nabla \cdot \xi + \mathbf{nn} : \nabla \xi) \times (q^2 - \nu B^2) / \int_T d^3x (B/q)]. \quad (33)$$

These give the minimizing $\hat{f}$ in terms of ξ , so equation (30) now represents a quadratic form in ξ alone, and otherwise involving only equilibrium quantities. In the hydromagnetic^{1, 5} fluid theory, where the pressure develops according to the adiabatic laws

$$d(p_{\parallel} B^2 / \rho^3) / dt = 0 \quad d(p_{\perp} / \rho B) / dt = 0 \quad (34)$$

the corresponding expression δW_D is

$$\begin{aligned} \delta W_D = \frac{1}{2} \int d^3x \{ & \mathbf{Q}^2 + (\nabla \times \mathbf{B}) \cdot \xi \times \mathbf{Q} + \frac{5}{3} p \\ & + (\nabla \cdot \xi)^2 + (\xi \cdot \nabla p_+) \nabla \cdot \xi \\ & + p_+ [\frac{1}{3} (\nabla \cdot \xi)^2 - 2 (\nabla \cdot \xi) \mathbf{nn} : \nabla \xi + 3 (\mathbf{nn} : \nabla \xi)^2] \\ & + p_- [-\xi \cdot \nabla (\mathbf{nn} : \nabla \xi) + 4 (\mathbf{nn} : \nabla \xi)^2 \\ & - \mathbf{n} \cdot \nabla \xi \cdot \nabla \xi \cdot \mathbf{n} + \xi \cdot \nabla \mathbf{n} \cdot \nabla \xi \cdot \mathbf{n} \\ & - (\mathbf{n} \cdot \nabla \xi)^2 + \mathbf{n} \cdot \nabla \xi \cdot (\xi \cdot \nabla \mathbf{n})] \}. \end{aligned} \quad (35)$$

We can now write (30) as

$$\delta W = \delta W_D + I - \frac{1}{2} \int d^3x \mathbf{nn} : \nabla \xi \{ p_+ (2 \nabla \cdot \xi + \mathbf{nn} : \nabla \xi) + 3 p_- \mathbf{nn} : \nabla \xi \}, \quad (36)$$

where

$$I = -\frac{1}{2} m \int \int \int (B/q) d\nu d\epsilon d^3x g_{\epsilon} \{ \lambda^2 - \nu^2 B^2 (-\nabla \cdot \xi + \mathbf{nn} : \nabla \xi)^2 \}. \quad (37)$$

(This expression for σW can be shown to be independent of the component of ξ parallel to $\mathbf{B}$ as it should be on physical grounds.)

COMPARISON THEOREMS

If our condition is necessary as well as sufficient for stability, it is easy to prove that stability under the

present particle theory implies stability under the CGL fluid theory. For by means of Schwarz' inequality

$$\begin{aligned} \lambda^2 \leq \int_T d^3x (B/q) [\nu^2 B^2 (\nabla \cdot \xi - \mathbf{nn} : \nabla \xi)^2 \\ + 2\nu B q^2 \mathbf{nn} : \nabla \xi (\nabla \cdot \xi - \mathbf{nn} : \nabla \xi) \\ + (\mathbf{nn} : \nabla \xi)^2 q^4] / \int_T (B/q) d^3x. \end{aligned} \quad (38)$$

If we now insert this inequality into (37) we find

$$I \leq \frac{1}{2} m \int \int \int (B/q) d\nu d\epsilon d^3x g_{\epsilon} \{ 2\nu B \mathbf{nn} : \nabla \xi (\nabla \cdot \xi - \mathbf{nn} : \nabla \xi) + 3q^2 (\mathbf{nn} : \nabla \xi)^2 \}. \quad (39)$$

When the right-hand side of (39) is expressed in terms of $p_{\perp}$ and $p_{\parallel}$, it becomes precisely the last integral on the right-hand side of (36). Hence,

$$\delta W \leq \delta W_D. \quad (40)$$

We can obtain an important inequality in the opposite direction when the equilibrium distribution function is isotropic ($g_{\nu} = 0$). In this case

$$p = p_{\perp} = p_{\parallel} = \frac{1}{3} m \int_0^{\infty} d\epsilon (2\epsilon)^{3/2} g(\epsilon, L), \quad (41)$$

and we may write

$$\begin{aligned} \lambda = 2\epsilon \int_T B d^3x (1 - yB)^{-1/2} [(1 - \frac{3}{2} yB) \mathbf{nn} : \nabla \xi \\ + \frac{1}{2} yB \nabla \cdot \xi] / \int_T B d^3x (1 - yB)^{-1/2}, \end{aligned} \quad (42)$$

where

$$y = \nu / \epsilon. \quad (43)$$

If we now take y and ϵ as variables in velocity space rather than ν and ϵ we find we may write I in terms of the moment p after an integration by parts in ϵ and obtain

$$I = I_1 + I_2$$

$$\begin{aligned} = (15/4) \int \int_0^{1/B_{\min}} d^3x dy \frac{Bp}{(1 - yB)^{1/2}} \left\{ \frac{\int_T B d^3x (1 - yB)^{-1/2} [(1 - \frac{3}{2} yB) \mathbf{nn} : \nabla \xi + \frac{1}{2} yB \nabla \cdot \xi]^2}{\int_T B d^3x (1 - yB)^{-1/2}} \right. \\ \left. - \frac{1}{2} \int d^3x [2p (-\nabla \cdot \xi + \mathbf{nn} : \nabla \xi)^2] \right\}, \end{aligned} \quad (44)$$

where $B_{\min}$ is the minimum value of B along a line of force. For this isotropic case we now have

$$\begin{aligned} \delta W = \frac{1}{2} \int d^3x [\mathbf{Q}^2 + (\nabla \times \mathbf{B}) \cdot \xi \times \mathbf{Q} \\ + (\xi \cdot \nabla p) (\nabla \cdot \xi)] + I_1. \end{aligned} \quad (45)$$

This result has been also obtained independently by M. Rosenbluth⁷ using another method. Since the integrand in I_1 is positive, we may take Schwarz' inequality in the opposite direction, perform the y integration, and obtain

$$\begin{aligned} \delta W \geq \delta W_H = \frac{1}{2} \int d^3x [\mathbf{Q}^2 + (\nabla \times \mathbf{B}) \cdot \xi \times \mathbf{Q} \\ + (\xi \cdot \nabla p) (\nabla \cdot \xi) + \frac{5}{3} p \langle \nabla \cdot \xi \rangle^2], \end{aligned} \quad (46)$$

where

$$\langle \nabla \cdot \xi \rangle = \left(\int_T d^3x \nabla \cdot \xi \right) / \int_T d^3x. \quad (47)$$

We may conclude that if for a $\gamma = 5/3$ hydromagnetic fluid we can show stability ($\delta W_H < 0$) then we may conclude the system will indeed be stable under our more refined particle picture.

COLLISIONS

In case collisions are not negligible, the situation is somewhat altered in that we lose most of the constants of the motion. Those of the type in (19), for which G is independent of f and v , remain. However the fact, that the equilibrium distribution function is now locally Maxwellian (as it must be for static equilibrium with collisions) enables us to proceed with the argument. We do not lose the property that particles stick to magnetic lines of force, however, since the size of the step away from a magnetic line of force after a collision goes to zero with m/e .

The Boltzmann $\mathcal{H}$ function

$$\mathcal{H} = \int \int \int d^3x (B/q) dv d\epsilon \theta(L) f(\epsilon, L) \ln f$$

$$\mathcal{H}_0 = + \mathcal{H} + \mathcal{H}^* + \dots \quad (48)$$

has the well-known property

$$\dot{\mathcal{H}} \leq 0. \quad (49)$$

We now assume that all regular, time-independent, phase functions have their equilibrium values at $t = -\infty$ and in particular obtain

$$\mathcal{H}(t = -\infty) = \mathcal{H}_0. \quad (50)$$

Now

$$\dot{\mathcal{H}}_0 = 0 \quad (51)$$

and, therefore,

$$\dot{\mathcal{H}} \leq 0. \quad (52)$$

But $d\mathcal{H}/dt$ is linear in $\hat{f}$ and hence a reversal in the sign of $\hat{f}$ leads to $d\mathcal{H}/dt \geq 0$. We conclude, therefore,

$$\mathcal{H} = 0 \quad (53)$$

and finally obtain

$$\hat{\mathcal{H}} \leq 0. \quad (54)$$

Now $\mathcal{E}$ is still a constant of the motion and we may use the expression for $\mathcal{H}$ to eliminate $\hat{f}$ from (25), obtaining expression (28) for this Maxwellian case with the additional positive term $-\mathcal{H}$ on the right-hand side. We could have used the theorem that⁸ $\mathcal{E} + \mathcal{H}$ is a minimum for the Maxwellian distribution with modulus θ , to arrive at this result. We minimize this expression with respect to $\hat{f}$ now with the constraints that the $\mathcal{H}$ function for each tube of force is constant to first order (see (53)) and the number of particles in each tube is constant. That is, using (16) and (20) we may minimize subject to the constraints

$$0 = \int \int \int_T d^3x dv d\epsilon (B/q) \epsilon [q^2 \nabla \cdot \xi g_e$$

$$+ (-\nabla \cdot \xi + \mathbf{nn} : \nabla \xi) (q^2 - vB) g_e - \hat{f}] \quad (55)$$

and

$$0 = \int \int \int_T d^3x dv d\epsilon (B/q) [q^2 \nabla \cdot \xi g_e$$

$$+ (-\nabla \cdot \xi + \mathbf{nn} : \nabla \xi) (q^2 - vB g_e) - \hat{f}]. \quad (56)$$

This leads to

$$\frac{5}{8} \int d^3x p \langle \nabla \cdot \xi \rangle^2 \quad (57)$$

for the value of the integral involving $\hat{f}^2$. (The meaning of $\langle \nabla \cdot \xi \rangle$ is the same as in (46).) It follows at once that

$$\delta W_{\text{coll}} \geq \delta W_H, \quad (58)$$

i.e., stability is not destroyed by the occurrence of collisions.

CHARGE NEUTRALITY

We conclude our presentation with a brief discussion of the charge neutrality condition which is also a consequence of the small m/e limit. This condition is

$$0 = \sum_i n^i e^i$$

$$= \sum_i e^i \int f^i (B/q) dv d\epsilon$$

$$= - \sum_i e^i \int f_e^i B q dv d\epsilon. \quad (59)$$

We must now minimize (25) with respect to $\hat{f}$ subject to the present constraint as well as (16). This leads to a coupled set of linear integral equations for the multipliers with which the constraints are introduced. We have not solved these equations and defer further discussion to future work.

ACKNOWLEDGEMENT

We are indebted to M. Rosenbluth for a valuable discussion.

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Mr. Kruskal presented Paper P/365, above, at the Conference and added the following remarks:

I should like to summarize the development of the theory of plasma stability and, in particular, to compare the work of Rosenbluth and Rostoker, as described in Paper P/349 presented to this Conference, with the Kruskal-Obermann theory as expounded in Paper P/365.

The main problem in the production of controlled thermonuclear energy is the confinement of a completely ionized plasma of hot nuclear fuel by means of strong magnetic fields. A major obstacle to such confinement, however, is the possible existence of instabilities which may quickly disrupt an otherwise satisfactory equilibrium configuration. Accordingly, it is a major task of theory to predict the stability of given configurations, as well as to devise ever better means for so doing.

In the earliest theoretical approaches the plasma was treated as a continuous hydrodynamic fluid described at each point of space-time by a few parameters, for which, together with the electromagnetic fields, one could write a complete set of partial differential equations determining the possible motions. This type of description is appropriate when there is a mechanism which keeps neighboring particles close together, so that a set of neighboring particles can form a coherent element of fluid. In many applications the frequent collisions between the particles constitute such a mechanism.

At first stability was treated by a number of investigators on the basis of the hydromagnetic equations, using the method of normal modes. This involves linearizing the equations for the perturbations of the equilibrium state under investigation, and looking for solutions of these linearized perturbation equations which are purely exponential in their time-dependence, with real or complex exponents. The equilibrium is unstable if growing exponential solutions exist.

Except for rather simple equilibrium states, however, this normal mode method is very difficult to carry through. Following the pioneering work of Lundquist,³ several groups of authors therefore developed an energy principle which greatly enlarges the class of stability problems which can be investigated practicably. This energy principle states that a static equilibrium is stable if and only if a certain associated homogeneous quadratic form is positive definite. The quadratic form represents physically the second-order variation in the potential energy of the system due to an arbitrary virtual displacement of the fluid.

The energy principle provides a very satisfactory general stability theory, at least for static equilibria, when there is some mechanism (almost necessarily collisions) that permits the plasma to be treated hydrodynamically. In controlled thermonuclear energy applications, however, the effects of collisions are negligible for processes that take place as quickly as the growth of most instabilities. There is, instead, a mechanism which is in a sense two-thirds effective

in keeping neighboring particles close together; namely, a strong magnetic field forces the charged particles to gyrate around a point which sticks to and moves with a line of force. As a result, the particles cannot disperse in the two directions perpendicular to the magnetic lines of force, but only in a direction parallel to the field lines.

Several groups of authors, notably Chew, Goldberger, and Low,¹ have shown how to treat the plasma in the mathematically appropriate way when collisions are not important. They employ the so-called collisionless Boltzmann equation with a term $(q/m)(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ in place of the usual collision term. The electric and magnetic fields satisfy the Maxwell equations and the current and charge density are expressed as sums over the different species of integrals over velocity space, utilizing the distribution function which satisfies the Boltzmann equation as a weighting function in the integrand.

One now wishes to obtain the limiting form of these equations as the radius of gyration of a particle in the magnetic field becomes very small compared to the characteristic length of the system under consideration, and the period of gyration becomes very small compared to a characteristic time. This limiting process may be formalized in various ways, the simplest of which is to treat the charge q as being very large. It is not trivial to carry this program through systematically. However, it has been shown that the resultant reduced system of equations is easier to handle than the original system. The reduced system, obtained in this fashion, may be used to investigate the stability of equilibria by the normal mode or equivalent methods. This is essentially the approach adopted by Chandrasekhar, Kaufman, and Watson. Unfortunately, compared to the hydromagnetic equations, it is more difficult to carry through the normal mode method with the reduced system when the equilibrium is not simple.

In the two papers which I wish to discuss here, (P/365 and P/349) an energy principle for determining stability is derived based on the reduced system. The methods used differ considerably from each other. The conclusions reached in each paper are in some respects more general than those in the other, but where they overlap they agree.

In the paper by Rosenbluth and Rostoker, the equilibrium state is assumed to have isotropic distribution functions to lowest order in the gyration radius. The first part of the paper is devoted to obtaining the equations of a normal mode, and it is shown that the first-order perturbations from equilibrium of all quantities are proportional to the exponential of ωt . The quantity ω , which may be complex, is the characteristic growth or frequency parameter of the mode. In this analysis, all the first-order perturbations are expressed in terms of the usual first-order vector ξ , which describes at each point the perpendicular displacement of the magnetic line of force. One obtains finally an integro-differential equation for ξ analogous to the differential equation for ξ obtainable from hydromagnetic theory. The presence

of the integrals, which are taken along equilibrium lines of force, is of course a consequence of the spatially non-local character of the treatment. The normal mode equations constitute an eigenvalue problem for ω .

The next step in the Rosenbluth-Rostoker method is to construct the second-order change in energy due to the first-order perturbation. This can be expressed as a quadratic functional W in ξ . For $\omega = 0$, the integrand of the functional contains three terms which are the same as in the hydromagnetic case because they represent the second-order change in the energy of the magnetic field. The fourth term in the integrand involves two integrals which, for each point x and each value of the dummy variable of integration, are line integrals taken along part of the equilibrium line of force through x .

It turns out that the quadratic functional W vanishes and is stationary for a vector field ξ which is not identically zero if and only if ξ is a solution of the eigen-value problem described earlier and has the eigenvalue $\omega = 0$. This suggests strongly that if W is negative for some ξ , then there is an instability. A proof is given that if W is *positive-definite* then the equilibrium is stable.

In the other paper, we (Oberman and Kruskal) have made no effort to obtain conditions for instability, but have sought the weakest conditions we could for stability. To do this we look for constants of the motion of the system which do not depend explicitly on the time and which are regular near the equilibrium configuration. Since any purely unstable motion has been arbitrarily close to the equilibrium state far enough back in time, such constants of motion must have the same values for an unstable motion as for equilibrium. That is, the first and second-order perturbations of these constants of motion must

vanish. This leads to severe restrictions on the first-order perturbations of the physical quantities; when they are so severe that these perturbations must vanish, stability is assured.

We do not assume that the equilibrium distribution functions are isotropic, but we do require of the equilibrium that the mass velocity vanish to lowest order in the gyration radius for each species of ion. We obtain finally a quadratic form in ξ , the positive-definiteness of which implies stability. In the case of isotropic equilibrium distributions it reduces exactly to the form W obtained by Rosenbluth and Rostoker.

Our method of using constants of the motion can be applied even when the collision terms of the Boltzmann equation are retained. There are then far fewer suitable constants available, but this is compensated for by the necessary restriction to those equilibria which are invariant during collisions to lowest order in the gyration radius. Such equilibria are those having Maxwellian distributions with constant temperature along lines of force. The final result turns out to be the same energy principle as before.

Both papers also give essentially the same comparison theorems. One of these is that W is bounded *above* by the simpler result derived from the standard hydromagnetic theory with two distinct pressures, one parallel and one perpendicular to the magnetic field, each governed by its own adiabatic equation of state. The other comparison theorem is that in the isotropic case, W is bounded *below* by the result of standard hydromagnetic theory with one scalar pressure. Because of these comparison theorems, fortunately, we have demonstrated in these two papers that the stability results previously obtained from standard hydromagnetic theories still have considerable significance when viewed in terms of our more accurate calculations.

Theoretical Structure of Plasma Equations

By M. N. Rosenbluth and N. Rostoker *

In high-temperature plasmas, collisions are very infrequent. Thus the charged particles travel on independent orbits determined by the electromagnetic field. At first sight this would seem completely different from a conventional fluid where particles are closely hemmed in by their neighbors. However, there can exist collective modes of motion in which the particles interact with each other by altering the fields.

In this paper a new method is developed for the solution of the linearized transport equation. By facilitating direct use of the properties of particle orbits, a considerable simplification is achieved. In particular, a variational expression is derived for determining stability which is rigorous in the limit of small Larmor radius.

Investigations of the stability of a fully ionized plasma are usually based on the magnetohydrodynamic equations^{1, 2} (MH approximation). The essential features of these equations are the assumptions of scalar pressure P and an adiabatic invariant

$$\frac{d}{dt}(P\varrho_m^{-\gamma}) = 0. \quad (1)$$

Chew, Goldberger, and Low³ have proposed a different approximation (CGL approximation) in which the pressure is not scalar and there are two adiabatic invariants

$$\frac{d}{dt}\left(\frac{P_{\perp}}{\varrho_m B}\right) = 0, \quad (2)$$

$$\frac{d}{dt}\left(\frac{P_{\parallel} B^2}{\varrho_m^3}\right) = 0. \quad (3)$$

$P_{\parallel}$ and $P_{\perp}$ are pressures parallel and perpendicular to the direction of the magnetic field $\mathbf{B}$. Neither the accuracy nor the conditions for the validity of these approximations are completely understood.

The transport equation provides a complete description of the plasma and a deductive basis for approximate descriptions. The MH approximation and

the CGL approximation consist of taking moments of the transport equation. Each moment equation is a differential equation that involves only macroscopic variables. The adiabatic invariants are introduced arbitrarily to terminate the set of moment equations. To investigate the stability of a particular plasma configuration, the moment plus adiabatic invariant equations are solved subject to suitable boundary conditions.

The essential feature of the present approach is that the transport equation is solved at the outset to give the perturbed distribution function in terms of the perturbed field variables. The macroscopic variables calculated from the solution must then satisfy Maxwell's equations. In the limit of small Larmor radius, these become integro-differential equations. This approach leads to more complex macroscopic equations, but it has the virtue of being correct when the MH and CGL approximations are not.

A convenient way of treating stability problems is to calculate the change in energy which results from a small perturbation. In the MH approximations this method leads to a variational principle.⁴ The energy change calculated by the present procedure of solving the transport equation also will be shown to give a variational principle. It is bounded below by the MH energy change and above by the CGL energy change.

SOLUTION OF THE TRANSPORT EQUATION

At sufficiently high temperatures the plasma can be described by the collision-free transport equation⁵

$$\partial f_i / \partial t + \mathbf{v} \cdot \nabla f_i + (\mathbf{F}_i / m_i) \cdot \nabla v_i = 0, \quad (4)$$

where

$$\mathbf{F}_i = q_i(\mathbf{E} + (1/c)\mathbf{v} \times \mathbf{B}).$$

$\mathbf{E}$ and $\mathbf{B}$ are macroscopic fields that satisfy Maxwell's equations. They are produced by the charge density

$$\varrho_e = \sum_i q_i \int f_i d^3v,$$

and the current density

$$\mathbf{j} = \sum_i q_i \int f_i \mathbf{v} d^3v.$$

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$f_i(\mathbf{x}, \mathbf{v}; t)$ is the distribution function for the i th particle species. (For present purposes, $i = 1$ for electrons and $i = 2$ for ions; the subscript i will be deleted except where it is essential.)

The stationary state whose stability is to be considered satisfies

$$\mathbf{v} \cdot \nabla f_0 + (q/mc)(\mathbf{v} \times \mathbf{B}) \cdot \nabla f_0 = 0. \quad (5)$$

The linearized time-dependent equation that determines the stability of this state is

$$\partial(\delta f)/\partial t + \mathbf{v} \cdot \nabla \delta f + (q/mc)(\mathbf{v} \times \mathbf{B}) \cdot \nabla \delta f = -\delta \mathbf{F} \cdot \nabla f_0/m. \quad (6)$$

This equation describes both the dynamical and statistical aspects of the problem.

The dynamical aspects of the stability problem have been clarified by a treatment⁶ which proceeds from the solution of the orbit equations

$$\begin{aligned} d\mathbf{v}'/dt' &= (q/m)(\mathbf{E} + (1/c)\mathbf{v}' \times \mathbf{B}), \\ d\mathbf{x}'/dt' &= \mathbf{v}'. \end{aligned} \quad (7)$$

To calculate macroscopic variables, such as charge density, current density, etc., it is essential to know the weighting function for the particles. A more convenient treatment of the statistical aspects of the problem based on a solution of the transport equation has been given by Chandrasekhar, Kaufman and Watson.⁷

In the present approach, Eq. (6) is solved by making time integrations along the unperturbed orbits. This procedure does not emphasize one of the dynamical or statistical aspects at the expense of the other. Equation (6) can be written as

$$d(\delta f[\mathbf{x}(t), \mathbf{v}(t); t])/dt = -\delta \mathbf{F} \cdot \nabla f_0/m,$$

so that

$$\delta f(\mathbf{x}, \mathbf{v}; t) = -(1/m) \int_{t_0}^{t'} \delta \mathbf{F} \cdot \nabla f_0 dt' + \delta f(t_0). \quad (8)$$

The integrand must be evaluated along the unperturbed orbit at $\mathbf{x}'(\mathbf{x}, \mathbf{v}; t')$, $\mathbf{v}'(\mathbf{x}, \mathbf{v}; t')$, where these functions satisfy

$$\begin{aligned} d\mathbf{v}'/dt' &= (\mathbf{v}' \times \mathbf{B})q/mc \\ d\mathbf{x}'/dt' &= \mathbf{v}', \end{aligned} \quad (9)$$

and the initial conditions

$$\begin{aligned} \mathbf{x}'(\mathbf{x}, \mathbf{v}; t) &= \mathbf{x}; \\ \mathbf{v}'(\mathbf{x}, \mathbf{v}; t) &= \mathbf{v}. \end{aligned}$$

The relevant physical problem defined by Eq. (8) is an initial-value problem. An arbitrary initial displacement is made from the state described by f_0 , and one wishes to know the subsequent time evolution. This type of problem has been solved by Laplace transforms.⁸ The equations for the Laplace transforms of the initial-value problem are the same as for an eigenvalue problem where $\delta f = f_1(\mathbf{x}, \mathbf{v})e^{pt}$, $\delta \mathbf{B} = \mathbf{B}_1(\mathbf{x})e^{pt}$, etc., except for some inhomogeneous terms from the initial conditions. The Laplace transforms

will have poles at the eigenvalues $p = p_n$. There are no additional singularities in which p has a positive real part. The significant time-dependence of the solution of the initial-value problem will be of the form $t^l e^{p_n t}$, where l is the order of the pole at $p = p_n$. For stability considerations it is sufficient to know whether p_n has a positive real part so that we shall only consider the eigenvalue problem. As long as p has a positive real part, there are no convergence difficulties in Eq. (8). Furthermore, in this case, the arbitrary function $\delta f(t_0)$ in Eq. (8) drops out if $t_0 \rightarrow -\infty$.

For an unperturbed magnetic field $\mathbf{B}(\mathbf{x})$ that is not constant, Eq. (5) and (9) cannot be solved exactly. The appropriate expansion parameters are $\lambda_i = a_i/L$ where $a_i = m_i v_{\perp} c / q_i B$ are the radii of gyration and $L \gg a_i$ is a characteristic length of the problem. (For example, $L = B/|\nabla B|$.) In the calculation of δf there are additional parameters: p/ω , where $\omega = qB/mc$ is the cyclotron frequency; p/ω_p , where $\omega_p = (4\pi N q^2/m)^{1/2}$ is the plasma frequency; and L_D/L , where $L_D = (\Theta/4\pi N q^2)^{1/2}$ is the Debye length. These parameters are assumed to be of the same order as λ . Each parameter is proportional to q^{-1} .

For Maxwell's equations we need to compute $\delta \mathbf{j}$, the perturbed current. Because $\delta \mathbf{j}$ is proportional to $q\delta f$, it would seem that δf must be calculated to order λ^{n+1} to determine $\delta \mathbf{j}$ to order λ^n . However, to calculate the pressure tensor

$$\delta \mathbf{P} = \sum_i m_i \int \mathbf{v} \mathbf{v} \delta f_i d^3 v, \quad (10)$$

to order λ^n requires δf only to order λ^n . As any solution for δf must satisfy the moment equation

$$p\mathbf{Q}_m \delta \mathbf{v} = -\nabla \cdot \delta \mathbf{P} + (1/c)[\delta \mathbf{j} \times \mathbf{B} + \mathbf{j} \times \delta \mathbf{B}], \quad (11)$$

$\delta \mathbf{j} \times \mathbf{B}$ can be inferred from $\delta \mathbf{P}$. This method is clearly simpler than the direct calculation of $\delta \mathbf{j}$.⁷

Our procedure is then as follows: $\delta \mathbf{E}$ is to be regarded as the fundamental variable. $\delta \mathbf{B}$ can be expressed in terms of it by Maxwell's equations. The distribution function is then calculated by Eq. (8) and the charge density (or alternately the component $\delta j_{\parallel}$) is computed directly from it in terms of $\delta \mathbf{E}$. The two components of current perpendicular to $\mathbf{B}$ are given by Eq. (11). The Maxwell equations $\nabla \times \delta \mathbf{B} = (4\pi/c)\delta \mathbf{j}$ and $\nabla \cdot \delta \mathbf{E} = 4\pi\delta \rho_e$ now provide the system to be solved for $\delta \mathbf{E}$. (It is often convenient to use the latter equation to eliminate $\delta E_{\parallel}$.) Equation (8) will be solved to order λ^0 , so that the resulting equations should be analogous to magnetohydrodynamics in which the charge q does not appear.

We now return to a brief discussion of the equilibrium state.

To solve Eq. (5) it is convenient to use a local cylindrical coordinate system with the z axis in the direction of $\mathbf{B} = B\mathbf{n}$. A solution of the form

$$f_0 = f(\mathbf{x}, \mathbf{v}) + \lambda g(\mathbf{x}, \mathbf{v}) + O(\lambda^2)$$

is substituted into

$$\mathbf{v} \cdot \nabla f_0 + \omega^{-1} \partial f_0 / \partial \phi = 0.$$

Coefficients of λ^n are set equal to zero. From the coefficient of λ^0 , $\partial f / \partial \phi = 0$. Therefore, $f = f(\mathbf{x}, v_\perp, v_\parallel)$. For present purposes we assume $f = f(\mathbf{x}, v^2)$, i.e., an initially isotropic pressure tensor. From the coefficient of λ it follows that

$$\mathbf{n} \cdot \nabla f = 0 \quad (12)$$

and

$$f_0 = f(\mathbf{x}, v^2) + (1/\omega)(\mathbf{v} \cdot \mathbf{n} \times \nabla f) + \lambda h(\mathbf{x}; v_\perp^2, v_\parallel). \quad (13)$$

The solution of Eq. (9) is only required to order λ^0 . To this order, the particle undergoes circular motion about a guiding center that follows a field line, i.e.,

$$\mathbf{v}' = V_\parallel \mathbf{n} + V_\perp [\mathbf{e}_1 \cos \int^{t'} \omega dt'' \pm \mathbf{e}_2 \sin \int^{t'} \omega dt''], \quad (14)$$

where

$$\begin{aligned} V_\parallel^2 &= v^2 - 2\mu B/m, \\ V_\perp^2 &= 2\mu B/m, \\ \mathbf{e}_1 &= R_1(\mathbf{n} \cdot \nabla) \mathbf{n} \quad \text{and} \quad \mathbf{e}_2 = \mathbf{n} \times \mathbf{e}_1. \end{aligned}$$

The magnetic moment $\mu = mV_\perp^2/2B$ and v^2 are constants of the motion.⁶ If B is constant along the orbit, $V_\perp = v_\perp$ and $V_\parallel = v_\parallel$ are constants. Otherwise, the particle orbit may have turning points because $V_\parallel = 0$ when

$$B(\mathbf{x}') = (v^2/v_\perp^2)B(\mathbf{x}). \quad (15)$$

δf is calculated by substituting Eq. (13) into Eq. (8); with the assumed exponential time-dependence,

$$\begin{aligned} -\delta f &= \frac{c}{p} \int_{t'=-\infty}^{t'=t} \frac{(\mathbf{v}' \times \nabla \times \delta \mathbf{E}) \cdot (\mathbf{B} \times \nabla f)}{B^2} dt' \\ &+ c \int_{t'=-\infty}^{t'=t} \frac{\delta \mathbf{E} \cdot \mathbf{B} \times \nabla f}{B^2} dt' \\ &+ \frac{2q}{m} \int_{t'=-\infty}^{t'=t} (\mathbf{v} \cdot \delta \mathbf{E}) \left. \frac{\partial f}{\partial v^2} \right|_c dt' \\ &+ \lambda \frac{q}{m} \int_{t'=-\infty}^{t'=t} (\mathbf{n} \cdot \delta \mathbf{E}) \frac{\delta h}{\delta v_\parallel} dt' + O(\lambda), \end{aligned} \quad (16)$$

where

$$\left. \frac{\partial f}{\partial v^2} \right|_c = \frac{\partial f}{\partial v^2} + \frac{1}{\omega} (\mathbf{v}' \cdot \mathbf{n} \times \nabla \frac{\partial f}{\partial v^2})$$

is the value of $\partial f / \partial v^2$ at the guiding center and remains constant during a Larmor period. As $\delta f \sim [(\delta E_\parallel / \lambda) + \delta E][1 + O(\lambda) + \dots]$, the charge density will be $\delta \rho_e \sim [1/L][(\delta E_\parallel / \lambda^2) + (\delta E_\perp / \lambda)][1 + O(\lambda) \dots] \gg \nabla \cdot \delta \mathbf{E}$. Therefore, $\delta E_\parallel / \delta E_\perp \sim \lambda$. The physical reason for this is that $\delta E_\parallel$ can cause a large charge density. If it is to be slowly varying it must be small. $\delta E_\perp$ cannot produce a correspondingly large charge density because the $\delta \mathbf{E}_\perp \times \mathbf{B}$ drifts are independent of the sign of q . It is convenient to introduce a two-component displacement vector $\delta \mathbf{x} = \mathbf{x} e^{pt}$ such that

$$\delta \mathbf{E} = (p/c) \mathbf{B} \times \delta \mathbf{x} + \delta \mathbf{E}_\parallel \quad (17)$$

Bearing in mind the above discussion, we treat $\delta E_\parallel$ as of order $\lambda \delta \mathbf{x}$. It is clear that the component of $\mathbf{x}$ parallel to the field $\mathbf{B}$ is irrelevant and we set it equal to zero in the subsequent algebra.

In the first term of Eq. (16)

$$\begin{aligned} (c/pB^2) (\mathbf{v}' \times \nabla \times \delta \mathbf{E}) \cdot (\mathbf{B} \times \nabla f) \\ = (V_\parallel/B) [\nabla \times (\delta \mathbf{x} \times \mathbf{B})] \cdot \nabla f + O(\lambda). \end{aligned}$$

From the vector identities

$$\begin{aligned} \nabla \cdot [(\delta \mathbf{x} \times \mathbf{B}) \times \nabla f] \\ = \nabla f \cdot [\nabla \times (\delta \mathbf{x} \times \mathbf{B})] - (\delta \mathbf{x} \times \mathbf{B}) \cdot \nabla \times \nabla f \\ = \nabla \cdot [\mathbf{B} (\delta \mathbf{x} \cdot \nabla f) - \delta \mathbf{x} (\mathbf{B} \cdot \nabla f)] \\ = \mathbf{B} \cdot \nabla (\delta \mathbf{x} \cdot \nabla f) + \delta \mathbf{x} \cdot \nabla \nabla \cdot \mathbf{B} \end{aligned}$$

and the relations

$$\nabla \cdot \mathbf{B} = \mathbf{B} \cdot \nabla f = \nabla \times \nabla f = 0,$$

the first term is reduced to

$$\int_{t'=-\infty}^{t'=t} V_\parallel \mathbf{n} \cdot \nabla (\delta \mathbf{x} \cdot \nabla f) dt'. \quad (16.1)$$

In the second term

$$\begin{aligned} (c/B^2) (\delta \mathbf{E} \cdot \mathbf{B} \times \nabla f) \\ = p(\mathbf{n} \times \delta \mathbf{x}) \cdot (\mathbf{n} \times \nabla f) = p(\delta \mathbf{x} \cdot \nabla f); \end{aligned}$$

after an integration by parts, the second term is

$$\delta \mathbf{x} \cdot \nabla f - \int_{t'=-\infty}^{t'=t} V_\parallel \mathbf{n} \cdot \nabla (\delta \mathbf{x} \cdot \nabla f) dt'. \quad (16.2)$$

In the third term

$$\begin{aligned} \mathbf{v}' \cdot \delta \mathbf{E} &= (p/c) (\mathbf{v}' \cdot \mathbf{B} \times \delta \mathbf{x}) + \mathbf{v}' \cdot \delta \mathbf{E}_\parallel \\ &= (mp/q) (\delta \mathbf{x} \cdot d\mathbf{v}'/dt') + (\mathbf{v}' \cdot \delta \mathbf{E}_\parallel); \end{aligned}$$

after an integration by parts, the third term is

$$\begin{aligned} 2p\delta \mathbf{x} \cdot \mathbf{v} (\partial f / \partial v^2) - 2p \int_{t'=-\infty}^{t'=t} [\mathbf{v}' \cdot (\mathbf{v}' \cdot \nabla) \delta \mathbf{x} \\ + p(\delta \mathbf{x} \cdot \mathbf{v}')] \partial f / \partial v^2|_c dt' \\ + 2q/m \int_{t'=-\infty}^{t'=t} (\mathbf{v}' \cdot \delta \mathbf{E}_\parallel) \partial f / \partial v^2|_c dt'. \end{aligned}$$

The integrands next must be averaged over a Larmor period. The quantity $\partial f / \partial v^2|_c$ is constant during this time to $O(\lambda)$, and the oscillatory parts of $\mathbf{v}'$ in Eq. (14) give zero except where a product $\cos^2$ or $\sin^2$ obtains. Thus, averaged over a period, we see that

$$\langle \mathbf{v}' \cdot \mathbf{v}' \rangle = \mathbf{I} V_\perp^2/2 + (V_\parallel^2 - V_\perp^2/2) \mathbf{n} \mathbf{n};$$

$\mathbf{I}$ is the unit dyadic, and $\mathbf{n}$ the unit vector in the direction of $\mathbf{B}$. The result for the third term is

$$\begin{aligned} -p \int_{t'=-\infty}^{t'=t} [(2V_\parallel^2 - V_\perp^2) \mathbf{n} \cdot (\mathbf{n} \cdot \nabla) \delta \mathbf{x} \\ + V_\perp^2 \nabla \cdot \delta \mathbf{x}] (\partial f / \partial v^2) dt' + 2p\delta \mathbf{x} \cdot \mathbf{v} \partial f / \partial v^2 \\ + (2q/m) \int_{t'=-\infty}^{t'=t} V_\parallel \delta \mathbf{E}_\parallel (\partial f / \partial v^2) dt'. \end{aligned}$$

The integrations now are to be carried out along the zero-order particle paths $d\mathbf{x}'/dt' = V_{\parallel} \mathbf{n}$. As $df/dt' = V_{\parallel} \mathbf{n} \cdot \nabla f = 0$, $\partial f/\partial v^2$ can be taken outside the integrals.

The result for the distribution function is

$$\begin{aligned} -\delta f &= \delta \xi \cdot \nabla f + 2p(\delta \xi \cdot \mathbf{v})(\partial f/\partial v^2) \\ &+ (2q/m)(\partial f/\partial v^2) \int_{t'=-\infty}^{t'=t} V_{\parallel} \delta \mathbf{E}_{\parallel} dt' \\ &- p(\partial f/\partial v^2) \int_{t'=-\infty}^{t'=t} [(2V_{\parallel}^2 - V_{\perp}^2) \mathbf{n} \cdot (\mathbf{n} \cdot \nabla) \delta \xi \\ &+ V_{\perp}^2 \nabla \cdot \delta \xi] dt' + O(\lambda) \dots \end{aligned} \quad (18)$$

Since $\delta \mathbf{B} = \nabla \times (\delta \xi \times \mathbf{B})$, the lines of force move so that each point on a line is displaced by the local value of $\delta \xi$ which is perpendicular to the line.⁹ Equation (18) is of the form $-\delta f = \delta \xi \cdot \nabla f + \delta \mathbf{v} \cdot \nabla_v f$. This means that particles are all displaced by the local value of $\delta \xi$ so that they remain "frozen" on the line of force.

A particular particle is identified by the fact that it has velocity $\mathbf{v}$ and position $\mathbf{x}$ at time t . The zero-order particle orbit $\mathbf{x}'(\mathbf{x}, \mathbf{v}; t')$ simply follows a field line $\mathbf{B}$ which forms a closed loop. The orbits will always be at least quasi-periodic in a finite system. That is, they will return arbitrarily close to the starting point. Thus,

$$\mathbf{x}'(t' + \tau) = \mathbf{x}'(t') \quad (19)$$

where

$$\tau = 2 \int_{l_1} dl / |V_{\parallel}|.$$

l is a coordinate that measures the length along the particle orbit. For particle orbits that have no turning points, l_1 and l_2 are determined by $B(x)$. They depend on $\mathbf{x}$, but not on $\mathbf{v}$. ($l_2 - l_1$) is the distance around the closed loop that brings one back to the same point, e.g., once around a torus. If the particle orbit has turning points, l_1 and l_2 are their positions. These quantities may be determined by solving Eq. (15). They depend on $\mathbf{x}$ and the velocity direction $\cos \theta = \mathbf{v} \cdot \mathbf{n}/v$, but not on the magnitude of $\mathbf{v}$.

The integrals in Eq. (18) can be reduced to integrals over a single period. That is, if $\Psi(t' + \tau) = \Psi(t')$, then

$$\int_{t'=-\infty}^{t'=t} \Psi(t') e^{pt'} dt' = \frac{1}{1 - e^{-p\tau}} \int_{t'=t-\tau}^{t'=t} \Psi(t') e^{pt'} dt'.$$

The limiting forms of Eq. (18) are easily obtained for the cases $p\tau \ll 1$ and $p\tau \gg 1$. They are as follows:

For $p\tau \ll 1$,

$$\begin{aligned} -\delta f &= \xi \cdot \nabla f \\ &- \frac{2}{\tau} \frac{\partial f}{\partial v^2} \int_{l_1}^{l_2} [(2V_{\parallel}^2 - V_{\perp}^2) \mathbf{n} \cdot (\mathbf{n} \cdot \nabla) \xi \\ &+ V_{\perp}^2 \nabla \cdot \xi] dl' / |V_{\parallel}| + O(p\tau) \dots \end{aligned} \quad (20)$$

For $p\tau \gg 1$,

$$\begin{aligned} -\delta f &= \delta \xi \cdot \nabla f + 2p(\delta \xi \cdot \mathbf{v}) \frac{\partial f}{\partial v^2} \\ &- [(2v_{\parallel}^2 - v_{\perp}^2) \mathbf{n} \cdot (\mathbf{n} \cdot \nabla) \delta \xi + v_{\perp}^2 \nabla \cdot \delta \xi] \frac{\partial f}{\partial v^2} \\ &+ O\left(\frac{1}{p\tau}\right) + O(\lambda). \end{aligned} \quad (21)$$

In both cases the equation $\nabla \cdot \delta \mathbf{E} = 4\pi\delta\rho_e$ shows that the contribution of the first integral in Eq. (18), while of order λ^0 , is of higher order in $p\tau$ and $1/p\tau$, respectively, and may be omitted.

MARGINAL STABILITY EIGENVALUE PROBLEM

The quantity $\delta \mathbf{E}$ satisfies Maxwell's equation

$$\mathbf{B} \times \nabla \times \nabla \times \delta \mathbf{E} + (p/c)^2 \mathbf{B} \times \delta \mathbf{E} = (4\pi p/c^2) (\delta \mathbf{j} \times \mathbf{B}), \quad (22)$$

where $\delta \mathbf{j} \times \mathbf{B}$ is related to the pressure tensor by Eq. (11). The pressure tensor can now be calculated to order $p\tau$ by substituting Eq. (20) for δf into Eq. (10). It is convenient to introduce as velocity variables the constants of the motion, v^2 and

$$\alpha = \frac{2\mu}{mv^2} = (1 - \cos^2 \theta)/B(\mathbf{x}). \quad (23)$$

In these variables

$$-\delta f = \xi \cdot \nabla f - 2v^2 \frac{\partial f}{\partial v^2} \frac{J(\mathbf{x}, \alpha)}{K(\mathbf{x}, \alpha)}, \quad (20.1)$$

where

$$\begin{aligned} J(\mathbf{x}, \alpha) &= \int_{l_1(\mathbf{x}, \alpha)}^{l_2(\mathbf{x}, \alpha)} \{ [1 - \frac{3}{2} \alpha B(l')] \mathbf{n} \cdot (\mathbf{n} \cdot \nabla) \xi \\ &+ \frac{\alpha B(l')}{2} \nabla \cdot \xi \} \frac{dl'}{[1 - \alpha B(l')]^{\frac{1}{2}}} \end{aligned}$$

and

$$K(\mathbf{x}, \alpha) = \int_{l_1(\mathbf{x}, \alpha)}^{l_2(\mathbf{x}, \alpha)} \frac{dl'}{[1 - \alpha B(l')]^{\frac{1}{2}}}.$$

The result for the pressure tensor is

$$\delta \mathbf{P} = -(\xi \cdot \nabla \mathbf{P}) \mathbf{I} + \delta P_{\perp} \mathbf{I} + (\delta P_{\parallel} - \delta P_{\perp}) \mathbf{nn}, \quad (24)$$

where

$$P = (4\pi/3) \sum_i m_i \int f_i(\mathbf{x}, v^2) v^4 dv$$

is the unperturbed pressure,

$$\delta P_{\perp} = -\frac{15}{4} P B^2(\mathbf{x}) \int_{\alpha=0}^{\alpha=1/B(\mathbf{x})} \frac{J(\mathbf{x}, \alpha) \alpha d\alpha}{K(\mathbf{x}, \alpha) [1 - \alpha B(\mathbf{x})]^{\frac{1}{2}}} \quad (24.1)$$

and

$$\delta P_{\parallel} = -\frac{15}{2} P B(\mathbf{x}) \int_{\alpha=0}^{\alpha=1/B(\mathbf{x})} \frac{J(\mathbf{x}, \alpha) [1 - \alpha B(\mathbf{x})]^{\frac{1}{2}} d\alpha}{K(\mathbf{x}, \alpha)} \quad (24.2)$$

To change from the independent variable $\delta\mathbf{E}$ to $\delta\boldsymbol{\xi}$, we can substitute $\delta\mathbf{E} = p(\mathbf{B} \times \delta\boldsymbol{\xi})/c + O(\lambda)$ and $\delta\mathbf{B} = \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}) + O(\lambda)$ into Eq. (22). For the marginal stability case, $p \rightarrow 0$, the eigenvalue equations are

$$\mathbf{B} \times \nabla \times \nabla \times (\mathbf{B} \times \boldsymbol{\xi}) - (\nabla \times \mathbf{B}) \times \nabla \times (\mathbf{B} \times \boldsymbol{\xi}) = 4\pi \nabla \cdot \delta\mathbf{P}. \quad (25)$$

There are two equations for the two components of $\boldsymbol{\xi}$. In the case of a cylindrical pinch, $\mathbf{n} \cdot \nabla \mathbf{B} = 0$ everywhere, and none of the particle orbits has turning points. For displacements $\boldsymbol{\xi} = \boldsymbol{\xi}_k e^{ikl}$, $J(\mathbf{x}, \alpha) = 0$ unless $k = 0$. For $k \neq 0$, $\delta\mathbf{P}$ reduces to a scalar so that magnetohydrodynamics is valid. In the case $k = 0$, singularities develop in the equations.¹⁰ A complete treatment of this would require carrying the expansions to higher order in λ .

ENERGY PRINCIPLE FOR STABILITY PROBLEMS

The eigenvalue equations for the marginal stability case can be derived from a variational principle. The appropriate functional is the energy change that results from the perturbation

$$W = \int \left\{ \frac{\delta\mathbf{E} \cdot \delta\mathbf{j}}{2p} + \frac{|\delta\mathbf{B}|^2}{8\pi} \right\} d^3\mathbf{x}. \quad (26)$$

This can be expressed as a functional of $\boldsymbol{\xi}$ using Eqs. (11) and (24). In the marginal stability case

$$W = W_0 + W_1, \quad (27)$$

where

$$\begin{aligned} W_0 = & \frac{1}{2} \int \left\{ -\boldsymbol{\xi} \cdot \nabla (\boldsymbol{\xi} \cdot \nabla P) \right. \\ & + \frac{1}{c} \mathbf{j} \cdot \boldsymbol{\xi} \times \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) \\ & \left. + \frac{1}{4\pi} |\nabla \times \boldsymbol{\xi} \times \mathbf{B}|^2 \right\} d^3\mathbf{x} \end{aligned} \quad (27.1)$$

and

$$W_1 = -\frac{1}{2} \int \{ (\delta P_{\parallel} - \delta P_{\perp}) \mathbf{n} \cdot (\mathbf{n} \cdot \nabla) \boldsymbol{\xi} + \delta P_{\perp} \nabla \cdot \boldsymbol{\xi} \} d^3\mathbf{x}. \quad (27.2)$$

A more explicit form for W_1 is obtained by substituting $\delta P_{\parallel}$ and $\delta P_{\perp}$ from Eqs. (24.1) and (24.2), and carrying out the volume integration over flux tubes; i.e., $d^3\mathbf{x} = dl \, d\Phi/B(l)$. Φ is a coordinate which labels a given flux tube. $d\Phi = B dS$ indicates the amount of flux contained therein. It is a convenient variable, as a particle orbit lies in a definite flux tube. The order of integration of the variables l , α can be interchanged, i.e.,

$$dl \int_{\alpha=0}^{\alpha=1/B(\mathbf{x})} d\alpha \rightarrow \int_{\alpha=0}^{\alpha=1/B_m} d\alpha \int_{l_1(\Phi, \alpha)}^{l_2(\Phi, \alpha)} dl, \quad (28)$$

where $B_m(\Phi)$ is the minimum value of B for a particular flux tube. The resultant expression for W_1 is

$$W_1 = \frac{15}{4} \int P \, d\Phi \int_{\alpha=0}^{\alpha=1/B_m} d\alpha [J(\Phi, \alpha)]^2 / K(\Phi, \alpha). \quad (27.3)$$

W_1 is clearly a positive definite quantity. The term W_0 obtains in the MH approximation as well as the CGL approximation. The difference between these approximations and the present result, which is correct to order λ , lies entirely in W_1 . The condition that W be a minimum with respect to arbitrary variations in $\boldsymbol{\xi}$ turns out to be just that Eq. (25), the zero frequency equation, is satisfied. Thus, if $W_{\min} = 0$, the stability is marginal, and we can conclude that $W_{\min} > 0$ is a necessary condition for stability.

A brief outline of the proof for sufficiency will now be given. Again we consider the change in energy which would result from a given perturbation $\delta\mathbf{E}$ with an assumed frequency p . In general, p may be complex, but has a positive real part. The energy change is then found to have oscillatory parts as well as a monotonically changing part. In order for a solution to exist, both parts of the energy change must vanish because there is assumed to be no external energy source. We restrict our attention to the monotonically changing part

$$W' = \int \left\{ \frac{1}{2} \left(\frac{\delta\mathbf{E} \cdot \delta\mathbf{j}^* + \delta\mathbf{E}^* \cdot \delta\mathbf{j}}{p + p^*} \right) + \frac{1}{2} \left(\frac{\delta\mathbf{B} \cdot \delta\mathbf{B}^*}{8\pi} \right) \right\} d^3\mathbf{x}. \quad (29)$$

As in the previous case, we use Eq. (11) to find $\delta\mathbf{j}_{\perp}$. The quantities $\delta\mathbf{j}_{\parallel}$, $\delta\mathbf{V}$, and $\delta\mathbf{P}$ are calculated by taking the appropriate moments of Eq. (18.) The result for W' is of the form

$$\begin{aligned} 2W' = & W_0(\boldsymbol{\xi}, \boldsymbol{\xi}^*) + W_1(\boldsymbol{\xi}, \boldsymbol{\xi}^*) + p^* p \int \frac{1}{2} Q_m \\ & \times \boldsymbol{\xi}^* \cdot \boldsymbol{\xi} d^3\mathbf{x} + W_1'(\boldsymbol{\xi}, \boldsymbol{\xi}^*; E_{\parallel}, E_{\parallel}^*; p, p^*) \end{aligned} \quad (29.1)$$

The first two terms are just the zero frequency limit, i.e., Eq. (27). The third term is a positive definite kinetic energy. The fourth term is a complicated correction to the potential energy. The major part of the proof consists in showing that $W_1' > 0$ for any $\boldsymbol{\xi}$, unless $p = 0$ and $\delta\mathbf{E}_{\parallel} = 0$, in which case $W_1' = 0$, by definition. After introducing the variables Φ and α as before, W_1' is expressed in the form

$$\begin{aligned} W_1' = & \int d\Phi \int dv \int d\alpha \int_{l_1}^{l_2} dl \{ X^*(\alpha, l, v) \\ & \times \int_{t=-\infty}^0 Y(\alpha, t, v) e^{pt} dt + (\text{complex conjugate}) \}. \end{aligned} \quad (29.2)$$

If we fix our attention on a particular v and α , there is a well-defined relation between t and l . We may then write X and Y as Fourier series in t because the orbits are periodic. The first two integrations can be performed explicitly, and the result displayed in terms of the expansion coefficients for $\boldsymbol{\xi}$ and $E_{\parallel}$. In this form it is straightforward to show that the contribution from all elements $dv \, d\alpha$ is positive.

We may therefore conclude that if $W_0 + W_1 > 0$ for all $\boldsymbol{\xi}$, then so is W' , and no instability is possible.

Thus the condition $W_{\min} > 0$, where W is given by Eq. (27), is a necessary and sufficient condition for stability.

RELATION TO APPROXIMATE METHODS

In this section we show that Eq. (27) for W , which is correct to order λ , is bounded below by the MH approximation and above by the CGL approximation. Physical reasons for this are given.

Lower Bound of W

According to the Schwarz inequality,

$$\int (J/K)^2 K d\alpha \geq [\int J d\alpha]^2 / \int K d\alpha.$$

Therefore, a lower bound for W_1 is

$$W_L = \frac{15}{4} \int P d\Phi \frac{[\int_{\alpha=0}^{\alpha=1/B_m} J(\Phi, \alpha) d\alpha]^2}{\int_{\alpha=0}^{\alpha=1/B_m} K(\Phi, \alpha) d\alpha}. \quad (30)$$

It is now possible to perform the α integrations. The result is discussed below. For the magneto-hydrodynamic approximation,⁴

$$W_{MH} = W_0 + W_2,$$

where W_0 is formally the same as Eq. (27.1) and

$$W_2 = \int P (\nabla \cdot \xi)^2 d^3 \mathbf{x} 2\gamma. \quad (31)$$

In this case ξ is defined by $\delta \mathbf{V} = p \xi$, where $\delta \mathbf{V}$ is the macroscopic velocity. Therefore, ξ has three components, whereas the ξ in W has only two components. However, W_0 is stationary with respect to variations in $\xi_{\parallel}$, and we may choose $\xi_{\parallel}$ so that W_2 is also minimized. The condition for this is

$$\nabla \cdot \xi = \int_{l_1}^{l_2} \nabla \cdot \xi_{\perp} \frac{dl}{B(l)} \bigg/ \int_{l_1}^{l_2} \frac{dl}{B(l)} \quad (32)$$

which is constant along a flux tube. By substituting this value for $\nabla \cdot \xi$ in Eq. (31), it can be shown that $W_2 = (3\gamma/5) W_L$. For a monatomic gas, $\gamma = 5/3$, so that

$$W \geq W_{MH}. \quad (33)$$

Upper Bound of W

This time the Schwarz inequality is applied to Eq. (27.3) in the form

$$\begin{aligned} & [J(\Phi, \alpha)]^2 / K(\Phi, \alpha) \\ & \leq \int_{l_1(\alpha)}^{l_2(\alpha)} \{ [1 - \frac{3}{2} \alpha B(l')] \mathbf{n} \cdot (\mathbf{n} \cdot \nabla) \xi + \frac{1}{2} \alpha B(l') \nabla \cdot \xi \}^2 \\ & \quad \times [1 - \alpha B(l')]^{-\frac{1}{2}} dl' \end{aligned} \quad (34)$$

If the right-hand side of this inequality is substituted for J^2/K in Eq. (27.3), the α integrations can be carried out and we obtain an upper bound for W_1 :

$$W_U = \frac{1}{2} \int P [3(\mathbf{n} \cdot \nabla \xi)^2 - 2(\mathbf{n} \cdot \nabla \xi) \nabla \cdot \xi + 2(\nabla \cdot \xi)^2] d^3 \mathbf{x} \quad (35)$$

When $\omega, \omega_p \gg p \gg 1/\tau$, Eq. (21) is applicable for δf . The pressure tensor has the same form as Eq. (24), with

$$\begin{aligned} \delta P_{\perp} &= P [\mathbf{n} \cdot \mathbf{n} \cdot \nabla \delta \xi - 2 \nabla \cdot \delta \xi], \\ \delta P_{\parallel} &= -P [2 \mathbf{n} \cdot \mathbf{n} \cdot \nabla \delta \xi + \nabla \cdot \delta \xi]. \end{aligned} \quad (36)$$

Using Eq. (27.2), we find that in this case the potential energy change is $W = W_0 + W_U$. (Equation (26) includes a term $\rho_m p^2 \xi^2$, which is clearly the kinetic energy; it vanishes in the marginal stability case and is omitted here.) Thus an upper bound for the marginal stability criterion is just the potential energy change for large p . The upper bound also has a direct interpretation in terms of the CGL approximation. Equations (2) and (3) are differentiated and linearized:

$$\begin{aligned} & \frac{B^2}{\rho_m 3} [p \delta P_{\parallel} + (\delta \mathbf{V} \cdot \nabla) P] + \frac{2P}{\rho_m 3} \mathbf{B} \cdot [p \delta \mathbf{B} + \delta \mathbf{V} \cdot \nabla \mathbf{B}] \\ & - \frac{3B^2 P}{\rho_m 4} [p \delta \rho_m + \delta \mathbf{V} \cdot \nabla \rho_m] = 0. \\ & \frac{1}{\rho_m B} [p \delta P_{\perp} + \delta \mathbf{V} \cdot \nabla P] - \frac{P}{\rho_m B^3} \mathbf{B} \cdot [p \delta \mathbf{B} + (\delta \mathbf{V} \cdot \nabla) \mathbf{B}] \\ & - \frac{P}{\rho_m^2 B} [p \delta \rho_m + \delta \mathbf{V} \cdot \nabla \rho_m] = 0. \end{aligned} \quad (37)$$

After substituting $\delta \mathbf{V} = p \delta \xi$, $\delta \mathbf{B} = \nabla \times \delta \xi \times \mathbf{B}$, and $p \delta \rho_m + \delta \mathbf{V} \cdot \nabla \rho_m = -\rho_m \nabla \cdot \delta \mathbf{V}$, Eq. (36) is obtained for $\delta P_{\perp}$ and $\delta P_{\parallel}$. In the CGL theory these expressions contain a $\xi_{\parallel}$ but $\delta \mathbf{E}$ and $\delta \mathbf{B}$ are unaffected by any component of $\delta \xi$ along the field lines, and the motion of particles is due entirely to these fields. Therefore, a more reasonable version of the CGL theory would appear to be obtained by considering $\xi_{\parallel}$ to be zero in Eq. (36). With this interpretation the upper bound Eq. (35), agrees with CGL.

PHYSICAL INTERPRETATION

The potential energy given by Eq. (27.3) can be given a more transparent form. To this end, we calculate the change in kinetic energy, T , which a particular particle will experience from the perturbation, by making use of the invariance of the action integral.⁶

$$\delta A = \delta \int_{l_1}^{l_2} V_{\parallel} dl = \delta \int_{l_1}^{l_2} \left[\frac{2}{m} (T - \mu B) \right]^{\frac{1}{2}} dl = 0. \quad (38)$$

The integration is carried out along the perturbed orbit rather than the unperturbed one as heretofore. The parameter μ is still a constant of the motion,⁶ but T is not.

After expressing the change in B and dl on the moving line in terms of ξ , one finds for the change in kinetic energy of a particle,

$$\delta T = 2T J(\Phi, \alpha) / K(\Phi, \alpha). \quad (39)$$

After some algebra, Eq. (27.3) becomes

$$W_1 = (5/4) \sum_{\text{particles}} (\delta T)^2 / T. \quad (40)$$

From Eqs. (30) and (31) we see that the MH approximation corresponds to

$$W_2 = (5/4) \sum_{\text{particles}} \langle \delta T \rangle^2 / T, \quad (41)$$

where the angular brackets indicate an average with respect to α over all particles of energy T on a line of force. Such an average is, of course, inherent in a fluid-type equation of state where all the particles on a line are supposed to remain in thermal equilibrium.

Similarly, the C-G-L upper limit can be written as

$$W_U = (5/4) \sum_{\text{particles}} (\delta T_U)^2 / T, \quad (42)$$

where δT_U is the energy change predicted by using, instead of Eq. (38), the equation

$$\delta(V_{\parallel} dl) = 0. \quad (43)$$

In other words, the energy change for a particle at a particular point on a field line is calculated as if the fields were behaving all along its orbit just as they are locally. This provides a physical understanding of the

meaning of the neglect of the third-moment pressure transport terms, which is the essence of the CGL theory. This approximation should agree with the high-frequency limit, where the past history is forgotten.

Thus, in the correct theory, one must average along each particle's orbit and then sum over all particles. These operations are replaced by a double average in the MH approximation and by a double sum in the CGL approximation. In view of this interpretation, the inequalities which we find are not surprising.[†]

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[†] It has been brought to the authors' attention that basically the same results as obtained in this paper have been derived independently from a completely different point of view, i.e., a thermodynamic argument by Kruskal and Oberman. This work also appears in the Proceedings of the Second Geneva Conference (P/365 this volume).

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Dynamics of a Rarefied Plasma in a Magnetic Field

By R. S. Sagdeyev, B. B. Kadomtsev, L. I. Rudakov and A. A. Vedyonov

The nature of the motion and properties of a high temperature plasma in a magnetic field is of particular interest for the problem of producing controlled thermonuclear reactions. The most general theoretical approach to such problems lies in the description of the plasma by the Boltzmann and Maxwell equations that connect the self-consistent electric and magnetic fields with the ion and electron distribution functions.

If the mean free path of the particles is appreciably larger than the characteristic dimensions of the system and if the time between collisions is greater than the appropriate characteristic time, then the collision term can be neglected in the kinetic equation or considered to be a small correction.

In the absence of collisions the plasma particles interact through long-range electromagnetic forces and the particle trajectories can be described by an equation of motion with self-consistent electric and magnetic fields. This is expressed mathematically by the fact that in the absence of collisions the solution to the kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + (e/M)(\mathbf{E} + c^{-1}\mathbf{v} \times \mathbf{B}) \cdot \partial f / \partial \mathbf{v} = 0 \quad (1)$$

can be represented by an arbitrary function of the first integrals of the characteristic equations for the motion of the individual particles

$$dt = d\mathbf{r}/\mathbf{v} = d\mathbf{v}/(e/M)(\mathbf{E} + c^{-1}\mathbf{v} \times \mathbf{B}).$$

The exact equations for the motion of a plasma in an electromagnetic field can only be solved in certain simple cases especially because the fields are influenced by the collective motion of all the particles. For a certain class of problems it is possible to work out a procedure for decreasing the number of variables and thus simplify the characteristic equations.

The physical system can be characterized by "fast" and "slow" variables. If we are investigating changes in the system over a long period of time, we can eliminate the fast variables and describe the state of the system in terms of the slow variables. The method

of eliminating the fast variables from the kinetic equation was developed by Beliaev.*

We shall eliminate the "fast" variables for the particular case when the electric and magnetic fields change slowly in time and space; i.e., the conditions $R_B/L \ll 1$ and $1/\omega T \ll 1$ are satisfied, where R_B is the Larmor radius, ω is the Larmor particle frequency and L, T are the characteristic size and characteristic time of interest. The net particle motion consists of the "fast" Larmor rotation and the "slow" drift of the guiding centers.

The procedure for excluding the Larmor phase; i.e., the "fast" variable, from the equation of motion by means of a transformation to new "drift" variables $(v_\perp, v_\parallel, \mathbf{R}, \alpha)$ was worked out by Bogoliubov and Zubarev.¹ Omitting the algebra, we shall present the final result, limiting ourselves to the zero and first order approximations in R_B/L and $1/\omega T$:

$$dv_\perp/dt = (v_\perp/2\omega)d\omega/dt \quad (2)$$

$$dv_\parallel/dt = -\epsilon_0 \cdot [-(e/M)\mathbf{E} + (v_\perp^2/2\omega)\nabla\omega + d\mathbf{w}/dt + v_\parallel d\epsilon_0/dt] \quad (3)$$

$$d\mathbf{R}/dt = \epsilon_0 v_\parallel + (1/\omega)\epsilon_0 \cdot [-(e/M)\mathbf{E} + (v_\perp^2/2\omega)\nabla\omega + d\mathbf{w}/dt + v_\parallel d\epsilon_0/dt], \quad (4)$$

where

$$d\alpha/dt \equiv -\omega, \quad \epsilon_0 \equiv \mathbf{B}/B, \quad \mathbf{w} \equiv c\mathbf{E} \times \mathbf{B}/B^2, \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_\parallel(\epsilon_0 \cdot \nabla) + (\mathbf{w} \cdot \nabla). \quad (5)$$

Since the characteristic equations are known, we can easily write the kinetic equation for the distribution function $f(v_\parallel, v_\perp, \mathbf{R}, \alpha, t)$ in terms of the drift variables:

$$\frac{\partial f}{\partial t} + (d\mathbf{R}/dt) \cdot \nabla f + (dv_\parallel/dt) \partial f / \partial v_\parallel + (dv_\perp/dt) \partial f / \partial v_\perp + (d\alpha/dt) \partial f / \partial \alpha = 0. \quad (6)$$

By expanding the resulting kinetic equation in powers of ω^{-1} , or more precisely, in powers of R_B/L and $1/\omega T$; we see that f_0 in $f = f_0 + f_1 + \dots$, does not depend on α and one obtains an equation which contains only the "slow" variables:

* Ed. note: S. T. Beliaev, *Kinetic Equation for a Rarefied Gas in Strong Fields*; AN SSSR, *Plasma Physics and Problems of Controlled Thermonuclear Reactions*, 3, 50, Moscow (1958).

$$\begin{aligned} & \partial f / \partial t + v_{\parallel}(\mathbf{e}_0 \cdot \nabla f) + (\mathbf{w} \cdot \nabla f) \\ & + \mathbf{e}_0 \cdot [(e/M)\mathbf{E} - (v_{\perp}^2/2\omega)\nabla\omega - d\mathbf{w}/dt]\partial f / \partial v_{\parallel} \\ & + (v_{\perp}^2/2\omega)(d\omega/dt)\partial f / \partial v_{\perp} = 0. \end{aligned} \quad (7)$$

If rare collisions are taken into account, a small term on the right-hand side of the kinetic equation appears.

The electromagnetic fields and the distribution functions are connected through Maxwell's equations. It is easy to show that to first order in ω^{-1} the velocity of the centre of gravity of an elementary volume of plasma, $n\mathbf{V} = \int f(\mathbf{v}, \mathbf{r}, t)\mathbf{v}d\mathbf{v}$, does not coincide with the mean drift velocity, $n\bar{\mathbf{v}} = \int (d\mathbf{R}/dt)f_0 v_{\perp} dv_{\perp} dv_{\parallel} d\alpha$, and is given by

$$n\mathbf{V} = \int (-\nabla \times (v_{\perp}^2/2\omega) \mathbf{e}_0 + d\mathbf{R}/dt)f_0 v_{\perp} dv_{\perp} dv_{\parallel} d\alpha.$$

With this expression, Maxwell's equations can be written

$$\nabla \cdot \mathbf{E} = 4\pi e \int (f_0^i - f_0^e) v_{\perp} dv_{\perp} dv_{\parallel} d\alpha, \quad (8)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (9)$$

$$\nabla \times \mathbf{E} = -(1/c)\partial \mathbf{B} / \partial t, \quad (10)$$

$$\begin{aligned} \nabla \times \mathbf{B} = & (4\pi e/c) \int [-\nabla \times (v_{\perp}^2/2\omega) \mathbf{e}_0 \\ & + d\mathbf{R}/dt](f_0^i - f_0^e) v_{\perp} dv_{\perp} dv_{\parallel} d\alpha. \end{aligned} \quad (11)$$

It may be seen from this equation that the plasma possesses diamagnetic properties.

The set of Eqs. (6)–(11) completely describes the plasma when collisions are neglected. However, it should be stressed that this system of equations follow from the first non-vanishing approximation to the exact system of plasma equations when this exact system is expanded in terms of a small parameter, ω^{-1} . This is the so-called drift approximation.

EQUATIONS FOR THE MACROSCOPIC MOTION OF THE PLASMA

Strictly speaking, conventional hydrodynamic considerations are valid when the plasma has a high density, low temperature and a corresponding collision mean free path which is much less than the characteristic dimensions of the plasma volume. However, Chew, Goldberger and Low² and also Watson and Brueckner³ have shown that it is possible to obtain a closed system of equations for the first few moments of the distribution function. These equations are formally analogous to the usual system of magnetohydrodynamic equations with a non-isotropic pressure tensor when collisions are neglected. These authors obtained the moment equations by expanding the kinetic equation in powers of $\omega^{-1} = eH/Mc$ or in powers of the mass to charge ratio, M/e .

The similarity in the results obtained for these two apparently opposite cases is explained by the fact that the magnetic field deflects the particles and symmetrizes their velocity distribution in the plane perpendicular to the magnetic field. In this sense, the effect of the magnetic field is analogous to the effect of collision.

The gas-dynamic equations for the ion and electron gases can also be deduced from the kinetic equation in the drift approximation,⁵ as will be shown below.

The moments of the distribution function are defined by the following integrals

$$n = B \int f d^3v$$

$$nu = B \int v_{\parallel} f d^3v$$

$$\bar{\mu} = MB \int (v_{\perp}^2/2B) f d^3v$$

$$p = MB \int (v_{\parallel} - u)^2 f d^3v$$

$$Q = MB \int (v_{\parallel} - u)^3 f d^3v,$$

where $d^3v \equiv d\mu dv_{\parallel} d\alpha$ and $\mu = v_{\perp}^2/2B$. Note that Bd^3v is the true volume element in velocity space.

If we assume that there is no heat flux Q along the magnetic field then we can multiply the equation for the drift motion (4) and the kinetic equation (5) by various powers of $v_{\parallel}$ and $v_{\perp}$ and integrate over the velocities. This procedure leads to a closed system of equations which describe the motion of the ionic component of the plasma

$$\partial n / \partial t + \nabla \cdot n(\mathbf{e}_0 u + \mathbf{u}) = 0 \quad (12)$$

$$\begin{aligned} Mdu/dt = & -(\omega/n)(\mathbf{e}_0 \cdot \nabla(p/\omega)) \\ & + \mathbf{e}_0 \cdot [e\mathbf{E} - (\bar{\mu}/n)\nabla B - Md\mathbf{w}/dt] \end{aligned} \quad (13)$$

$$d(pB^2/n^3)/dt = 0 \quad (14)$$

$$d(\bar{\mu}/n)/dt = 0 \quad (15)$$

$$\begin{aligned} d\mathbf{R}/dt = & u\mathbf{e}_0 + (1/\omega)\mathbf{e}_0 \cdot [- (e/M)\mathbf{E} + u d\mathbf{e}_0/dt \\ & + d\mathbf{w}/dt + (p/nM)(\mathbf{e}_0 \cdot \nabla)\mathbf{e}_0 + (\bar{\mu}\nabla B)/n]. \end{aligned} \quad (16)$$

The equations for the electron component differ from these equations only in sign of the charge and in the value of the mass. It should also be noted that the equations of motion (13) and (16) can be written in terms of the an isotropic pressure tensor,

$$p_{ij} = \begin{pmatrix} \bar{\mu}B & 0 & 0 \\ 0 & \mu B & 0 \\ 0 & 0 & p \end{pmatrix}.$$

The relations which govern the pressure variation are given by Eqs. (14) and (15), which correspond to the adiabatic law with $\gamma = 3$ and 2 for the longitudinal and transverse motions, respectively. The transverse adiabatic invariant is the ion (or electron) magnetic moment. Maxwell's equations (8)–(11) can also be rewritten by introducing the moments of the distribution function

$$\nabla \cdot \mathbf{E} = 4\pi e(n_i - n_e) \quad (8a)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (9a)$$

$$\nabla \times \mathbf{E} = -(1/c)\partial \mathbf{B} / \partial t \quad (10a)$$

$$\begin{aligned} \nabla \times \mathbf{B} = & (4\pi e/c) \{ -c \nabla \times (\bar{\mu}_i + \bar{\mu}_e)\mathbf{e}_0/e \\ & + n_i \bar{\mathbf{v}}^i - n_e \bar{\mathbf{v}}^e \}. \end{aligned} \quad (11a)$$

In order to describe the motion along the lines of force of the magnetic field by means of these equations, a special condition must be satisfied; namely, there must be no heat flux along the magnetic field

lines of force. If this condition is violated then the equation $d(pB^2/n^3)/dt = 0$ is not valid, and the whole system of equations of motion (12)–(16) are inapplicable. In these cases a rigorous treatment has to be carried out on the basis of the more complicated system of Eqs. (7)–(11).

HYDRODYNAMICS OF A LOW PRESSURE PLASMA

The problem of determining the equilibrium state and stability of a magnetically confined plasma is perhaps one of the most interesting and important problems in plasma physics. If the magnetic field is very strong this problem can be attacked by using the magnetohydrodynamic equations with a non-isotropic pressure tensor. However, if the plasma is confined for a time greater than the collision time then the longitudinal and transverse pressures will be equalized. The pressure can then be regarded as isotropic in calculating the equilibrium state of the plasma. The anisotropy of the pressure should be taken into consideration only while investigating the stability for increments of time shorter than the collision time. However, it can be shown⁴ that the anisotropy of the perturbation pressure for an isotropic equilibrium pressure can only improve the conditions for stability. Therefore, when considering the stability of an isotropic plasma, one can use the ordinary hydrodynamic equations: the stability conditions obtained in this way are somewhat over-stringent.

We shall now consider the hydrodynamic description of a plasma when its pressure is appreciably lower than the magnetic pressure, i.e., $\beta \equiv 8\pi p/B^2 \ll 1$. In this case the magnetohydrodynamic equations can be simplified. Indeed, in such cases, the magnetohydrodynamic wave velocity $c_A = (B^2/4\pi\rho)^{1/2}$ is much greater than the sound velocity $c_s = (8p/\rho)^{1/2}$. Therefore if we do not specifically consider the hydrodynamic waves we may take c_A to be infinite. This means that any perturbation of the plasma leading to a transverse displacement of the line of force is propagated instantaneously along the line of force and all the plasma along the line starts to move at once. The equation for this motion may be obtained by expanding the hydrodynamic equations in a power series in β . Let us assume that $p = p_0 + \epsilon p_1 + \dots$, $\rho = \rho_0 + \epsilon \rho_1 + \dots$, $\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1 + \dots$, $B = \epsilon^{-1/2} (B_0 + \epsilon B_1 + \dots)$, substitute these expansions into the magnetohydrodynamic equations, compare coefficients of the same powers of ϵ and then put $\epsilon = 1$. The following system of equations is obtained by this procedure:

$$[\nabla \times \mathbf{B}_0] \times \mathbf{B}_0 = 0, \quad \nabla \cdot \mathbf{B}_0 = 0 \quad (17)$$

$$\partial \mathbf{B}_0 / \partial t = \nabla \times [\mathbf{v}_0 \times \mathbf{B}_0] \quad (18)$$

$$\begin{aligned} \rho_0 \partial \mathbf{v}_0 / \partial t + \rho_0 (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0 + \nabla p_0 \\ = (1/4\pi) ([\nabla \times \mathbf{B}_1] \times \mathbf{B}_0 + [\nabla \times \mathbf{B}_0] \times \mathbf{B}_1) \end{aligned} \quad (19)$$

$$\partial \rho_0 / \partial t + \nabla \cdot \rho_0 \mathbf{v}_0 = 0 \quad (20)$$

$$\partial (p_0 / \rho_0^\gamma) / \partial t + (\mathbf{v}_0 \cdot \nabla) (p_0 / \rho_0^\gamma) = 0. \quad (21)$$

By retaining only the zero-order terms we eliminate the other equations and can drop the subscript zero. If we do not consider force-free fields we obtain from Eq. (17)

$$\nabla \times \mathbf{B}_0 = 0$$

$$\nabla \cdot \mathbf{B}_0 = 0; \text{ i.e., } \mathbf{B} = \sum_i J_i \nabla \phi_i,$$

where the J_i are the external currents and the ϕ_i are the corresponding scalar potentials which, generally speaking, are not single valued.

For simplicity we shall suppose that the currents do not change with time and that the conductors are immobile. Then $\partial \mathbf{B} / \partial t = 0$ and from Eq. (18) we obtain

$$\mathbf{v} = \epsilon_0 v_{||} + \epsilon_0 \nabla \phi / B,$$

where $v_{||}$ is arbitrary an function of r , t and ϕ is a function of r and t which satisfies the condition $\epsilon_0 \cdot \nabla \phi = 0$. Inserting this into Eq. (19) we must consider Eq. (19) to be an equation defining B_1 . Furthermore, as is well known, the condition for the existence of a solution must be fulfilled: the left-hand side of Eq. (19) must be orthogonal to the solutions of the homogeneous equation conjugate to the equation $L\mathbf{B}_1 \equiv [\nabla \times \mathbf{B}_1] \times \mathbf{B}_0 = 0$. This condition leads to

$$\rho \partial v_{||} / \partial t + \rho \epsilon_0 (\mathbf{v} \cdot \nabla) \mathbf{v} + \epsilon_0 \cdot \nabla p = 0 \quad (22)$$

$$\int \nabla \cdot [(\rho/B^2) \nabla (\partial \phi / \partial t) - (\rho/B) \epsilon_0 \times (\mathbf{v} \cdot \nabla) \mathbf{v} - (1/B) \epsilon_0 \times \nabla p] dl / B, \quad (23)$$

where the integral is taken along any line of force. Together with Eqs. (20) and (21), these equations describe the behaviour of a low pressure plasma in the zeroth approximation. It should be noted that this approximation appears to be inadequate if the magnetic lines of force are not closed but cover the whole surface. Indeed, in this case, ϕ must be constant on the surface, whereas $\mathbf{v}$ retains only two degrees of freedom.

If we put $\mathbf{v} = 0$ in Eqs. (22) and (23) then conditions for equilibrium are obtained which can be transformed into the form

$$\epsilon_0 \cdot \nabla p = 0, \quad \nabla p \times \nabla U = 0; \quad U \equiv - \int dl / B, \quad (24)$$

i.e., the pressure p must be constant on the surface $U = \text{constant}$.

Equations (2) through (23) can be used for investigating stability. We shall put $p = p_0 + p'$, $\rho = \rho_0 + \rho'$, $\mathbf{v} = \partial \boldsymbol{\eta} / \partial t$, $\boldsymbol{\eta} = \boldsymbol{\eta}_{||} + \boldsymbol{\eta}_{\perp} = \eta_{||} \epsilon_0 + [\epsilon_0 \times \nabla \phi] / B$, where p_0 , ρ_0 are the equilibrium pressure and density and p' , ρ' , $\boldsymbol{\eta}$ are small quantities. Substituting this into Eqs. (20)–(23) and neglecting higher order quantities, we obtain linearized equations which describe small deviations from equilibrium. It is possible to show that these equations can be obtained from the variational principle $\delta \int L dt = 0$, where $L = T - V$, $T = \frac{1}{2} \int \rho_0 (\partial \boldsymbol{\eta} / \partial t)^2 d\mathbf{r}$ is the kinetic energy and V is the potential energy of the plasma. If we introduce curvilinear coordinates (ξ_1, ξ_2, ξ_3) so that the lines ξ_3 coincide with the (closed) lines of magnetic force, then V can be written as follows

$$V = -\frac{1}{2} \int B (g/g_{33})^{1/2} (\boldsymbol{\eta}_\perp \cdot \nabla p_0) (\boldsymbol{\eta}_\perp \cdot \nabla U) d\xi_1 d\xi_2 \\ + \frac{1}{2} \int \gamma p_0 (\nabla \cdot \boldsymbol{\eta})^2 d\mathbf{r}, \quad (25)$$

where g_{33} is a component of the metric tensor and $g = \det g_{ik}$. For plasma stability it is necessary and sufficient that the potential energy V be positive definite; i.e., its minimum value be positive. Minimizing Eq. (25) with respect to $\boldsymbol{\eta}_\perp$ we obtain

$$V_{\min} = -\frac{1}{2} \int B (g/g_{33})^{1/2} \{ (\boldsymbol{\eta}_\perp \cdot \nabla p_0) (\boldsymbol{\eta}_\perp \cdot \nabla U) \\ + (\gamma p_0/U) (\boldsymbol{\eta}_\perp \cdot \nabla U) \} d\xi_1 d\xi_2 \quad (26)$$

which shows that a necessary and sufficient condition for the stability of a low pressure plasma is†

$$\nabla p_0 \cdot \nabla U + (\gamma p_0/U) (\nabla U)^2 < 0. \quad (27)$$

This condition is a hydrodynamic analogue to the stability criterion obtained from the kinetic equation by Tserkovnikov⁵ for a plasma in a magnetic field with a cylindrical symmetry. It is also similar to the condition for the convective stability of a nonuniformly heated gas in a gravitational field, where the condition that the drop in the gas temperature with height should be within a certain limit is analogous to the condition that the plasma pressure must not rise too fast with increasing U .

If the plasma has a sharp boundary the first term in Eq. (27) is much larger than the second term and, consequently, the condition for stability is expressed

by $\nabla p_0 \cdot \nabla U < 0$. This condition was obtained earlier by Rosenbluth and Longmire⁶ from an energy principle.

INSTABILITY OF A PLASMA IN A MAGNETIC FIELD WITH AN ANISOTROPIC ION VELOCITY DISTRIBUTION

If the equilibrium pressure is nonisotropic another kind of instability appears in the plasma. In order to investigate this type of instability in its pure form we shall consider a homogeneous plasma that is stable when the pressure is isotropic. This case can again be analyzed with the nonisotropic magnetohydrodynamic equations, but a more exact analysis would involve the use of the kinetic equations. Both calculations lead to qualitatively consistent criteria for stability but the quantitative results are somewhat different. The situation here is quite analogous to the Langmuir plasma oscillation problem.

Let us first make the assumptions that we can neglect the particle collisions, that the wavelength of the perturbation is much larger than the mean ion Larmor radius and that the interval of time is much less than the Larmor frequency. Under these conditions, the drift approximation (Eqs. (4)–(11)) can be used.

The unperturbed distribution function is chosen to be of the form

$$f_0(v_\perp^2, v_\parallel^2) = [n_e M^{3/2} / 2T_\perp \pi^{3/2} (2T_\parallel)^{1/2}] \times \exp - \{ (M/2)(v_\parallel^2/T_\parallel + v_\perp^2/T_\perp) \}.$$

On linearizing the initial equations for a small perturbation and performing a Fourier coordinate transformation and a Laplace time transformation, it is easy to obtain the characteristic equation for $p = p(k_y, k_z)$:

$$\frac{p}{k_z} = \left(\frac{T_\parallel}{M} \right)^{1/2} \left[\frac{\beta_\perp [(T_\perp/T_\parallel) - 1] - (1 + \beta_\perp - \beta_\parallel) (k_z/k_y)^2 - 1}{(\pi/2)^{1/2} \beta_\perp T_\perp/T_\parallel} \right] \quad (28)$$

where the z axis is in the direction of the unperturbed field B_0 and $\beta_\perp, \beta_\parallel$ are defined by

$$\beta_\parallel \equiv 8\pi n_0 T_\parallel / B_0^2 \\ \beta_\perp \equiv 8\pi n_0 T_\perp / B_0^2.$$

The perturbations increase exponentially with time if

$$\beta_\perp [(T_\perp/T_\parallel) - 1] > (1 + \beta_\perp - \beta_\parallel) (k_z/k_y)^2 + 1. \quad (29)$$

In the limiting case $B_0 \rightarrow 0$, the instability condition is $T_\perp > T_\parallel$ and for $k_y = 0$ the characteristic equation is

$$(p/k_z)^2 = (B_0/4\pi n_0 M) (1 + \beta_\perp - \beta_\parallel). \quad (30)$$

This corresponds to the Alfvén magnetohydrodynamic branch. When $\beta_\parallel > 1 + \beta_\perp$, the plasma is unstable and, when $B_0 \rightarrow 0$, this instability condition reduces to $T_\parallel > T_\perp$.

We shall now discuss the physical interpretation of the instabilities which arise from perturbations of a plasma with a nonisotropic temperature. Consider a density perturbation in an otherwise homogeneous plasma. At points where the density is high the magnetic field is decreased because of the diamagnetism of the anisotropic plasma. There also appears a gradient in the magnetic field and a curvature of the magnetic field lines as a result of the perturbation. If the curvature is small ($k_z \ll k_y$) and $T_\perp > T_\parallel$, then the main force acting on the perturbed plasma will be $-\mu \nabla \cdot \mathbf{B}$. This force tends to increase the amplitude of the perturbation. If $T_\perp < T_\parallel$, then the force $-\mu \nabla \cdot \mathbf{B}$ is compensated by the longitudinal pressure and the amplitude of the density perturbation does not continue to grow.

If the curvature of the perturbed force lines is large so that $k_z \gg k_y$ (k_y is negligible), then for $T_\parallel > T_\perp$ there are large uncompensated centrifugal forces $-\rho \mathbf{R}/R^2$, where $\mathbf{R}$ is the radius of curvature of the perturbed line of force. It is not difficult to

† For a magnetic field with axial symmetry this condition has also been obtained by a slightly different method.⁴

see that these forces tend to increase the amplitude of the initial perturbation.

All the calculations described heretofore are based on linearized equations and, naturally, they give no information about the limiting amplitude of the undamped oscillations of an anisotropic plasma.

We have carried out the analysis to second order approximation for a number of simple limiting cases and have shown that the appearance of the instability described above leads to a transfer of kinetic energy from the transverse to the longitudinal motion when $T_{\perp} > T_{\parallel}$, and in opposite direction if $T_{\perp} < T_{\parallel}$. Therefore, it is reasonable to assume that the instability tends to develop until the longitudinal and transverse energies are equalized; i.e., until the transverse and longitudinal temperatures are approximately equal.

STABILITY OF A PINCHED CYLINDRICAL PLASMA USING THE KINETIC EQUATION

In this section the problem of the stability of a high-temperature, pinched, cylindrical plasma is treated. We shall assume that all the conditions for the validity of the drift approximation are satisfied. For simplicity we shall neglect the electron temperature in comparison to the ion temperature. With this assumption, we can use the starting equations of the previous section.

We shall consider that in the equilibrium state, the plasma cylinder is uniform along the z axis and along the azimuth ϕ . Outside the cylinder $r = a$ the non-zero components of the field are $H_{\phi e}^{(0)}(r) = 2I/rc$, $H_{ze}^{(0)} = \text{constant}$. Inside the cylinder, we have $B_{\phi i}^{(0)} = 0$ and $B_{zi}^{(0)} = B_0 = \text{constant}$. The indices i, e relate to the fields internal and external to the cylinder, respectively. The ion distribution function f_0 is defined at the beginning of the previous section. We neglect the electric field E_z , assuming that it is eliminated by the motion of the electrons along the z axis. This is true when the condition $\partial/\partial t \ll \omega_{pe}$ is satisfied, where ω_{pe} is the Langmuir electron frequency.

On the surface of the plasma cylinder the following condition should be satisfied:

$$8\pi p_{\perp} = H_{\phi e}^2 + H_{ze}^2 - (B_z)_i^2,$$

where

$$p_{\perp} = \int (Mv_{\perp}^2/2)(f_0 + f_1) d\mathbf{v}.$$

The motion of the boundary obeys the drift equation $\mathbf{v}_{dr} = c\mathbf{E} \times \mathbf{B}/B^2$. The perturbed quantities are taken to be of the form $F(r, \phi, z, t) \sim F(r)\exp i(kz + m\phi + \omega t)$. One then obtains the following dispersion relation in the usual way:

$$\frac{I_m'(\alpha)}{\alpha I_m(\alpha)} \left(1 - \frac{(m + kh_e)^2}{kK_{m-1}(k)/K_m(k) + m} \right) = h_i^2(1 + \beta_{\perp} + (k\pi/B_0^2)) \int \frac{v_{\perp}^4 d^3v}{\omega_0 + kv_{\parallel}} \frac{\partial f_0}{\partial v_{\parallel}}, \quad (31)$$

where I_m and K_m are Bessel functions of an imaginary argument. The parameter α is determined by the relation

$$\alpha^2 = \frac{k^2(1 + \beta_{\perp} - \beta_{\parallel}) - (\omega/c_A)^2}{1 + \beta_{\perp} + (\pi k/B_0^2) \int v_{\perp}^4 (\omega + kv_{\parallel})^{-1} \partial f_0 / \partial v_{\parallel} d^3v}, \quad (32)$$

where

$$c_A = (B_0^2/4\pi n_0 M)^{1/2},$$

$$\text{and } h_i = B_0/H_{\phi e}^{(0)}(a),$$

$$h_e = H_{ze}^{(0)}/H_{\phi e}^{(0)}(a).$$

The form of this equation has much in common with the dispersion equation obtained in the magneto-hydrodynamic approximation by Shafranov.⁷

We shall now consider a number of particular cases.

1. $T_{\perp} = T_{\parallel}$: In this case the instability criterion coincides precisely with the criterion obtained by Shafranov for $\gamma = 2$. However, the dependence of the perturbed quantities on the wave vector will differ from the dependence obtained from the hydrodynamic approximation.

2. $I = 0$, $H_{\phi e}^{(0)} = 0$, $T_{\perp} \neq T_{\parallel}$: The magneto-hydrodynamic treatment did not show any instability under these conditions. In our treatment, however, it is possible to have an instability in a plasma cylinder confined by a longitudinal magnetic field when there is no axial current. The instability criterion can be written explicitly when the perturbations along the axis are greater than the radius of the magnetically confined plasma. The plasma column will be unstable if

$$H_{ze}^2 + B_{zi}^2 + 4\pi n_0(T_{\perp} - T_{\parallel}) < 0. \quad (33)$$

This condition can be fulfilled only if $T_{\perp} < T_{\parallel}$.

In the limiting case of short wavelength perturbations, an instability may occur with $T_{\perp} > T_{\parallel}$. The criterion for the existence of such an instability is given by

$$[\alpha(\omega = 0)]^2 < 0. \quad (34)$$

3. The vibrational branch: Because of the thermal motion there is a damping of magnetosonic waves in the cylinder analogous to Landau damping. For small k , when the damping is small, the imaginary part of ω is

$$\text{Im } \omega = \frac{8\pi n_0 T}{B_i^2} \left(\frac{H_e}{B_i} \right)^2 \times \frac{(\alpha c_A)^2 k (\pi M/2T)^{1/2} \exp - (M/2T)(\omega/k)^2}{2(m+k)(2m+1)}. \quad (35)$$

The instability associated with temperature anisotropies could be observed in experiments on so-called adiabatic heating when only the transverse temperature decreases with decreasing magnetic field and the longitudinal temperature remains constant. This situation might arise at high temperatures where collisions can be neglected.

In experiments on the fast compression of a plasma column the compression time can be less than the mean time between ion collisions. This automatically leads to a temperature anisotropy that can make the condition for stability more rigid.

NON-LINEAR ONE-DIMENSIONAL MOTION OF A RAREFIED PLASMA

We shall now investigate the motion of a rarefied plasma far from an equilibrium state. We shall restrict ourselves to the case when all the variables will depend only on one space coordinate x perpendicular to a magnetic field directed along the z axis $\mathbf{B} = (0, 0, B(x))$. The system of hydrodynamic equations (12)–(16) are valid subject to the conditions that the space and time gradients are small, i.e., the Larmor radius and period are small compared to the appropriate linear dimensions and time. However, generally speaking, these conditions for the arbitrary motion of a plasma will not be fulfilled after a certain time.

Indeed, the formal similarity between the system of Eqs (12)–(16) and the ordinary gas-dynamic equations with $\gamma = 2$ enables us in the case of a rarefied plasma in a magnetic field to immediately generalize Riemann's solution corresponding to the formation of a shock wave. Steep velocity and temperature gradients in an ordinary shock front lead to a considerable increase of the entropy due to viscosity and heat conduction. However, in a rarefied plasma the formation of discontinuities ought to take place in a different manner since collisions are of no importance.

A hypothesis concerning the possible existence of a shock wave was advanced in an earlier paper⁸ where the non-conservation of the ion magnetic moment in a field with large gradients was considered as a mechanism leading to non-adiabaticity.

The rigorous treatment of this problem is very complicated since the motion even of a single charged particle in an arbitrary nonuniform field cannot be determined without the aid of a computer.

The simplest idealization of the processes inside the wavefront would be to consider the change in the Larmor energy across a magnetic discontinuity. It appears, however, that the non-adiabatic increase of the Larmor energy of the individual ions does not imply that the random energy is also non-adiabatically increased since there is a grouping of the Larmor phases of the ions.

This grouping of phases leads to a local oscillation of the ions. In a self-consistent problem this might also give rise to oscillations. Damping can in principle take place in the absence of collisions, e.g., due to the Landau mechanism or on account of phase mixing in a nonuniform magnetic field, and the damping length will determine the width of the shock wave. With the curvature of the velocity profile increasing, the condition $T \gg \omega^{-1}$ is violated before the condition $L \gg R_B$ since $T \sim L[(B^2/4\pi\rho) + u^2]^{-1/2}$. Therefore the wavelength λ of the oscillation may be expected to be of the order $\lambda \sim [(B^2/4\pi\rho) + u^2]^{1/2} \omega_{pi}^{-1}$. For

a low pressure plasma this becomes $\lambda \sim B/[(4\pi\rho)^{1/2} \omega_{pi}] = c_A/\omega_{pi}$, where $\omega_{pi}^2 = 4\pi n e^2/M$. However, it may be shown by using the exact equations for the ion and electron motion that for vanishing plasma pressure, the assumption of plasma quasi-neutrality and electron adiabaticity is sufficient to ensure the applicability of Riemann's solution, instead of the condition $\omega T \gg 1$ (the condition $L \gg R_B$ is then fulfilled identically since $p_1 = 0$).

In ordinary gas dynamics the only type of stationary motion is a stationary shock wave. In a rarefied plasma there exists still another solution which is a travelling magnetosonic wave with a finite amplitude.

If the thermal motion is neglected the equation connecting the wave velocity with the amplitude of the magnetic field has the form $u^2 = (B_1 + B_0)^2/16\pi\rho_0$ where the field amplitude B_1 of the wave cannot exceed $3B_0$.

The correction to the velocity due to thermal motion is[‡]

$$u_1 = \frac{\alpha^2[(u_0^2 - \gamma)^{1/2} - 3u_0/4]}{2u_0^2 - \gamma - 2u_0(u_0^2 - \gamma)^{1/2}}, \quad (36)$$

where

$$u_0 = B_0/(4\pi\rho_0)^{1/2}, \quad \gamma \equiv (B_1 + B_0)B_0/2\pi\rho_0, \quad \alpha^2 \equiv 2T_0/M$$

This equation shows that the transition to small amplitude waves leads to absurd results since the denominator vanishes. This fact is probably associated with the existence of a damping in the linear approximation analogous to Landau damping.

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[‡] Ed. note. In another version of this paper the term $3u_0/4$ is written $3u_0/2$.

Stability of a Linear Pinch

By Bergen R. Suydam *

It is now well known that a trapped longitudinal magnetic field has a stabilizing influence on a linear pinch. Once such a stabilized pinch is set up, however, diffusion will lead to mixing of the initially crossed fields; the torsion of the field lines will diminish and the plasma may ultimately become unstable. It is the purpose of this paper to study continuous plasma and field distributions in order to see at what point instability might be expected.

A variational principle has been given¹ which applies very nicely to the problem at hand. Briefly, one subjects the plasma to a displacement ξ and calculates the resulting change in the total energy, δW , of the hydromagnetic system. Stability then hinges on whether or not some ξ can produce a diminution of the energy. If we define the vector $\mathbf{Q} = \nabla \times [\xi \times \mathbf{B}]$ it turns out that the change in energy is

$$\delta W = \int d^3x \{ \mathbf{Q} \cdot \mathbf{Q} - 4\pi \mathbf{J} \cdot [\mathbf{Q} \times \xi] + \gamma p (\nabla \cdot \xi)^2 + (\nabla \cdot \xi)(\xi \cdot \nabla p) \} \quad (1)$$

where γ is the specific heat ratio and p is the pressure. The integration is taken over the complete volume. If a displacement exists which makes δW negative, we have instability. In the linear pinch we are dealing with axial symmetry and the components of the magnetic field in cylindrical coordinates are $\mathbf{B} = (0, B_\theta, B_z)$. Moreover, it is assumed that B_θ , B_z and p are functions of r alone. It is then possible to analyze ξ in terms of displacements of the form

$$\xi = [\xi_r(r), \xi_\theta(r), \xi_z(r)] \exp i(kz + m\theta). \quad (2)$$

The integration with respect to θ and z can be carried out and δW can be minimized with respect to ξ_θ and ξ_z by purely algebraic means. When this computation has been carried out we find

$$\delta W = \int_{r_1}^{r_2} \left\{ \frac{(r f \xi' + g \xi)}{m^2 + (kr)^2} + (f^2 - h) \xi^2 \right\} 2r dr, \quad (3)$$

where we have set

$$\begin{aligned} f &\equiv kB_z + mB_\theta/r \\ g &\equiv kB_z - mB_\theta/r \\ h &\equiv (8\pi J_z/r)B_\theta \\ \xi &\equiv d\xi'/dr, \end{aligned}$$

and have dropped the subscript r on ξ for simplicity.

In order that δW be a minimum with respect to functions ξ , the displacement must be a solution of the Euler-Lagrange equation

$$\xi'' + P\xi' + Q\xi = 0,$$

where

$$\begin{aligned} P &\equiv 3/r + 2f'/f - 2k^2r/[m^2 + (kr)^2] \\ Q &\equiv -[(kr)^2 + (m^2 - 1)]/r^2 - 2k^2g/f[m^2 + (kr)^2] \\ &\quad - 8\pi k^2 p'/rf^2. \end{aligned} \quad (6)$$

When ξ is chosen to be a solution of Eq. (5), the integrand of Eq. (3) is a perfect differential and we have

$$\delta W = 2 \left[\frac{r^2 f \xi (r f \xi' + g \xi)}{m^2 + (kr)^2} \right]_{r_1}^{r_2}. \quad (7)$$

Equation (5) must be solved subject to certain boundary conditions which we take to be

$$\begin{aligned} \xi(0) &\text{ is finite,} \\ \xi(R) &= 0. \end{aligned} \quad (8)$$

In choosing the second boundary condition we have placed a perfectly conducting wall at $r = R$ in order to benefit from its stabilizing influence.^{2, 3, 4} However, it will turn out that the wall has no stabilizing effect on the modes we shall study.

Our previous experience with the theory of instabilities⁵ leads us to be particularly wary of "fluted" displacements which interchange magnetic field lines without bending them. The bending of field lines requires energy while interchanging them does not. The purpose of the trapped axial field was to stabilize the plasma by twisting the field lines so that any arbitrary displacement will bend some of them. Nevertheless, some displacements will bend some field lines less than others and it seems reasonable to expect that those displacements which bend the lines the least will be the most dangerous.

Now the magnetic field lines describe a set of spirals with a pitch

$$\mu \equiv B_\theta/rB_z \quad (9)$$

which, in general, varies from layer to layer. The lines of the ξ field (2), on the other hand, describe a set of spirals whose pitch is constant. If these two sets of spirals should match over a finite region of

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space then a displacement is possible which does not bend magnetic field lines. If, on the other hand, μ' is not zero anywhere, then it is nevertheless always possible to choose k such that the two spiral systems match at a particular radius. When this happens, displacements are possible which bend magnetic field lines very little in the neighborhood of this point. Accordingly, we shall assume that the worst choice of k, m is such that

$$j = kB_z + mB_\theta/r \quad (10)$$

vanishes at some point in $(0, R)$. Let the point where $j = 0$ be denoted by $r = a$.

Now we note that $r = a$ is a regular singular point of Eq. (5). The theory of such singularities tells us that the solutions to Eq. (5) can be written in the form

$$\xi = (r - a)^\nu \times \text{Power series in } (r - a),$$

where ν is a root of the indicial equation

$$\nu^2 + \nu + M^2 = 0, \quad (11)$$

with

$$M^2 \equiv \frac{8\pi p'}{rB_z^2} \left(\frac{\mu}{\mu'} \right)^2 \Big|_{r=a}.$$

Thus we have

$$\nu_{1,2} = \frac{1}{2}(1 \pm (1 - 4M^2)^{1/2}). \quad (12)$$

If the roots are real we have $4M^2 \leq 1$. Now the boundary conditions (8) determine ξ uniquely (except for a normalization factor) in $(0 - a)$ and in $(a - r)$. Therefore, on either side of $r = a$ there will be an admixture of the more singular of the two solutions; i.e.

$$\xi = (r - a)^{\nu_1} \{1 + \text{higher order terms in } (r - a)\}, \quad (13)$$

where ν_1 is given by Eq. (12) with the minus sign. For this choice of ξ the integral (3) diverges and ξ is an improper function.

We can, however, consider the following proper displacement: ξ is the (properly normalized) solution of Eq. (5) in the ranges $0 \leq r \leq a - \epsilon$ and $a + \epsilon \leq r \leq R$. ξ is constant for $a - \epsilon < r < a + \epsilon$. If ϵ is chosen so small that the first term of Eq. (13) dominates the power series, then this first term can be substituted into Eq. (1) and it follows that δW is always positive.

If the roots are complex, it is convenient to write Eq. (12) in the form

$$\nu_{1,2} = -\frac{1}{2}(1 \pm i\beta) \\ \beta \equiv (4M^2 - 1)^{1/2}.$$

In terms of β , the displacement ξ is given by

$$\xi = |r - a|^{-1/2} \cos \left\{ \frac{1}{2} \beta \log |r - a| + \phi \right\} \\ \times [1 + \text{higher order terms}], \quad (15)$$

where ϕ is a constant phase angle determined by the boundary conditions. Again $\xi(r)$ is an improper function at $r = a$. We can circumvent this difficulty by choosing a displacement given by Eq. (15) over

the range $0 \leq r \leq a - \epsilon$ and setting $\xi = \text{constant}$ for $a - \epsilon \leq r \leq a$. The range $a \leq r \leq R$ is treated in a similar fashion. The quantity ϵ is to be chosen so that the leading term in Eq. (15) dominates at $r = a - \epsilon$. When this choice of ξ is made we obtain

$$\int_0^a \left\{ \frac{(rj\xi' + g\xi)^2}{m^2 + (kr)^2} + (j^2 - h)\xi \right\} 2rdr \\ = \frac{a^3 B_z^2 \mu'^2}{2(1 + (a\mu)^2)} \left[1 - 2M^2 \right. \\ \left. + (1 - 2M^2) \cos 2\psi + \beta \sin 2\psi \right], \quad (16)$$

where ψ is defined by

$$\psi \equiv \frac{1}{2} \beta \log \epsilon + \phi \quad (17)$$

and can be made anything we please (modulo 2π) by a suitable choice of ϵ . But the bracketed expression on the righthand side of Eq. (16) oscillates between the values 1 and $(1 - 4M^2)$ as ψ varies. If $(1 - 4M^2)$ is negative, it is possible to choose ϵ so that (16) is negative. Similarly, the integral taken from a to R can be made negative and we have found that complex roots imply instability.

The result of our investigation can be stated as a theorem:

A necessary condition that the $m \neq 0$ modes of a linear pinch be stable is that

$$(r/4) (\mu'/\mu)^2 + 8\pi p'/B_z^2 \geq 0 \quad (18)$$

at every point in the plasma.

The method by which we obtained this theorem from the Euler-Lagrange equation suggests the likelihood that the above inequality might also be a sufficient condition for stability. However, there are two major difficulties which will be discussed in the following paragraphs.

The foregoing analysis, leading to our theorem, suggests the importance of extremely localized mixing at any point of instability since the unstable modes we have found are those for which the radial displacement ξ is very small except in the immediate neighborhood of $r = a$, where the B and ξ fields interlace. Therefore, it is of interest to inquire: suppose a small region is unstable, in the sense that the inequality stated in our theorem is violated, what then happens?

The answer to such a question is very difficult to give, but an estimate has been made in the following manner: A displacement ξ of the unstable type is chosen. This leads to new values for p, B_z, B_θ and ρ in the neighborhood of $r = a$. Mixing is now simulated by replacing p, ρ, B_θ, B_z by the values obtained by averaging over θ , and we ask whether the new distribution is more or less stable using the above theorem as a criteria.

The result seems to go qualitatively as follows: The mixing of a small unstable region leads to a distribution which is less unstable on the inside and more unstable on the outside. Thus, if some interior shell were unstable, it would mix until stable and

this, in turn, would upset the stability of the next shell which would proceed to mix and so on. In this fashion such an instability would propagate towards the surface. If, however, a layer near the surface is given excess stability, the outward progression of the mixing should be stopped. This excess stability of the surface layers ought to be insured if the B_z field were so programmed that μ' is made quite large in this region. The simplest programming appears to be one which would reverse B_z in the vacuum after the plasma has pinched.

The inequality appearing in our theorem is a necessary condition for stability. It might also be argued that this is a sufficient condition since it was obtained from the Euler-Lagrange equation of (3). There are, however, two difficulties to be overcome before this can be asserted. The first is that we did not truly minimize δW with respect to k , but rather

used a heuristic *principle of minimum bending*. However, the principle seems physically reasonable and the objection does not seem to be very serious. The second difficulty is quite deep and arises because the Euler-Lagrange equation is a necessary condition, but in no way guarantees a minimum.

Some progress has been made in clarifying the second difficulty. Note that the condition that $\xi(0)$ be finite determines a solution of the Euler-Lagrange equation, and that $\xi(R) = 0$ determines another solution. It has been possible to prove the following theorem:

The necessary and sufficient condition for stability is that neither of the above-mentioned solutions to the Euler-Lagrange equation has a zero in the open interval $(0, R)$.

This makes further progress possible by application of the Sturmian theory to Eq. (5).

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The Stability of a Constricted Gas Discharge

By R. J. Tayler *

One of the most obvious methods of attempting to confine a hot plasma is to use the pinch effect of the self magnetic field of an axial current. This is illustrated in Fig. 1. A cylindrical system is most easily studied theoretically, although it is probably necessary to close the system into a torus, or similar shape, to avoid heat and particle losses at electrodes. Although some consideration has been given to toroidal geometry, most complete results have been obtained for infinite cylindrical systems and only these will be discussed here. This paper is to be regarded as a progress report on an unfinished problem and an indication is given of aspects of the problem which, it is hoped, will be studied soon.

The original idea of using the pinch effect was that of constricting a gas discharge to an extremely small radius when the only heat losses from it would be radiation losses, principally bremsstrahlung.¹ It was hoped that power would eventually be achieved by passing a high enough current to reach thermonuclear temperatures, and by containing the system for a long enough time.² At an early stage, it became clear that this simple idea must be modified because of the presence of instabilities (see Fig. 2). These were observed³ and predicted^{4,5} at about the same time. When attempts were made to understand the instabilities and to predict methods of stabilisation, it was found that a high degree of constriction did not appear to be possible if the discharge were to be stable.^{6,7} Thus, study is now made of rather fat discharges in a large tube.⁸

It should be noted that a linear stability theory, such as was considered in Refs. 6 and 7, does not immediately prove that a highly constricted discharge cannot be used. Although such a discharge is unstable, non-linear stabilising forces may prevent it from approaching the walls of its containing vessel. At the same time, the mass motions of the discharge may lead to additional heating, which would be an advantage at high temperatures when Joule heating becomes relatively inefficient.⁹ The non-linear problem has not received adequate investigation to date. What evidence there is suggests that the non-linear forces are not very effective unless the plasma is fairly close to stability.^{10,11}

Observations are, unfortunately, very easy only for completely unstable discharges. Thus, the early work at Harwell was done in Pyrex tubes and the entire current channel could be photographed.³ Such a photograph is shown in Fig. 3. Generally the most developed instability appeared to be a helix of pitch 45°: this result is remarkably independent of the particular apparatus used. A very detailed analysis of instabilities has been made by Allen, using a racetrack torus.¹² When stabilisation is seriously considered, the tube has a conducting wall and visual observations can be made only through slits. In this case, the presence of the slits may effect stability locally.¹³ Probe and microwave measurements are difficult to interpret, unless their time resolution is very good, and visual measurements may be ambiguous because the most frequent occurrence is for the light to be given out by impurities, which may not occupy the same volume as the main current channel.

The full problem of discharge stability, even for a cylindrical system, is extremely complicated. In particular, a discharge will not be in simple hydrostatic equilibrium and, even in the absence of instabilities, the finite electrical conductivity will cause gradual changes in the magnetic field configurations. In most of the work described in this paper, it is assumed that steady motions and field penetration occur so slowly that, in the stability problem, the discharge can be supposed ideally conducting and at rest. Two theoretical models are considered here. The first approach is extremely naïve: plasma dynamics is completely neglected and only the forces exerted by the electromagnetic field are considered. The second approach is to use Maxwell's equations combined with a simple one-fluid hydrodynamics—classical hydromagnetics. A more sophisticated procedure would be to start from Boltzmann's equation for the several types of particle present and to try to derive a hydrodynamic system. As an example of this procedure there is the work of Watson and co-workers¹⁴⁻¹⁶ and of Chew, Goldberger and Low.¹⁷ In the simplest problems that have been considered, essentially similar results have been obtained from the Kinetic Theory, Hydromagnetic and even the Magnetostatic approach.

The remainder of the paper is arranged as follows. In the next section the magnetostatic or extensible

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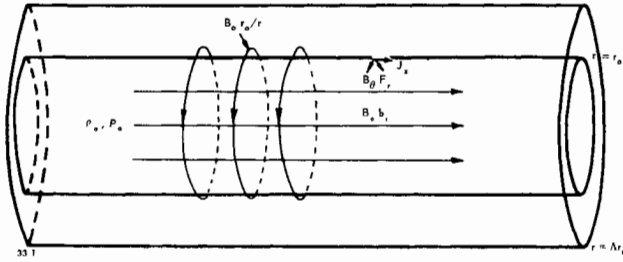


Figure 1. Equilibrium configuration of the pinched discharge. The discharge is pinched to a radius r_0 by the inward radial force $F_r = j_z B_\theta$; the discharge may contain an axial field $B_z b_i$ and be surrounded by conducting walls of radius Λr_0 .

wire model of the discharge is studied. The following two sections are based on the hydromagnetic equations: in one, the gradual loss of stability of the stabilised pinch,^{7,18-20} due to field diffusion, is discussed, and, in the other, another type of stable discharge configuration is introduced. In the final section there is a discussion of further problems which must be studied. Gaussian units are used throughout for the electrical quantities.

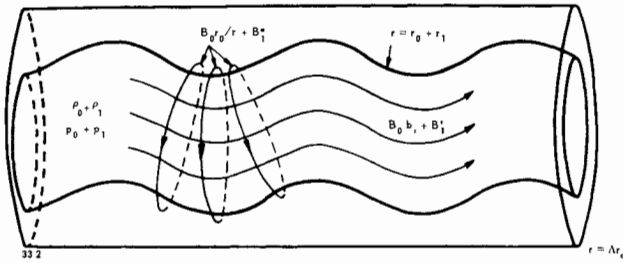


Figure 2. Perturbed configuration of the pinched discharge. When the surface of the discharge is distorted as shown, the azimuthal field lines crowd together on the inside of the kink, leading to a force tending to increase the perturbation. The tension in the axial field lines and image currents in the walls tend to resist the perturbation.

THE WRIGGLING DISCHARGE

The first model of the discharge considered is one in which all the plasma dynamics is neglected and only electromagnetic forces are considered. Thus the discharge is replaced by an extensible wire carrying a current. When this model was introduced, interest was concentrated on the highly constricted discharge and the discharge was treated as a very thin wire. Given perturbed configurations of the wire were considered and the forces acting on the configuration were calculated in a magnetostatic approximation. First calculations were made for the outward, instability-producing forces on a free discharge;⁴ later calculations were made of restoring forces due to conducting walls.¹⁰ The results obtained from this model are much better than might be expected and for the linear terms the stability criteria are in good agreement with those found for a hydromagnetic model. In addition it is quite easy to calculate non-linear terms in the forces and to obtain an estimate of

the amplitude to which given perturbations can grow before the net force is inward. This is a somewhat academic calculation because as soon as non-linear terms are considered configuration-changing forces are introduced and all the modes of instability interact. However, it does give a first clue to the importance of non-linear forces and suggests that they cannot be very effective unless the plasma is fairly close to stability; that implies that it is not highly constricted.

The results for one particularly simple configuration are as follows. Suppose a discharge of radius r_0 is in a tube of radius Λr_0 and carries a current J . The perturbed discharge takes a helical form given by the equations

$$x = A \cos kz, \quad y = A \sin kz. \quad (2.1)$$

The outward (self) force and inward (image) force are expanded for small values of Ak and $1/\Lambda$ and take the form:

$$\begin{aligned} \frac{r_0 c^2 F_\rho^{\text{self}}}{J^2} = & AkX_0 [-\ln X_0 + \ln 2 - \gamma + \frac{1}{2}] \\ & - A^3 k^3 X_0 [-\ln X_0 - \ln 2 - \gamma + \frac{5}{8}] \\ & + A^5 k^3 X_0 \left[-\ln X_0 + 4 \ln 2 - \frac{243}{64} \ln 3 - \gamma + \frac{23}{32} \right] \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} \frac{r_0 c^2 F_\rho^{\text{image}}}{J^2} = & - AkX_0 \left[\frac{K_0(\Lambda X_0) + K_2(\Lambda X_0)}{I_0(\Lambda X_0) + I_2(\Lambda X_0)} \right] \\ & - 2A^3 k^3 X_0 \left[\frac{K_1(2\Lambda X_0) + K_3(2\Lambda X_0)}{I_1(2\Lambda X_0) + I_3(2\Lambda X_0)} \right] \\ & - A^5 k^5 X_0 \left\{ \frac{13}{64} \left[\frac{K_0(\Lambda X_0) + K_2(\Lambda X_0)}{I_0(\Lambda X_0) + I_2(\Lambda X_0)} \right] \right. \\ & \quad \left. + \left[\frac{K_1(2\Lambda X_0) + K_3(2\Lambda X_0)}{I_1(2\Lambda X_0) + I_3(2\Lambda X_0)} \right] \right. \\ & \quad \left. + \frac{243}{64} \left[\frac{K_2(3\Lambda X_0) + K_4(3\Lambda X_0)}{I_2(3\Lambda X_0) + I_4(3\Lambda X_0)} \right] \right\}, \end{aligned} \quad (2.3)$$

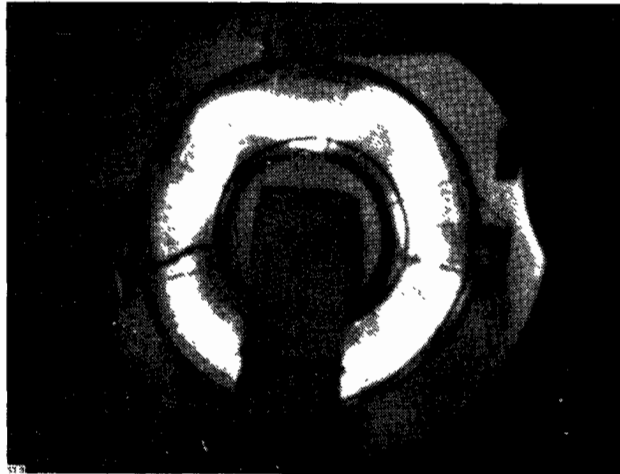


Figure 3. An unstable discharge in a pyrex torus

where F_ρ is the radial force, γ is Euler's constant 0.5772..., I_n and K_n are modified Bessel functions of the first and second kinds and X_0 is a non-dimensional wave number kr_0 . The correcting terms to (2.2) and (2.3) are of order $(Ak)^7$ and $1/\Lambda^2$.

Results have been calculated from (2.1) and (2.2) for several values of Λ and they are shown for one value of Λ ($\Lambda = 20$) in Fig. 4. It can be seen from this figure that the change in wave number, X_0 , between a perturbation which is completely stable and one for which the outward force vanishes when it is halfway across the tube, is very small. Values of wavelengths and amplitudes, for which the net force vanishes, are shown in Table 1 where $R_0 = \Lambda r_0$ is the tube radius.

Table 1. Critical Values of Amplitudes for Which Outward Force Vanishes

$1/\Lambda X_0$	1.49	1.46	1.39	1.29	1.14
A/R_0	0.0	0.2	0.4	0.6	0.8

Fuller details of this work are given in Ref. 10.

It would also be possible to include the effect of an axial magnetic field inside the discharge channel, by introducing a tension into the wire; however, this problem has been studied only on the hydromagnetic model. It is possible to show that an external magnetic field cannot completely stabilise helical perturbations of the type considered here.

In more recent work, by Curtis and Roberts,¹¹ an attempt has been made to follow the time development of the wriggling discharge by using an electronic computer. The model of this section is further simplified, by replacing the exact static forces by an approximation better suited to numerical work. The equation of motion

$$\rho \ddot{\mathbf{x}} = \mathbf{F}^{\text{self}} + \mathbf{F}^{\text{image}} \quad (2.4)$$

is then solved.

The parameters taken were: tube radius, R_0 ; length of discharge, $10 R_0$; and number of points along discharge channel, either 100 or 300. The discharge radius was not included as an explicit parameter, as the logarithmic term in the self force was replaced by a constant; this is roughly equivalent to saying that the correct value of the force is used for some mean wavelength of instability. The value of the logarithm taken corresponded to a discharge fatter than the one considered in the static problem above ($\Lambda \cong 10$).

Because of the finite number of points along the discharge, there is a cut-off in possible perturbation wavelengths. Some smoothing had also to be introduced into the integration procedure, because of the propagation of rounding-off errors, and this introduced a further wavelength cut-off. This cut-off might be considered in some sense equivalent to the introduction of an axial magnetic field which stabilises short wavelengths. With these conditions the discharge moved across about 50% of the tube

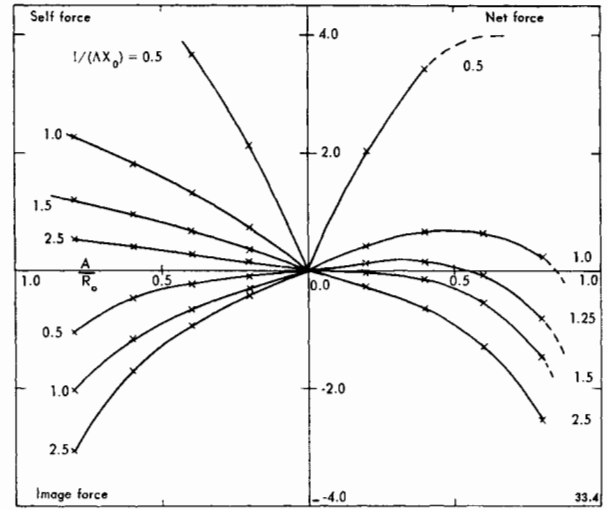


Figure 4. Forces on helical discharge $\Lambda = 20$. The outward self force. The inward image force and the net force, on a helical current channel, are shown as functions of the amplitude and wavelength of the helix

radius. Although the fatter discharge should be more stable than the results given in Fig. 4, it appears to be even more stable than would be expected. The full interpretation of these results has not yet been made.

THE STABILISED PINCH

In this section the stability problem is studied, with the hydromagnetic equations as starting point. These are, for an ideally conducting fluid of pressure p , density ρ , and ratio of specific heats γ , carrying electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ and current and charge densities $\mathbf{j}$ and Q ,

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{\mathbf{j} \times \mathbf{B}}{c}, \quad (3.1)$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v}, \quad (3.2)$$

$$\frac{1}{p} \frac{dp}{dt} = \frac{\gamma}{\rho} \frac{d\rho}{dt}, \quad (3.3)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}, \quad (3.4)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (3.5)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3.6)$$

$$\nabla \cdot \mathbf{E} = 4\pi Q \quad (3.7)$$

and

$$\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} = 0. \quad (3.8)$$

It is supposed that an infinite cylinder of conducting fluid, of radius r_0 , is surrounded by a vacuum containing a magnetic field $B_0(0, r_0/r, b_e)$ [in (r, θ, z) co-ordinates]. In turn, the vacuum is surrounded by a conducting wall at radius Λr_0 . The stability of a

system in which none of the equilibrium quantities depend explicitly on θ and z is investigated by considering departures from equilibrium in which any variable q has the form

$$q = q_0 + q_1(r, t) e^{i(m\theta + kz)}. \quad (3.9)$$

By solving the complete set of equations (3.1) to (3.8) for perturbations of the form (3.9), it has been shown by several authors^{7,18,19} that there exist stable configurations of this system, if the magnetic field inside the conducting fluid is constant and purely axial, having the value $(0, 0, B_0 b_i)$. In this case disturbances corresponding to $m=0$ and $m=1$ are the most difficult to stabilize but there are sets of values of (b_i, b_e, Λ) for which complete stability is obtained. These results are illustrated in Fig. 5. Chandrasekhar, Kaufman and Watson have obtained the same stability criteria, for a plasma in which collisions are neglected and Boltzmann's equation is taken as the starting point.²⁰

In fact, such discontinuous magnetic fields will not be realised in practice. Even if they could be obtained, the finite conductivity of the discharge would lead to a gradual interpenetration of the axial and azimuthal fields and to changes in the discharge radius. As it has also been shown⁷ that the discharge is unstable if the current density in the plasma is uniform and b_i and b_e are equal, it appears that at some stage in the interpenetration stability is lost. The field diffusion proceeds at a rate determined by a penetration time, $\tau_p = 4\pi\sigma r_0^2/c^2$ where σ is the relevant electrical conductivity. If instabi-

lities occur, a characteristic instability time τ_i is the time taken by a sound wave crossing the plasma. For high temperature plasmas $\tau_p \gg \tau_i$. Because of this inequality, it seems possible to divide the problem into two parts; *a*, the penetration problem, in which stability is assumed and the rate of field diffusion is calculated; and *b*, the stability problem, in which stability is investigated assuming that the plasma is static and ideally conducting.

Penetration

The penetration problem has not been studied in great detail. Neglecting the complications due to plasma motions and anisotropic electrical conductivity, a simple calculation has been made of the diffusion of an axial field out of a plasma and of an azimuthal field into it.²¹ With the added assumption that the axial magnetic flux and the axial current remain constant, the equations for the time and space dependence of the plasma magnetic fields become:

$$B_\theta/B_0 = R + 2 \sum_{p=1}^{\infty} \frac{J_0(\alpha_p) J_1(\alpha_p R)}{\alpha_p [J_2(\alpha_p)]^2} \exp - \alpha_p^2 T, \quad (3.10)$$

$$B_z/B_0 = b_{e0} + (b_{i0} - b_{e0})/\Lambda^2 + 2(b_{i0} - b_{e0}) \times \sum_{q=1}^{\infty} \frac{J_1(\beta_q) J_0(\beta_q R)}{\beta_q \{ [J_1(\beta_q)]^2 + \Lambda^2 [J_0(\beta_q)]^2 \}} \exp - \beta_q^2 T \quad (3.11)$$

where: $J_1(\alpha_p) = 0$;

$$\frac{J_0(\beta_q)}{J_1(\beta_q)} = - \frac{2}{(\Lambda^2 - 1) \beta_q}; \quad (3.12)$$

$T = t/\tau_p$; $R = r/r_0$; b_{i0} and b_{e0} are the initial values of b_i and b_e ; and the J_n are Bessel functions.

Results have been calculated for the special case of $\Lambda = 2$ and $b_{e0} = 0$ and these are shown in Figs. 6 and 7. The radial dependence of both axial and azimuthal fields is shown for several values of the time. It can be seen that quite considerable field penetration can occur in time $5 \times 10^{-3} \tau_p$ and, in fact, it will be seen that times less than this are of importance in stability problems. However, for a very hot plasma, even $10^{-3} \tau_p$ will be greater than τ_i . Although an actual discharge does not have a constant scalar conductivity, the results of Figs. 6 and 7 are probably qualitatively correct in showing that a rather small fraction of the relevant penetration time suffices for noticeable field diffusion. The results can be generalised slightly; if σ is time- but not space-dependent, exactly the same equations (3.10), (3.11) hold, provided that a new time coordinate is introduced, defined by

$$T' = \int_0^T \frac{\sigma_0}{\sigma(T'')} dT'' \quad (3.13)$$

where σ_0 is the initial value of the conductivity.

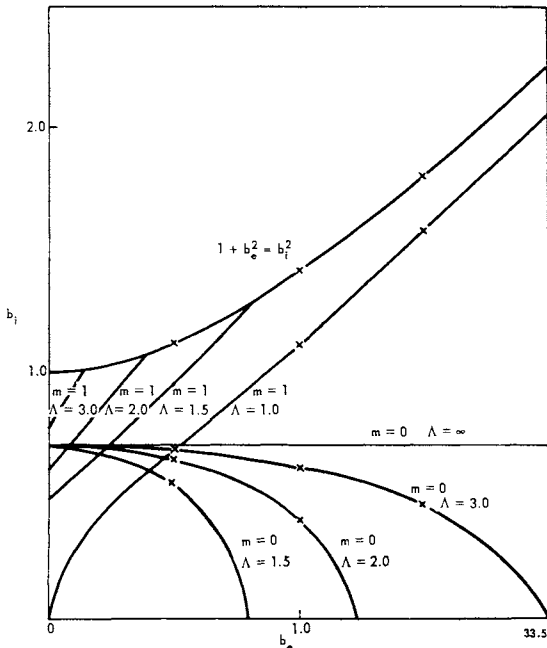


Figure 5. Stability diagram for surface current. For several values of wall radius Λ the regions of stability for $m=0$ and $m=1$ are shown. Stability occurs in the region above the curve marked with the given values of m and Λ and below the curve $1 + b_e^2 = b_i^2$. Complete stability for given Λ occurs when a point lies in the stability region for $m=0$ and $m=1$.

Stability

The stability problem has been treated by an energy principle method due to Bernstein, Frieman, Kruskal, and Kulsrud.²² This has been developed from one first used by Lundquist;²³ similar results have also been used by Hain, Lüst and Schlüter.²⁴ Bernstein *et al.* show that if a perturbation of a plasma is considered in which the element initially at $\mathbf{r}(0)$ moves to $\mathbf{r}(0) + \boldsymbol{\xi}$, the stability of the plasma against this perturbation can be determined by examining the sign of the change δW in the potential energy of the system. If δW is minimised with respect to all perturbations $\boldsymbol{\xi}$ satisfying certain conditions on the plasma-vacuum interface, and the minimum δW is negative, then the system is unstable. Otherwise it is stable. The energy principle will not be described in detail here; the results for the pinched discharge, which will be briefly discussed, are given fully in Refs. 25 and 26.

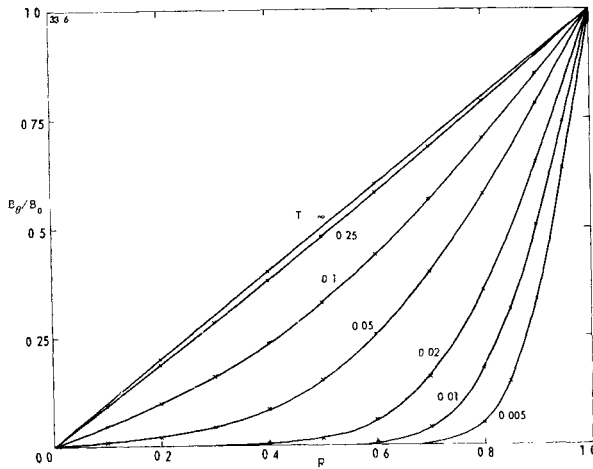


Figure 6. The diffusion of azimuthal magnetic field into a cylindrical conductor. The field is shown as a function of the dimensionless radius R , for several values of the dimensionless time T

It is supposed that in the central region of the plasma there is only an axial field $B_0 b_1$, but that axial and azimuthal fields are mixed in the outer fraction ϵ of the discharge radius. For mathematical simplicity, the field profiles taken are not those given by the solution of the penetration problem (Figs. 6 and 7). Instead it is supposed that both B_z and B_θ/r are linear with radius in the current layer. Thus

$$\mathbf{B} = \{0, B_0 r x / r_0, B_0 [b_1 + (b_e - b_1)x]\}, \quad (3.14)$$

where

$$r = r_0(1 - \epsilon) + r_0 \epsilon x. \quad (3.15)$$

For small ϵ , it can be seen from Figs. 6 and 7 that this approximation to the interpenetrating fields is probably quite good; in particular it appears justified to take the same ϵ for both B_θ and B_z . The field and pressure profiles for one particular set of values of b_1 , b_e , Λ and ϵ are shown in Fig. 8.

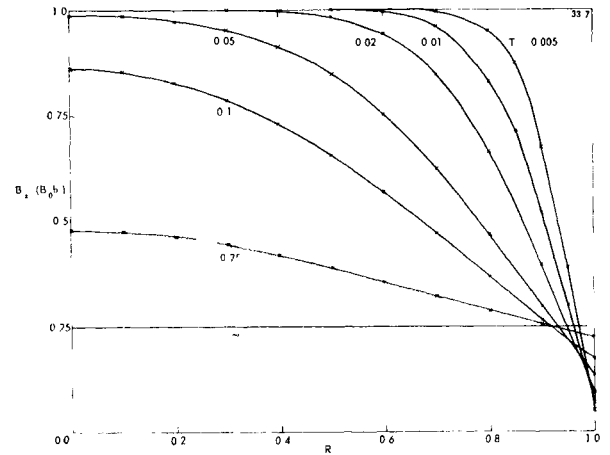


Figure 7. The diffusion of axial magnetic field out of a cylindrical conductor. The field is shown as a function of the dimensionless radius R , for several values of the dimensionless time T

Because of the absence of dependence on θ and z in the equilibrium quantities, any perturbation can be resolved into the harmonic form (3.9) and each pair of values of m and k can be considered separately. It is easy to minimise δW with respect to ξ_θ and ξ_z . When this is done, it is found that $\nabla \cdot \boldsymbol{\xi}$ vanishes, except at a discrete set of points, unless the pitch of the equilibrium magnetic field is constant in some region. This does not occur in the present problem. When $\nabla \cdot \boldsymbol{\xi} = 0$ is introduced into the expression for δW , all explicit dependence on the equilibrium plasma pressure is removed; the pre-eminence of incompressible perturbations is the reason why the results obtained in the last section are valid. Another feature that deserves mention is that for values of m greater than zero, the perturbation giving the lowest energy has an infinite shear at one point within the plasma.

With the forms for the magnetic fields (3.14), (3.15) and the ξ_θ and ξ_z minimisations performed, δW becomes, apart from a numerical constant:

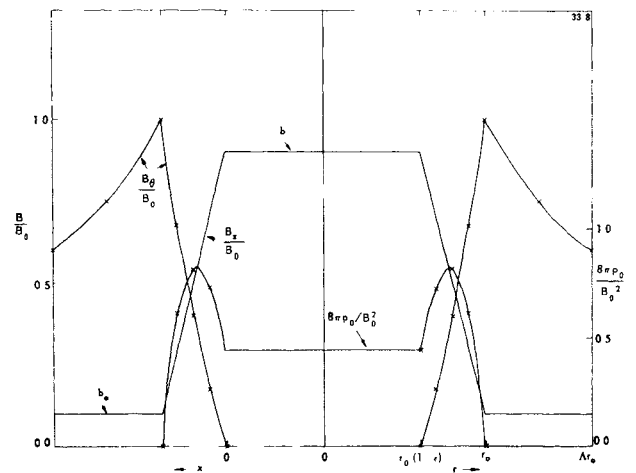


Figure 8. Sketch of discharge configuration with partially interpenetrated fields. The normalised magnetic fields (B_θ/B_0 , B_z/B_0) and plasma pressure ($8\pi p_0/B_0^2$) are shown as functions of the radius r

$$\begin{aligned}
\delta W = & -B_0^2 \xi_1^2 \frac{(m + b_e X_0)^2}{X_0} \\
& \times \left[\frac{K_m(X_0) I_m'(\Lambda X_0) - I_m(X_0) K_m'(\Lambda X_0)}{K_m'(X_0) I_m'(\Lambda X_0) - I_m'(X_0) K_m'(\Lambda X_0)} \right] \\
& + B_0^2 \xi_0^2 b_i^2 X_0 (1 - \epsilon) \frac{I_m[X_0(1 - \epsilon)]}{I_m'[X_0(1 - \epsilon)]} \\
& + B_0^2 \int_{r_0(1-\epsilon)}^{r_0} r dr \left\{ \xi^2 \left(\frac{m B_\theta}{r} + k B_z \right)^2 - \frac{2 B_\theta}{r^2} (r B_\theta)' \right. \\
& \quad \left. + \frac{1}{m^2 + k^2 r^2} \left(\frac{m B_\theta}{r} - k B_z \right)^2 \right\} \\
& + 2 r \xi \xi' \frac{1}{m^2 + k^2 r^2} \left(k^2 B_z^2 - \frac{m^2 B_\theta^2}{r^2} \right) \\
& + r^2 \xi'^2 \frac{1}{m^2 + k^2 r^2} \left(\frac{m B_\theta}{r} + k B_z \right)^2 \Big]. \quad (3.16)
\end{aligned}$$

In equation (3.16), K_m, I_m are modified Bessel functions, ξ is the radial component of ξ and ξ_0 and ξ_1 its values at the inner and outer boundaries of the current sheet, $X_0 = k r_0$, and the prime denotes differentiation with respect to r in the integral and with respect to the argument of the Bessel functions elsewhere. The first term in δW is the vacuum contribution, the second comes from the plasma centre and the third from the current sheet. If ϵ tends to zero, it can be shown the third term approaches the value -1 . In this case, the condition that δW vanishes reduces to the stability criterion for the stabilised pinch given in Ref. 7.

The expression for δW is minimised for ξ 's of the form

$$\xi = \xi_0 [1 + \epsilon \sum a_n x^n] \quad (3.17)$$

so that

$$\xi_1 = \xi_0 [1 + \epsilon \sum a_n]. \quad (3.18)$$

With the form (3.17) for ξ and (3.14) for the magnetic field, the integral in (3.16) can be evaluated either exactly or as a power series in ϵ .

One particular problem has first been considered. It is assumed that initially

$$\epsilon = 0, \quad \Lambda = 2, \quad b_e = 0, \quad b_i = 0.9 \quad (3.19)$$

and that field diffusion occurs subject to the following three conditions:

- (i) B_0 remains constant,
- (ii) Λ remains constant,
- (iii) The total axial flux remains constant.

Perturbations have been considered with $m = 0$ and $m = 1$; these were the types of perturbations most difficult to stabilise with surface currents. At first, trial functions are considered with only one arbitrary parameter; afterwards the number of parameters is increased.

In the case of $m = 0$, it is easy to see that δW is monotonically increasing with X_0 ; thus we need only determine its sign at $X_0 = 0$. For $m = 1$, no such

simple result holds. For $m = 0$, one-parameter trial functions have been considered for several values of n and ϵ . The resulting minimum values of δW are shown in Fig. 9. It can be seen that for each value of ϵ considered, δW has its minimum value for $n = 3$. It is also easy to show algebraically that the minimum δW , for one-parameter trial functions, approaches the asymptotic value shown as $n \rightarrow \infty$. It is possible to find the value of ϵ for which the minimum point on the curve, corresponding to those shown in Fig. 9, lies at $\delta W = 0$. This is the critical value at which stability ceases in the one-parameter approximation. Corresponding critical values of ϵ have been found for trial functions with up to four parameters and these are shown in Table 2. It can be seen that the value of ϵ changes only very slightly in going from one to four parameters. These results suggest that the critical value of ϵ converges well as more parameters are added, and the trial functions ξ also appear to be converging.

Table 2. Values of ϵ for Which Stability Is Lost for One-, Two-, Three- and Four-Parameter Trial Functions; $m = 0$

n	3	3, 4	2, 3, 4	2, 3, 4, 5
ϵ_{crit}	0.157	0.156	0.150	0.150

In the case of $m = 1$ it is more difficult to obtain complete results because the value of X_0 which gives the minimum δW is unknown. Results must therefore be obtained for several values of X_0 and the minimum value deduced. For one-parameter trial functions, it is found that δW has its minimum value for $n = 1$, and for such trial functions the critical value of ϵ at which stability ceases has been found. Corresponding values of ϵ have also been found for two- and three-parameter trial functions. These results are shown in Table 3; they are not as accurate as those in Table 2 because of the amount of inter-

Table 3. Values of ϵ for Which Stability Is Lost for One-, Two- and Three-Parameter Trial Functions; $m = 1$

n	1	2, 3	1, 2, 3
ϵ_{crit}	0.25	0.19	0.19

polation involved. However, the critical values for the two- and three-parameter cases are very close, both being between 0.185 and 0.190. It is clear that, in this case, the one-parameter trial function does not give an accurate idea of the critical value of ϵ ; it is impossible to judge whether closeness of the two- and three-parameter values is a sign of convergence, without also doing calculations for $n = 4$ and above. It seems likely, for this special problem, that the ϵ at which stability is lost is close to 0.15 and that the axisymmetric modes are the least stable.

In addition to this special problem, qualitative stability criteria for general field configurations have been obtained, using only one-parameter trial functions. For the case of $m = 0$, calculations have been

made for $\epsilon = 0.1$ and 0.2 and for several values of Λ . A one-parameter trial function with $n = 3$ was taken and, to this approximation, stability curves were drawn in the b_i, b_e plane. These correspond to the curves for $\epsilon = 0$ in Fig. 5 and they are themselves shown in Figs. 10 and 11. The upper curve in these figures is obtained from the condition that, for given b_e , there is a maximum value of b_i for which a plasma can be contained. The equation of this curve is

$$1 + b_e^2 + \frac{2}{3}\epsilon - \frac{1}{6}\epsilon^2 = b_i^2. \quad (3.20)$$

These stability curves should probably be a good approximation to the actual stability curves; in any case they always overestimate the region of stability.

Finally some calculations have been made with one-parameter trial functions with $m = 1$ and $\Lambda = 2$. The results of Table 3 indicate that stability curves obtained from these results cannot be more than qualitatively correct. Nevertheless, we have illustrated in Fig. 12a how, on the one-parameter approximation, the region of stability for both $m = 0$ and $m = 1$ shrinks as ϵ increases.

Although the results obtained above indicate that stability is lost for quite a small value of ϵ , the temptation to consider only the terms in the energy that

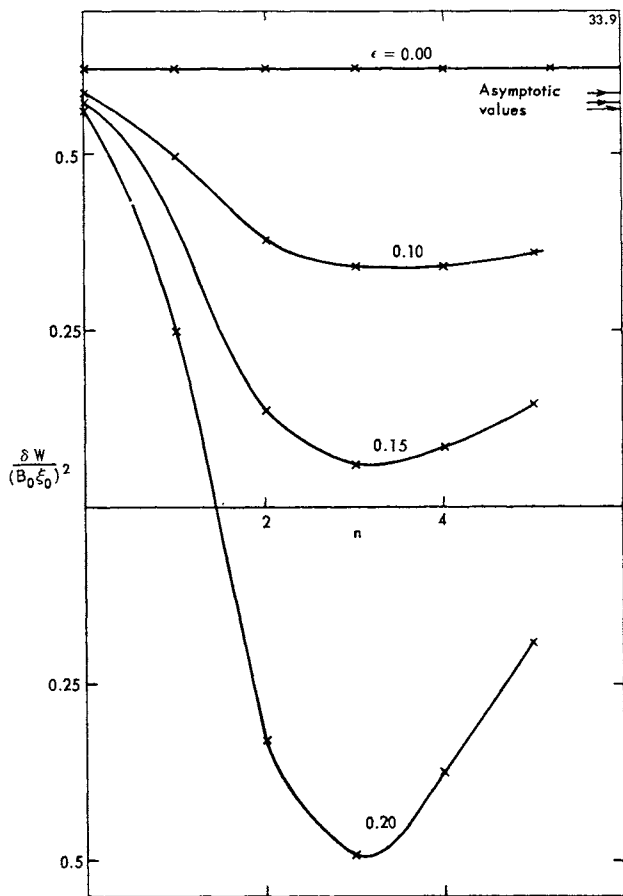


Figure 9. Minimum values of δW for one-parameter trial functions; $m = 0$. The value of $\delta W/B_0^2 \xi_0^2$ is plotted as a function of n for four values of ϵ . It can be seen that the one-parameter trial function with $n = 3$ gives the lowest value of δW

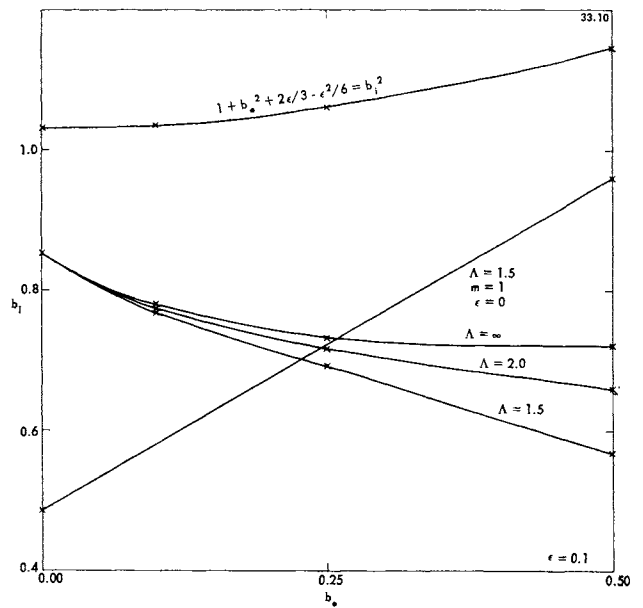


Figure 10. Stability diagram for one-parameter trial functions; $m = 0.1$, $m = 0$. In the one-parameter approximation, the region of stability is shown for $m = 0$, $\epsilon = 0.1$ and several values of the wall radius Λ . Stability occurs in the region above the curve marked with the given value of Λ and below the curve marked with the given value of Λ and below the curve marked with the given value of Λ . The stability curve for $m = 1$, $\epsilon = 0$ and $\Lambda = 1.5$ is also shown

are linear in ϵ must be resisted. For the special problem, the critical value of ϵ for axisymmetric perturbations, obtained by linearising in ϵ , is 0.333 compared with 0.157 obtained above.

All the results reported here have been obtained by hand computation and it is unlikely that the problem will be studied any further by this method. Miss S. J. Roberts is at present coding the problem for an electronic computer; it is hoped that when this is done, more general field configurations and trial functions will be able to be considered. It is also hoped that further evidence will be obtained about the convergence of the variational procedure.

Recent Developments in the Theory of the Stabilised Pinch

Since the previous section was written it has been shown that the convergence problem mentioned in the last paragraph is a real one. Rosenbluth²⁷ and Suydam²⁸ have studied the mathematical character of the third Euler equation and have shown that, because of a singularity in this equation, the *stabilised pinch* is unstable against surface instabilities for any depth of penetration of the current. This singularity occurs at the radius in the plasma where the magnetic field helix coincides with the perturbation helix. It can be asked whether this singularity is physically real; in particular what happens to it when particle motions and dissipative processes are considered? A first attempt at a study of this problem has recently been made by Hubbard²⁹ at Harwell.

Hubbard assumes that every particle is acted on not by a point magnetic field but by the field that has

been averaged over some averaging length (δ) and he has replaced the singular term in the differential equation by its value averaged over this distance. He has solved the resulting differential equation on the Mercury computer, for many values of the averaging length and the depth of the current layer, and has found stability diagrams corresponding to Fig. 12a. One such stability diagram is shown in Fig. 12b. In the limit of zero averaging length he has found results agreeing with those of Rosenbluth and Suydam. For finite values of the averaging length he has calculated the critical depth of field penetration at which stability ceases. He has also given arguments for choosing the averaging length and a discussion as to whether it is determined by the electrical conductivity, viscosity or particle gyration. He believes that the Larmor radius of the ions provides a suitable averaging length in many cases and his results sometimes show greater stability than the conventional stabilised pinch.

This work is at the moment based on physical intuition rather than on a rigorous mathematical foundation. He has merely modified the final second order differential equation and has supposed that marginal stability is still determined by the vanishing of the disturbance growth rate. In fact, if finite viscosity and electrical conductivity are included in the original equations, the order of the mathematical problem is considerably increased and complex growth rates occur.

GENERAL STABLE DISCHARGE CONFIGURATIONS

The conventional stabilised pinched discharge is not an obviously stable configuration; as has already been seen, very stringent conditions have to be satisfied by the parameters b_i , b_e and Λ for stability to be obtained. In the energy-principle formulation it is seen that the change in potential energy is positive in the vacuum and the centre of the plasma, and negative in the current layer and, for stability, an accurate

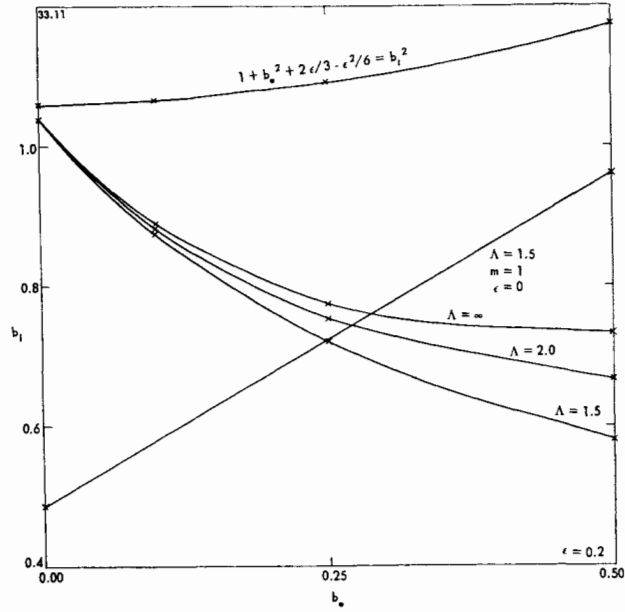


Figure 11. Stability diagram for one-parameter trial functions; $\epsilon = 0.2$, $m = 0$. In the one-parameter approximation, the region of stability is shown for $m = 0$, $\epsilon = 0.2$ and several values of the wall radius, Λ . Stability occurs in the region above the curve marked with the given value of Λ and below the curve $1 + b_e^2 + \frac{2}{3}\epsilon - \frac{1}{6}\epsilon^2 = b_i^2$. The stability curve for $m = 1$, $\epsilon = 0$, and $\Lambda = 1.5$ is also shown

balance has to be obtained. In this section the possible existence of obviously stable configurations is investigated. These are configurations in which the change in energy is positive in all parts of the discharge, for all perturbations, or even more restrictively ones in which all terms in the energy are positive everywhere. This problem has been studied at Harwell by Laing.³⁰

Suppose an ideally conducting plasma fills a tube of radius Λr_0 . If the plasma contains a magnetic field ($0, B_\theta, B_z$), the change in its potential energy W due to a perturbation $\xi e^{i(m\theta + kz)}$, after minimisation with respect to ξ_θ and ξ_z , is

$$\delta W = \int_0^{\Lambda r_0} r dr \left\{ \left(\frac{mB_\theta}{r} + kB_z \right)^2 - 2B_\theta (B_\theta/r)' - 4B_\theta^2/r^2 \right\} \xi^2 + \frac{\left\{ \left(\frac{mB_\theta}{r} - kB_z \right) \xi - \left(\frac{mB_\theta}{r} + kB_z \right) r \xi' \right\}^2}{(m^2 + k^2 r^2)}, \quad (4.1)$$

where ξ is the radial component of the displacement ξ and differentiation is denoted by a prime. As the conducting fluid fills the entire tube, the radial perturbation ξ must vanish at the radius Λr_0 . Using this condition, an alternative form can be obtained for δW by performing an integration by parts on the term in $\xi \xi'$ in (4.1).

Thus,

$$\delta W = \int_0^{\Lambda r_0} [A \xi^2 + B \xi'^2] dr, \quad (4.2)$$

where

$$\begin{aligned} rA &= (mB_\theta + krB_z)^2 + (mB_\theta - krB_z)^2/(m^2 + k^2 r^2) \\ &\quad - 2B_\theta^2 - 2m^2 k^2 r^2 (B_\theta^2 + B_z^2)/(m^2 + k^2 r^2)^2 \\ &\quad - k^2 r^3 (B_\theta^2 + B_z^2)'/(m^2 + k^2 r^2), \\ B &= r(mB_\theta + krB_z)^2/(m^2 + k^2 r^2). \end{aligned} \quad (4.3)$$

If the integrand in expression (4.1) is to be positive at all points in the discharge, the coefficient of ξ^2 must be positive. Thus,

$$(mB_\theta + krB_z)^2 \geq 2B_\theta (rB_\theta)'. \quad (4.4)$$

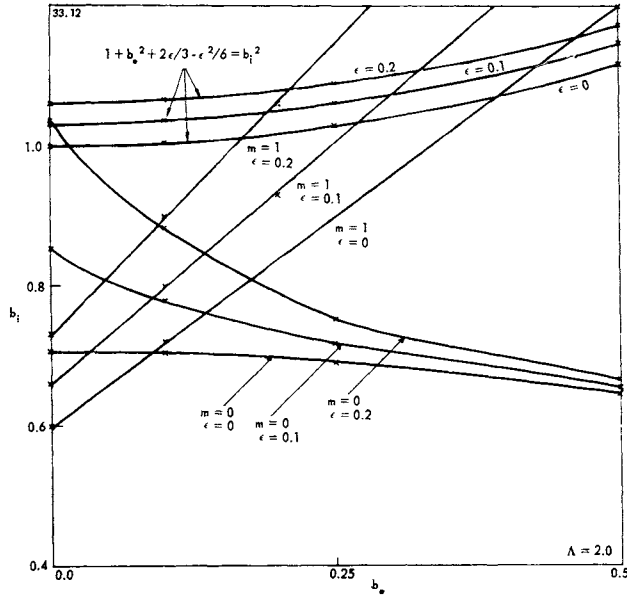


Figure 12a. Stability diagram for one-parameter trial functions; $\Lambda = 2$. Curves are drawn for three values of ϵ and for $m = 0$ and $m = 1$. It can be seen how the region of total stability, in this approximation, decreases as ϵ increases

This condition has previously been obtained from the perturbed differential equations in Ref. 7. If Eq. (4.4) is to be satisfied for all perturbations m, k , the right-hand side of the equation must be negative or zero at all values of r . Thus,

$$I_z I_z' \leq 0, \quad (4.5)$$

where I_z is the total axial current within radius r . This condition cannot be satisfied everywhere; in particular, at the discharge centre the current and its gradient must be in the same direction.

If the integrand in expression (4.2) is to be positive at all points in the discharge, A must be greater than zero. Using the hydrostatic equation $\nabla p = \mathbf{j} \times \mathbf{B}/4\pi$, this condition becomes

$$8\pi r p' d \geq a B_\theta^2 + 2b B_\theta B_z + c B_z^2, \quad (4.6)$$

where

$$\begin{aligned} a &= m^2[-1 + 1/(m^2 + X^2) + 2X^2/(m^2 + X^2)^2], \\ b &= mX[-1 + 1/(m^2 + X^2)], \\ c &= X^2[-1 - 1/(m^2 + X^2) + 2m^2/(m^2 + X^2)^2], \\ d &= X^2/(m^2 + X^2), \\ X &= kr. \end{aligned} \quad (4.7)$$

It can be shown that the worst perturbation for the inequality (4.6) is that with $k = 0$ and $m = 1$. If the inequality is to be satisfied for this perturbation,

$$r p' \geq B_\theta^2/6\pi. \quad (4.8)$$

If the inequality (4.8) is satisfied at all values of the radius, the plasma cannot be contained by the magnetic field. Equation (4.8) has been obtained by minimising the right-hand side of inequality (4.6) for all possible values of B_θ, B_z .

There remains the possibility that the pressure can increase outwards near the centre of the discharge and that the current and its gradient are in opposite directions in the outer region. Thus δW is divided up, so as to use the form (4.2) in the central region and the form (4.1) outside. There is also an additional term in δW , from the integration by parts on the interface $r = r_0$. Thus

$$\begin{aligned} \delta W &= \int_0^{r_0} (A \xi^2 + B \xi'^2) dr \\ &+ \int_{r_0}^{\Lambda r_0} r dr \left[\left\{ \left(\frac{m B_\theta}{r} + k B_z \right)^2 - 2 B_\theta (B_\theta/r)' \right. \right. \\ &- 4 B_\theta^2/r^2 \left. \right\} \xi^2 + \left\{ \left(\frac{m B_\theta}{r} - k B_z \right) \xi \right. \\ &- \left. \left. \left(\frac{m B_\theta}{r} + k B_z \right) r \xi' \right\}^2 / (m^2 + k^2 r^2) \right] \\ &+ \left[\frac{(k^2 r^2 B_z^2 - m^2 B_\theta^2) \xi^2}{m^2 + k^2 r^2} \right]_{r=r_0}. \end{aligned} \quad (4.9)$$

It is now insisted that all parts of this expression are positive. If the surface term is to be positive for all m and k , B_θ must vanish on $r = r_0$; thus the total axial current inside $r = r_0$ vanishes. In addition, there can be no axial current outside $r = r_0$; otherwise the inequality (4.5) would once again fail. The final requirements are

$$\begin{aligned} r p' &\geq B_\theta^2/6\pi, \text{ for } 0 < r < r_0 \\ B_\theta &= 0, \quad p' = -(B_z^2)'/8\pi \text{ for } r_0 < r < \Lambda r_0. \end{aligned} \quad (4.10)$$

A completely stable configuration satisfying the conditions (4.10) has been found by Laing.³⁰ The

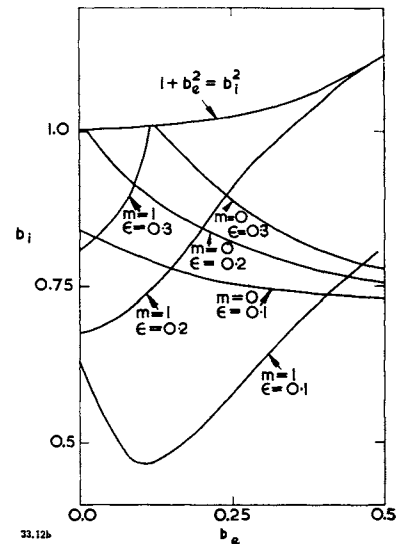


Figure 12b. Stability diagram with averaged fields $\delta = 0.05$ $\Lambda = 2$. The $m = 1$ stability curves are drawn for several values of ϵ and for the averaging distance (δ) 5 per cent of the discharge radius. The $m = 0$ curves refer to zero averaging lengths as there is no singularity in this case

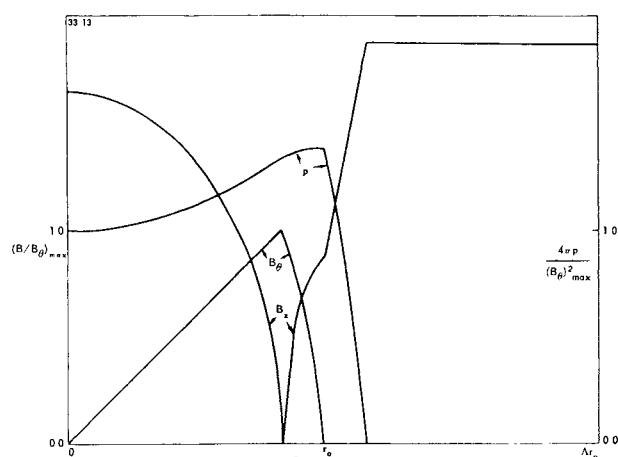


Figure 13. A stable configuration with reserved current. Curves are shown of the radial variation of B_θ , B_z and p for an "obviously" stable discharge configuration

values of magnetic fields and pressure for this configuration are shown in Fig. 13. It is not necessary for the axial magnetic field to drop to zero where the current has its maximum, as it does in this example; the example was obtained by assuming that B_θ has a simple analytical form. It is hoped that this type of configuration will be studied further and that neighbouring stable configurations can be found, which do not satisfy the very restrictive conditions of this section.

When the theoretical results of this section were obtained, it was not realised that Burkhardt and Lovberg³¹ had obtained a configuration similar in principle to the ones described above. In their case the reversed axial current arose unexpectedly and the discharge appeared to be stable.

FURTHER PROBLEMS

Plasma Motions

The only problem in which plasma motions have been considered in detail is that of the wriggling discharge. However, there is no reason to suppose that plasma motions are absent, even in an essentially stabilised discharge. Three such motions are: the initial contraction; motion of expansion or contraction during field diffusion; and steady oscillations, which may occur even for a stable discharge. The presence of these motions may lead to new types of instabilities. Many problems arising from the presence of equilibrium motions are essentially non-linear in the velocity; certainly, perturbations no longer grow or decay harmonically with time.

The oscillation problem can be studied in a linear way, as a first approximation. If it is imagined that a force is instantaneously applied to counteract the plasma acceleration, at the time in its motion when it is momentarily at rest, then effectively the problem is the same as that of a plasma supported against gravity by a magnetic field. A simple problem of this type has been studied by Kruskal and Schwarzschild.⁵ In their problem the plasma was very

unstable, because there was no magnetic field in the plasma. In the presence of crossed magnetic fields, which are required to stabilise the initial configuration, many of the Rayleigh-Taylor types of instability are also removed.

The next approach to the oscillation problem considers the plasma to be oscillating in one mode and considers small time dependent perturbations on this. The study of the stability of such a system leads to differential equations of a generalised Hill's type. Some progress has been made on one simple problem along these lines. The problem of the stability of a plane plasma, which is executing rigid body oscillations between plane conducting walls, can be reduced, in a certain approximation, to the solution of two simultaneous Mathieu equations. This is still only a very poor approximation to the real problem, in which there may be interaction between very many modes on equal terms and the oscillation energy may be passing back and forth between them.

Finite Dissipation

It has been stated that, on qualitative grounds, it is plausible to treat the electrical conductivity as infinite in stability calculations. Similar arguments can be made in support of the neglect of viscosity and thermal conduction in some problems. However, even if numerically the importance of dissipative processes looks small, it must be realised that their introduction completely alters the mathematical character of the stability problem. One particular problem, in which viscosity may be important, has already been mentioned; in the energy principle method the lowest energy, for non-symmetrical perturbations, is obtained for a perturbation ξ with an infinite velocity shear. As the energy can be made arbitrarily close to this minimum, with a perturbation with large but finite shear velocity, the result is probably virtually un-

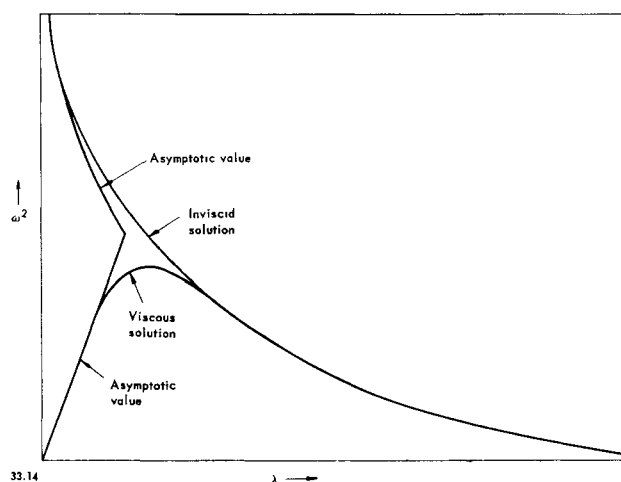


Figure 14. Influence of viscosity on discharge stability. A qualitative plot is given of the relationship between growth rate ω and wavelength λ for the $m = 1$ instability of a plasma carrying a surface current. This shows the inviscid solution and the effect of the introduction of viscosity. The asymptotic solutions at small wavelength intersect at a point which gives a first approximation to the wavelength of maximum instability

altered for small enough viscosity; but this requires further investigation. For axisymmetric perturbations the trouble does not arise and, for sufficiently symmetrical systems and perturbations, a generalised form of energy principle still appears to be valid. A solution has been given to one problem involving viscosity in Ref. 6. For the earliest problem of the instability of the pinch effect studied by Kruskal and Schwarzschild⁵ there is an infinite instability growth rate at zero wavelength. The introduction of any viscosity, however small, reduces this growth rate to zero. This is illustrated in Fig. 14. An attempt is also being made to study the stability of a plasma, with finite electrical conductivity carrying a uniform axial current and a uniform axial magnetic field. With no axial field, the problem is simple; with an

axial field, it is only easy to obtain asymptotic results and, at present, no general results on the effect of finite electrical conductivity have been deduced.

Two other factors are likely to become much more important to us in the future: the geometrical factor, which arises both through toroidal shape and through incomplete symmetry produced by gaps in conducting walls; and the problem of the range of validity of the hydromagnetic equations. It is true to say that the problem of gas discharge stability is still in its early stages and much remains to be done.

ACKNOWLEDGEMENTS

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Cusped Geometries

By J. Berkowitz, K. O. Friedrichs, H. Goertzel, H. Grad, J. Killeen and E. Rubin *

INTRODUCTION

We are interested in the containment and thermonuclear possibilities of a large family of stable magnetohydrodynamic free-boundary equilibrium configurations. The free boundary is a mathematical idealization in which there is a perfectly conducting plasma containing no magnetic field, separated at an interface (surface current) from a vacuum magnetic field.¹ The plasma pressure p is balanced by the magnetic pressure $H^2/8\pi$. Although the original motivation for studying this model was its mathematical simplicity, the sharp separation was soon found to offer practical advantages in performance provided that it can be realized experimentally.

It can be shown that any finite plasma configuration of the free boundary type, bounded by a smooth interface, is unstable.² However, there do exist a large number of absolutely stable configurations bounded by cusped surfaces. These have been shown to be stable for arbitrary perturbations, even of finite amplitude, regardless of which equations are assumed to govern the plasma motion.² The basic two-dimensional prototype is shown in Fig. 1. The magnetic field is generated by four line currents alternating in direction, coupled with the plasma sheet currents which flow in alternate directions on adjacent segments. There is a one-parameter family of plasmas of increasing size (Fig. 2) for a given coil configuration. The shapes can be computed by conformal mapping. The limiting shape for small dimensions is a hypocycloid, $x^{2/3} + y^{2/3} = \text{constant}$, and the largest configuration which completely contains the plasma is given by the equation

$$\theta = -\pi/4 - \int_1^{a(s)} t^{-1} (t-1)^{1/2} (16/9 - t)^{-1/2} dt,$$

where $a \equiv [(16/9) \cos \pi s]^2$, $\tan \theta$ is the slope and s is the arclength. This equation holds for the portion of the curve in the upper part of the first quadrant; the complete shape is found by symmetry. The shapes can also be computed numerically for current arrangements more realistic than line currents.

Among the theoretical questions that should be asked about a given configuration relative to its

utility as a thermonuclear device are its stability, rate of loss of particles and of energy, and experimental feasibility. To answer all but the first question, the original idealization must be modified to take into account a continuous transition from plasma to vacuum. The picture we have in mind is still that of a *thin* transition zone of minimum thickness separating a vacuum (or near vacuum) from an almost field-free plasma; but, whenever possible, we shall try to relate this simplified model to the more general case of completely intermixed plasma and field. Of special interest for comparison purposes is the opposite extreme to the free boundary case, *viz.*, a plasma of such low density that the plasma currents have no effect at all on the magnetic field of the external coils. One such case is the picket fence³ (Fig. 3a) which has the free boundary analogue shown in Fig. 3b. This interface is also explicitly computable using conformal mapping.

The major advantage of the cusped configuration is its stability. The chief disadvantage, at least of the basic geometry of Fig. 1, is the large rate of loss of particles. One can generalize the magnetic mirror concept and apply it to the motion of a particle approaching a cusp even though the magnitude of the magnetic field does not increase towards the cusp, but is a constant on the interface. Nevertheless, because of the extremely large effective mirror ratio (see the Appendix), it is found that the particle losses increase rapidly with temperature instead of decreasing as do the conventional mirror losses. For preliminary experiments with relatively low temperature plasmas the cusps are advantageous. At (eventual) thermonuclear temperatures a *crossed field* inserted in the plasma would be very effective in reducing losses. The prototype, shown in Fig. 4, has a uniform longitudinal field inside the plasma and a cusped field outside.

Another possible disadvantage of a cusped geometry is the uneconomical use of the magnetic field if the plasma is to be compressed. A possibility for ameliorating this situation is to compress the plasma from within, *cf.* Fig. 5. The tunnel created by such an internal conductor is stable against all deformations.² More conventionally, field shaping by the use of appropriately designed conductors and by the use of

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iron in the magnetic circuit can be effective. This question of magnetic efficiency is not entirely one-sided since most of the other proposed configurations require significant internal magnetic fields in the plasma to obtain stability.

SUMMARY OF GEOMETRIES

There exists a large variety of three-dimensional stable cusped configurations suggested by the prototypes depicted in Figs. 1, 3b, 4 and 5. The spindle shape of Fig. 6a is obtained by rotating the two-dimensional shape of Fig. 1 around an axis through two cusps. The resulting figure has a circular line cusp and two point cusps. The coil arrangement consists of two oppositely oriented Helmholtz coils. This three-dimensional free boundary has been computed numerically and does not deviate much from the rotated two-dimensional shape.

By placing a number of spindles (of the type shown in Fig. 6a) end to end, joining and widening

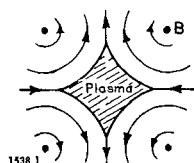


Figure 1. Field-plasma configuration in a two-dimensional cusped geometry

the point cusps, we obtain the configuration shown in Fig. 6b. The infinite periodic limiting case can also be described as a rotation of Fig. 3b about its longitudinal axis. Fig. 6c is obtained from Fig. 6b by bending it into a torus. The toroidal shape, Fig. 6d, is similarly obtained by rotating Fig. 1 about a horizontal axis in its plane; the coil system consists of four ring currents.

Figure 6e shows how a crossed field might be introduced into systems of the type shown in Figs. 6a or 6b; the field due to the axial wire is allowed to penetrate into the plasma. The addition of an axial surface current (as in a pinch discharge) to Fig. 6b modifies the shape somewhat; the cross-section mid-way between two circular cusps (i.e., under the

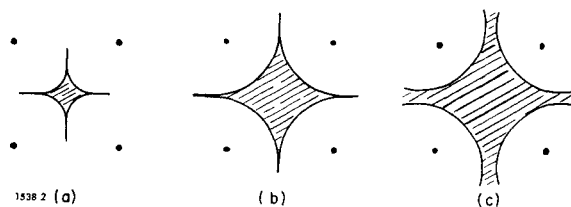


Figure 2. Plasma-field configuration in some limiting cases

external coil) becomes narrower. This can become unstable for a sufficiently large axial current. Specifically, since the interface has negative curvature there are two asymptotic directions separating directions of positive and negative curvature.

Instability is reached when the axial current rises sufficiently to make the total current cross an asymptotic direction (see Fig. 7). An axial, pinch-type current can be introduced in Fig. 1, i.e., a current on the interface directed into the plane of the figure. The asymmetric Fig. 8a results. As the axial current is increased it is likely that a transition will occur to the configuration shown in Fig. 8b.

PARTICLE LOSSES

Particles can be lost from a system by diffusion across the magnetic field and, in open-ended systems, lost along the magnetic field (end losses). The latter is our concern here. It is illuminating to compare two extreme cases, *collision-dominated* mirror losses and *nonadiabatic-dominated* cusp losses. We shall find that there is a continuous transition from one to the other.

First consider the elementary containment argument⁴ for the axially symmetric geometries shown in Figs. 9a and 9b. One assumes that the motion is adiabatic, i.e., the magnetic moment μ is a constant. This is the ratio of the gyration energy, W_L , corresponding to the component of velocity perpendicular to $\mathbf{B}$, to the magnitude of $\mathbf{B}$. To good approximation it is constant if the magnetic field seen by the particle changes only by a small amount in one gyro oscillation. The gyration energy increases as the particle moves toward larger B , but it cannot exceed the constant values, W , of the total kinetic energy.

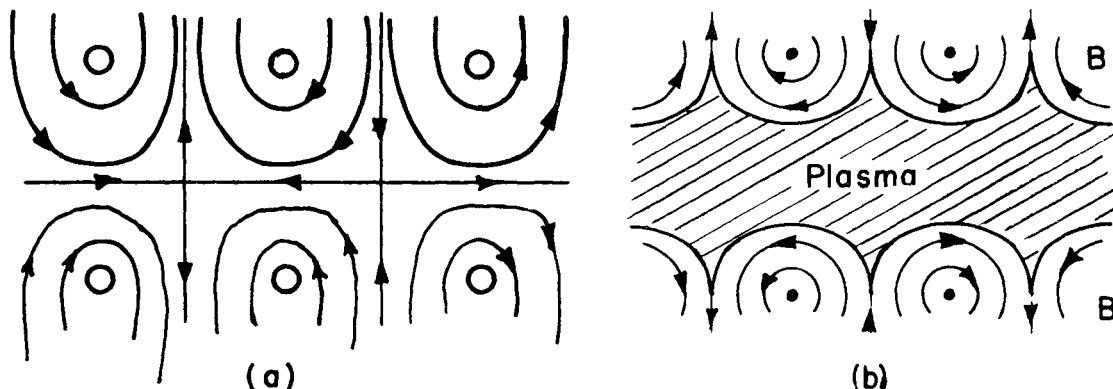


Figure 3. Picket fence geometry

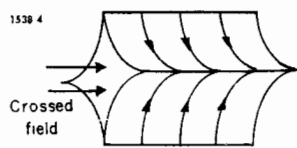


Figure 4

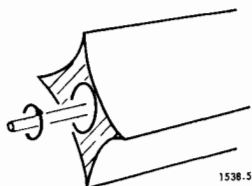


Figure 5

Consequently, reflection occurs when B reaches the value W/μ . Furthermore, the guiding center of the particle is constrained to lie on a fixed flux tube. This allows us to conclude that successive reflections will occur at exactly the same value of z (Fig. 9). All particles with $W_{\perp}/W$ less than a certain critical value (this defines the *loss cone*) are instantly lost, and the others are contained indefinitely. Under the assumption that the configuration is exactly axially symmetric, there are two mechanisms which can shift a particle into the loss cone. These two mechanisms are caused by collisions and alteration of the magnetic moment and are independent of each other. Either mechanism can be dominant; it is even possible for each one to be dominant for different classes of particles in a single system. For example, in Fig. 9b, an orbit can be quite accurately adiabatic for a particle whose gyro radius is small compared to its distance from the axis; the loss mechanism is collisions. On the other hand, the motion will be strongly nonadiabatic if the orbit passes close to the origin, and, for these orbits, collisions can be neglected. This effect can also be present in Fig. 9a (and it is *always* present in Fig. 9b) if the plasma pressure is comparable to the magnetic pressure near the center of the machine, thereby reducing the magnetic field in the center to a relatively low value. More precisely, we suppose that on the axis the *mirror ratio* (ratio of the maximum to minimum value of B) is large. A particle whose turning point is close to the position of maximum B will have a very small value of $W_{\perp}/W$ when it reaches the center of the machine. It can move a large axial distance in one gyro-period and therefore be strongly nonadiabatic.

An extreme case of nonadiabaticity is encountered in the cusped geometry. However, we shall first describe it in entirely different terms. We start

with the simplest model, *viz.*, with a sharp discontinuity separating field-free plasma from vacuum. At thermonuclear temperatures the mean free path is essentially infinite compared to the dimensions of the apparatus. A particle trajectory consists of straight lines joined by cycloidal arcs (Fig. 10a); the limiting case of zero gyro radius is the billiard ball model, Fig. 10b. In the latter case there are no losses; every particle is turned back before it reaches the cusp unless it is on the axis and aimed directly at the cusp. In the former case, we (tentatively) suppose a particle to be lost if it reaches the cusp. It is found that those particles which are aimed close enough to the cusp axis are lost; the others are reflected back. But it is easy to see that the aspect with which a reflected particle approaches the cusp at the opposite end of the machine has very little correlation with its behavior in the first cusp. To good approximation we may assume that the velocity distribution is isotropic, i.e., the loss cone is refilled as fast as particles are removed. This is in fact a general property of any configuration with a very large mirror ratio. It is possible to treat the motion of a particle when it is near a cusp by studying a suitable adiabatic invariant μ^* instead of the magnetic moment μ (see the Appendix). The mirror ratio appropriate to the adiabatic invariant μ^* has the order of the square root of the ratio of the gyro radius to the apparatus dimension. Although μ^* is approximately constant while the particle is near the cusp, it loses almost all memory of its initial value in crossing from one cusp to the next.

The loss rate computed on the basis of an isotropic velocity distribution⁵ is an upper estimate. It is a good estimate in a strongly nonadiabatic device. In other cases, the loss rate is governed by the rate at which particles diffuse into the loss cone by collisions and by nonadiabaticity.

The precise computation of particle losses is a very complex problem involving the solution of a self-consistent field problem for the transition zone between the plasma and field, the merging of two such zones at the cusp, and the interaction between these boundary layers and the space charge surrounding the plasma. We shall present a sequence of models of increasing complexity to estimate the loss rate.

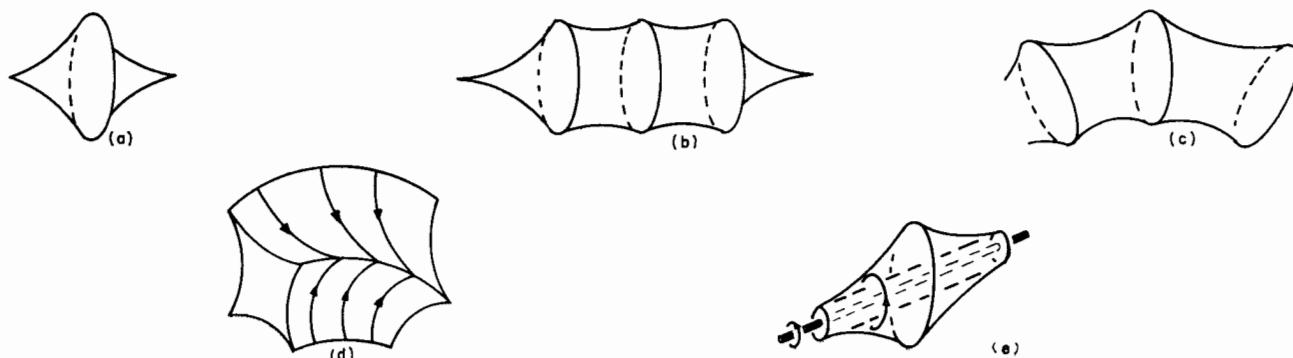


Figure 6

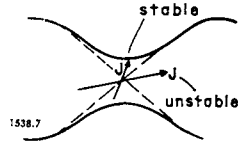


Figure 7

The boundary layer is formed of particles which belong to the plasma, i.e., which follow orbits as shown in Fig. 10a, and *free* particles which spiral nearby without entering the plasma proper. This combination produces a certain electromagnetic field which we must find, at least to some approximation. It can be shown that the loss rate per unit length of the cusp is given by

$$\nu = \int_{u>0} r u f du dv dw = \langle r \rangle \int_{u>0} u f du dv dw \quad (1)$$

where $f(u, v, w)$ is the particle velocity distribution (u is directed toward the cusp) and $r(u, v, w)$ is the range of a particle (see Fig. 10a), i.e., the distance travelled in the direction of the cusp in one encounter with the boundary layer. The length $\langle r \rangle$ can be interpreted as the width of a "hole" near the cusp through which particles stream freely.

To start, we ignore the electric field and assume that the magnetic field jumps discontinuously. The trajectories can be evaluated explicitly. For an isotropic, monoenergetic (i.e., fixed speed $u^2 + v^2 + w^2 = V^2$) distribution we compute

$$\langle r \rangle = \frac{2}{3} \pi \lambda, \quad \nu = \frac{1}{6} \pi n V \lambda, \quad \lambda = mV/eB; \quad (2)$$

where n is the number density. At a given temperature, electrons and ions are lost at the same rate; the electrons move faster, but the "hole" $\langle r \rangle$ is smaller by the same factor. This picture must be made self-consistent with respect to the current and the charge. The latter is far more important and its effects can be approximated by recomputing the orbits in the presence of a constant electric field E whose value is then adjusted to make the mean penetration depth of ions and electrons equal (the Debye length is assumed to be small compared to the boundary layer thickness). The result is, approximately,

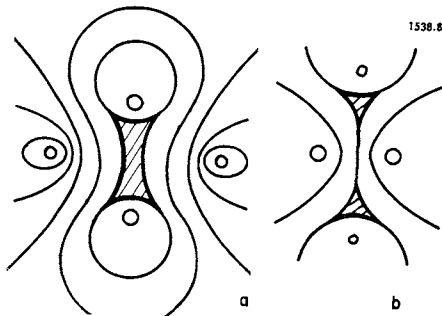


Figure 8

$$U = E/B = (4/9\pi)V_-$$

$$\langle r_+ \rangle = (9\pi/8)(V_+/V_-) \lambda_+$$

$$\nu_+ = (9\pi/32) (nV_+^2/V_-) \lambda_+$$

$$\langle r_- \rangle = \frac{2}{3} \pi \lambda_-$$

$$\nu_- = \frac{1}{6} \pi n V_- \lambda_- \quad (3)$$

For deuterons, $M/m = (V_-/V_+)^2 \sim 3600$, the ion loss rate is cut down by a factor 35 and the electron rate increases by only a few per cent. The boundary layer is now approximately self-consistent, but the difference in loss rates must be equalized by external space charge effects. The precise mechanism is vital since a possible factor 35 is at stake. If the space charge field does not penetrate into the plasma it merely accelerates the lost ions but returns electrons to the plasma, thereby equalizing the net loss rate at the lower value, ν_+ . The possibility of space charge penetration into the plasma can only be discussed realistically if the magnetic field is allowed to permeate the plasma. A rough analysis of this problem yields the result (mainly as a consequence of the very large mirror ratio) that the equalization occurs almost entirely by reducing the electron loss rate without appreciably altering the ion loss rate.

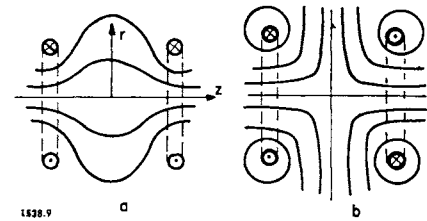


Figure 9

A more serious consideration is the possibility that some of the electrons which are turned back to the cusp may miss the cusp and become "free" electrons as previously described. These electrons will be trapped in the external space charge field and will oscillate between cusps in the outer reaches of the boundary layer, allowing the ion penetration to increase and thereby increasing $\langle r \rangle_+$ and ν_+ . This situation can be analyzed to some extent and arguments can be given which indicate that the effect is not large. However, we merely summarize by stating that the correct loss rate is somewhere between the two values, ν_+ and ν_- , quoted in (3), with a fair chance that it will turn out to be closer to the smaller value. An alternative description is that the loss rate is proportional to the thickness of the boundary layer which is somewhere between the electron and ion gyro radii, but may very well be closer to the former. The boundary layer thickness can possibly be controlled experimentally, e.g., as shown in Fig. 11. An arrangement is shown in which electrons might be stripped from the outer edges of the boundary layer.

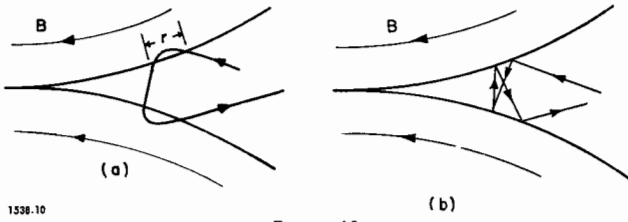


Figure 10

The Maxwell-averaged loss rate, using the more optimistic value ν_+ of Eq. (3) is,

$$\langle \nu \rangle = (9\pi/640) (nkT/eB). \quad (4)$$

From this we find the containment time of the spindle shape in Fig. 6a to be

$$\tau = 10^{-7} (R^2 H / T) \quad (5)$$

where τ is in seconds, H is in gauss, R is the radius in cm and T is the temperature in electron volts. The density is determined by the temperature when the magnetic pressure equals the gas pressure. For example, if $H = 10,000$ gauss, $R = 100$ cm, $T = 10$ ev, we have a containment time of one second. In the thermonuclear range, $H = 100,000$ gauss, $R = 100$ cm, $T = 10,000$ ev yields $\tau = 10$ milliseconds.

The general case in which the field and plasma are intermingled (as in Fig. 9b) is extremely complicated. The outer layers have a relatively small mirror ratio and are collision dominated so that the loss rate is relatively low and electron losses are preferred. There is possibly an intermediate layer in which the electrons are adiabatic and the ions are not, since the latter have larger gyro radii. The loss rate in this layer could show a preference for either ions or electrons depending on the parameters. An inner layer might be nonadiabatically dominated for both ions and electrons with the electron losses predominating.

The space charge problem is formidable. Whereas rough estimates of loss rates can still be made, it would be advisable to look to experiment for guidance.

Next we briefly consider the effect of a crossed field (see Fig. 4) and first take the case of a sharp discontinuity in the magnetic field. Computations involving generalized adiabatic invariants such as those described in the Appendix have been made to compute the loss rate through a cusp. The important point to realize is that this computation (which yields a lower loss rate than without the crossed field) applies only to the plasma layer which is within one gyro radius of the vacuum. The inner particles

suffer no losses at all. More precisely, the losses from the outer layer can only be replaced by diffusion across the magnetic field. This considerably lower loss rate then becomes the dominant one. In the case of a gradual transition, instead of a discontinuity of the magnetic field, the situation is, of course, very complex but the general qualitative features can easily be ascertained. The mirror reflection argument can be carried out largely as in the case without the crossed field. Moreover, the magnetic field never vanishes and nonadiabatic effects are considerably reduced.

We can summarize the effect of the crossed field as follows. In the general case it reduces nonadiabatic losses to the generally lower level of collisional losses and, in the sharp boundary case, to the still lower level of diffusional losses across the magnetic field.

THERMONUCLEAR POSSIBILITIES

For simplicity we consider only the geometry of Fig. 6a. The state of the plasma is given by three parameters, e.g., volume, pressure and temperature. Actually we use the radius R of the circular cusp section in cm, the magnetic field H in gauss, and the temperature T in electron volts. The energy loss through the cusp in watts is then (from Eq. (4))

$$P_C = 0.08 RTH. \quad (6)$$

An estimate of the ohmic losses is

$$P_R = 10^{-4} RH^2 \quad (7)$$

and the thermonuclear output (energy of charged particles only) for a deuterium-tritium mixture at 20,000 ev is ⁶

$$P_T = 10^{-15} R^3 H^4. \quad (8)$$

Clearly, P_T can be made to dominate $P_C + P_R$ by making R and H large (the ratio of bremsstrahlung losses to thermonuclear power depends on temperature alone).

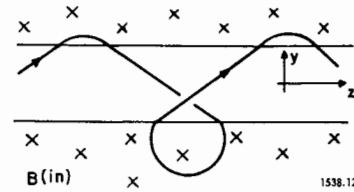


Figure 12

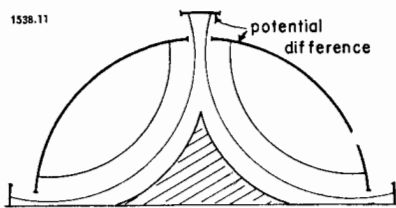


Figure 11

Let us define "breakeven" by the condition $P_T = 3(P_C + P_R)$. A possible set of breakeven figures is $H = 10^5$, $R = 40$, $P_C = 6 \times 10^9$, $n \sim 10^{16}$ ions/cm³. The power output would be comparable in size to a conventional large generating station. We envision a cycle somewhat as follows. Let $\mathcal{E}$ represent the original energy of the plasma. At some later time (in this example, about one millisecond) when $\mathcal{E}$ has been reduced by one-half due to cusp losses, the thermonuclear input has reached $3\mathcal{E}/2$ bringing the total

plasma energy to $2\mathcal{E}$ and the plasma pressure to double its original value. An adiabatic expansion against the magnetic field then transfers the amount $\mathcal{E}$ into the generator which supplies the field. The remaining energy, including the considerably greater energy carried by the neutrons, could be removed by a heat cycle of lower efficiency. Of course, these numerical results should be considered only as representing the correct order of magnitude.

A number of points should be mentioned here. One property of cusp losses is that, in a given device, high speed particles are lost more quickly and this tends to deplete the Maxwell tail. In particular, this applies to the charged reaction products. On the other hand, the hot reaction products, in modifying the boundary layer, serve to reduce the loss rate of the original "cold" plasma and, in particular, of its Maxwell tail. The use of the optimistic loss rate formula (4) is probably much more than offset by the gain that would arise from the use of crossed fields.

Essentially all of the methods of initiating and heating a plasma can be used in a cusped geometry. Four parallel discharges alternating in direction might be set up in a longitudinal magnetic field to initiate the field configuration shown in Fig. 4; or the coils in any of the cusped configurations could be pulsed after preionization or after creation of the plasma by a shockwave; or the plasma could be inserted in an already existing vacuum magnetic field of an appropriate type by an arrangement of plasma guns. A particularly intriguing possibility is to pulse the magnetic field with enough energy to bring the plasma to thermonuclear temperature. The plasma will undergo damped oscillations. Although the initial acceleration would probably be too high to maintain stability,² the acceleration would very likely drop to a stable value within a cycle or so. Other confinement geometries, e.g., the stabilized B_z -pinch, are not sufficiently stable to allow such violent treatment. Roughly speaking, in exchange for a high degree of stability we compromise somewhat in particle losses.

APPENDIX. THE CUSP AS A MIRROR

The theory of adiabatic invariants of a Hamiltonian which is slowly varying in time can be modified so as to apply to a Hamiltonian which has an "approximately ignorable" coordinate. This allows the containment arguments which have been discussed to be extended to an axially symmetric system of the type shown in Fig. 9a, even when the gyro radius is comparable to the radial dimensions, provided only that it is small compared to the axial length (z is the approximately ignorable coordinate). In this case, the adiabatic invariant μ^* is $W_\perp/\omega^*$, where ω^* is the frequency of oscillation of the particle in the "frozen" or cylindrically symmetric Hamiltonian (with a truly ignorable coordinate corresponding to the instantaneous value of z), instead of the magnetic moment which is $W_\perp/\omega$ (the gyro frequency ω equals B except for the constant factor e/m).

Exactly the same remarks apply to two-dimensional problems. For a particle approaching a line cusp the frozen Hamiltonian is that of two parallel plates, Fig. 12. With no electric field, ω^* has the value

$$1/\omega^* = 1/\omega + d/\pi |v_\perp|, \quad (9)$$

d is the separation between the plates (which varies slowly with z) and $W_\perp = \frac{1}{2}mv_\perp^2$. With the adiabatic invariant $\mu^* = W_\perp/\omega^*$, we can analyze the motion of a particle in either a cusp or a conventional mirror. The mirror ratio would be the ratio of ω^* at the cusp (viz. ω) to its value elsewhere; for large d , this ratio is essentially the ratio of the gyro radius to d/π . A mirror ratio can be assigned to the cusp as a whole by estimating at what value d the particle makes its last encounter with the magnetic field before crossing over to the next cusp. This mirror ratio turns out to be of the order of $M = \frac{1}{2}\sqrt{a}$, where a is the ratio of the mean range $\langle r \rangle$ to the length of the apparatus. The corresponding correction to the loss computation, i.e., the measure of anisotropy of the velocity distribution, is the small number $(1 + \ln 8M)/8M$.

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Magnetohydrodynamic Stability

By J. Berkowitz, H. Grad and H. Rubin *

Our primary concern is the hydromagnetic stability of free-boundary equilibria.¹ These are configurations in which a perfectly conducting field-free fluid is separated at an interface (surface charge and current) from a vacuum electromagnetic field ($\mathbf{E}, \mathbf{B}$) (Fig. 1). Two examples were treated by Kruskal and Schwarzschild:² one, a conducting fluid supported against gravity at a plane interface by a uniform magnetic field (Taylor instability); the other, an infinite cylindrical conducting fluid surrounded by an azimuthal magnetic field (pinch instability). In full generality, we consider a free-boundary equilibrium with an interface of arbitrary shape in the presence of an arbitrary conservative force field. The configuration of external fixed conductors is arbitrary. The external magnetic constraints may be either constant fluxes or constant currents. The conducting fluid may be taken incompressible or compressible with any equation of state, dissipative (viscous or heat conducting), non-dissipative, or even governed by the Boltzmann equation. Stability is considered both with respect to small amplitude and finite amplitude perturbations.

It is possible to solve the problem in such general terms because the stability problem separates into an electromagnetic problem which can be treated as essentially static if displacement current is ignored, and a fluid problem for which stability is treated by conventional thermodynamic considerations. The stability criterion is expressed in terms of geometric properties of the interface alone and is independent of the constraints or of the equations which govern the fluid. A number of absolutely stable equilibria are given (Section 5).

This work has been extended to the case of conducting fluid and magnetic field which are intermingled, instead of separated at an interface.³ Certain aspects of this more general problem are treated here (Section 9), and a class of stable axially symmetric configurations is described. The more difficult problem of stability in the presence of periodically varying magnetic fields is solved for a special geometry (Section 10).

For a Hamiltonian system with finitely many degrees of freedom, the type of argument used in this stability

analysis yields categorical answers. For such a system, an absolute minimum of a monotone potential represents a stable equilibrium in the sense that a small perturbation will never cause more than a small deviation from equilibrium, and a finite perturbation, a correspondingly finite deviation. If the equilibrium state corresponds to a relative minimum of the potential, there exists a maximum perturbation of the energy below which the system remains in some neighborhood of the equilibrium. The existence of the relative minimum (but no estimate of the maximum stable perturbation) can be ascertained by examining the second variation of the potential function. From the second variation, we may obtain more than a stability criterion: the eigenvalues computed with the kinetic energy as unit form are the frequencies (stable) or the growth constants (unstable) of small amplitude disturbances.

For a fluid, the infinity of degrees of freedom introduces complexities. Consider the example of a container of water under gravity (Fig. 2). There exists a potential and an energy integral $K + P$ (kinetic plus potential). In Fig. 2a, P_0 is the absolute minimum of P . For an energy perturbation ε , we have $K + P = P_0 + \varepsilon$. We conclude without further reference to the equations of motion that for all time

$$0 \leq K \leq \varepsilon; P_0 \leq P \leq P_0 + \varepsilon. \quad (1)$$

The first inequality states that the kinetic energy is never large. The second states that, in some sense, the configuration never deviates very much from equilibrium; for example, that no more than a certain small amount of water can exceed a given height. It is intuitive that the same conclusion applies to Fig. 2b, viz. that for a given ε only a certain amount of water can be lost over the rim. This conclusion does not follow from energy considerations alone and the inequalities (1) cannot be proved even for restricted values of ε . In principle, it may be possible for a fluid particle to gain large kinetic energy after passing over the rim and to pass this energy back to the water in the container. Because of this almost unsurmountable mathematical difficulty, the stability of a fluid system with respect to finite amplitude perturbations is treated in two parts: the first, mathematically precise, will show that a given equilibrium minimizes

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an appropriate potential among all neighboring states; the second will be a heuristic evaluation of this result in the light of some "practical" criterion for stability.

The point of this variational approach is to eliminate almost all reference to the equations of motion or their solutions. We use only the integrals and constants of the motion, energy and others. Although we would like to compare the potential of the equilibrium state with the potentials attainable by possible motions of the system, we compare it with the potentials of *all* mathematical configurations compatible with the constraints (i.e., the constants of the motion). The method could fail to indicate stability when it in fact exists; one possibility is that the class of admissible configurations is too large and no minimum exists (as for an incompressible fluid if the fluid volume is forgotten as a constant of the problem);

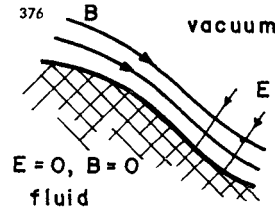


Figure 1

or it can be too conservative if neighboring configurations exist which are not dynamically accessible as solutions of the equations of motion.

A positive second variation of the potential for all admissible infinitesimal perturbations is evidence for stability. This finding makes plausible the existence of a relative minimum but does not ensure it and is therefore a less conservative criterion.

An entirely different kind of stability analysis is to linearize the equations of motion and separate an exponential time factor (normal-mode analysis). It is often possible to relate the second-variation formulation of the stability problem to the normal-mode analysis as in the case of finitely many degrees of freedom. The existence of an unstable mode is better evidence of instability than the mere existence of a perturbation which makes the second variation negative, but the normal mode analysis is no better in predicting stability than the second variation.

The stability of fluid magnetic equilibria will be considered from all three viewpoints: minimization of an appropriate potential with regard to finite per-

turbations, analysis of the second variation and normal mode analysis. For brevity we touch only upon the major points. A detailed presentation will be given elsewhere.

1. CONSTANTS OF THE MOTION

For the electromagnetic field we use Maxwell's equations (in rationalized MKS units) without displacement current:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \quad \begin{cases} \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{B} = \mu \mathbf{J} = 0. \end{cases} \quad (2)$$

At the interface, moving with fluid velocity $\mathbf{u}$, we have the boundary conditions

$$B_n = 0, \quad (\mathbf{E} + \mathbf{u} \times \mathbf{B})_t = 0; \quad (3)$$

the subscripts n, t denote normal and tangential components, respectively. Only u_n actually enters in (3). At a fixed conducting boundary, we have

$$B_n = \text{constant in time}, \quad E_t = 0. \quad (4)$$

The electromagnetic force $\mathbf{J} \times \mathbf{B}$ is normal to the interface and exerts a pressure of magnitude $B^2/2\mu$. The electric force $q\mathbf{E}$ and also the displacement current are relativistically small; moreover the two must be dropped or kept together to keep the electromagnetic momentum equation in conservation form. The system (2) and the boundary conditions (3), (4) are Galilean invariant and are even preferred to the Lorentz invariant Maxwell system for use here with non-relativistic fluid equations.

Under the boundary conditions (3), (4) the solution of the system (2) is determined at each instant by the position of the interface without regard to its velocity.⁴

If $B_n = 0$ at all boundaries, fixed or fluid, the magnetic field is determined by a finite number of parameters, either fluxes or mmfs (currents)

$$\Phi_r = \int_{S_r} \mathbf{B} \cdot d\mathbf{S}, \quad \text{or } I_r = \mu^{-1} \int_{C_r} \mathbf{B} \cdot d\mathbf{x}. \quad (5)$$

The magnetic energy is then given by

$$\begin{aligned} M &= \int \frac{1}{2} \mu^{-1} B^2 dV = \sum_r \frac{1}{2} I_r \Phi_r \\ &= \sum_{r,s} \frac{1}{2} I_r L_{rs} I_s = \sum_{r,s} \frac{1}{2} \Phi_r \lambda_{rs} \Phi_s, \end{aligned} \quad (6)$$

where the inductance matrix L_{rs} and the susceptance λ_{rs} are defined by

$$\Phi_r = \sum L_{rs} I_s, \quad I_r = \sum \lambda_{rs} \Phi_s. \quad (7)$$

From the integrated form of the first equation of (2),

$$\begin{aligned} \frac{d\Phi_r}{dt} &= \frac{d}{dt} \int_{S_r(t)} \mathbf{B} \cdot d\mathbf{S} \\ &= - \int_{C_r(t)} (\mathbf{E} + \mathbf{u} \times \mathbf{B})_t \cdot d\mathbf{x} = 0, \end{aligned} \quad (8)$$

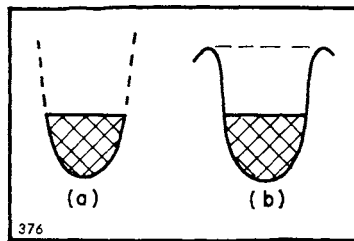


Figure 2

where C_r is the boundary of S_r . We conclude that the fluxes are constants of the motion. An example is a toroidal plasma surrounded by a vacuum magnetic field contained within a concentric copper torus. Two fluxes (or mmfs) completely determine the instantaneous magnetic field. In any deformation of the plasma, the two fluxes are constant.

More generally, when magnetic lines intercept a part of the boundary ($B_n \neq 0$), the magnetic field is determined by the given stationary values of B_n and by such fluxes Φ_r as may arise from particular closed circuits which do not link a boundary; these fluxes again are constants of the motion.

Constant current generators may be introduced by violating the conditions (4). For example, we may introduce slits in the fixed conductors and apply suitable potentials across them. With $E_t = 0$ elsewhere, the potential across the slit will be equal to the line integral in (8), destroying $\Phi_r = \text{constant}$ but permitting $I_r = \text{constant}$. More generally, we could distribute the voltage arbitrarily around the conductor so long as the total emf is sufficient to keep the current constant. Such a distributed emf can be consistent with the boundary condition $B_n = \text{constant}$ and, as before, the magnetic field is uniquely determined by the given values of B_n and the fixed currents I_r (instead of Φ_r).⁴ This result holds even for a mixture of constant-current and constant-flux constraints. In most of the subsequent analysis, we shall take the general case: there exist n independent fixed circuits, the first m of constant-current type and the remainder of constant-flux type:

$$I_r = \text{constant}, (r = 1, 2, \dots, m); \Phi_r = \text{constant}, (r = m + 1, \dots, n). \quad (9)$$

The rate of change of magnetic energy in a moving domain $V(t)$ as computed from (2), (3), (4) is

$$\begin{aligned} \frac{dM}{dt} &= \frac{d}{dt} \int_{V(t)} \frac{1}{2} \mu^{-1} B^2 dV \\ &= \oint_{S(t)} [\frac{1}{2} \mu^{-1} B^2 u - \mu^{-1}] \mathbf{E} \times \mathbf{B} \cdot d\mathbf{S} \\ &= - \oint_{S_1(t)} \frac{1}{2} \mu^{-1} B^2 \mathbf{u} \cdot d\mathbf{S} - \oint_{S_2(t)} \mu^{-1} \mathbf{E} \times \mathbf{B} \cdot d\mathbf{S} \end{aligned} \quad (10)$$

where S_1 denotes the interface and S_2 the fixed boundary; the Poynting vector contribution on S_1 reverses the sign of this integral. It can be shown that only the constant current circuits of (9) contribute to the second integral; namely,

$$\begin{aligned} \mu^{-1} \int_{S_2} \mathbf{E} \times \mathbf{B} \cdot d\mathbf{S} &= - \sum_{r=1}^m I_r \frac{d\Phi_r}{dt} \\ &= - \frac{d}{dt} \sum_{r=1}^m I_r \Phi_r. \end{aligned} \quad (11)$$

If we set

$$M^* = M - \sum_{r=1}^m I_r \Phi_r \quad (12)$$

we have in the general case (9)

$$\frac{dM^*}{dt} = - \oint_{S_1} \frac{1}{2} \mu^{-1} B^2 \mathbf{u} \cdot d\mathbf{S}. \quad (13)$$

In the special case $B_n = 0$ on all boundaries we have from (6)

$$M^* = \sum_{r=m+1}^n \frac{1}{2} I_r \Phi_r - \sum_{r=1}^m \frac{1}{2} I_r \Phi_r. \quad (14)$$

If the fluid is incompressible, we have the equations

$$\nabla \cdot \mathbf{u} = 0; \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \varrho^{-1} \nabla p + \nabla \phi = 0 \quad (15)$$

from which we obtain the rate of change of energy,

$$\frac{d}{dt} \int_{V(t)} \frac{1}{2} \varrho u^2 dV = - \int_{S(t)} (p + \varrho \phi) \mathbf{u} \cdot d\mathbf{S}, \quad (16)$$

where ϱ is the mass density, $\mathbf{u}$ the fluid velocity, p the pressure, and ϕ an external potential energy per unit mass. For any other kind of fluid, taking heat flow to be zero across the interface (radiation is neglected), we have

$$\frac{d}{dt} \int_{V(t)} \varrho (e + \frac{1}{2} u^2 + \phi) dV = - \int_{S_1(t)} u_i P_{ij} dS_j, \quad (17)$$

where e is the internal energy per unit mass, P_{ij} the stress tensor. The electromagnetic stress is normal to the interface and must be counterbalanced by the fluid stress (otherwise the acceleration would be infinite); hence, the fluid stress reduces to its normal component p_n and

$$p_n = B^2/2\mu \text{ (on } S_1), \quad (18)$$

whence

$$\frac{dK}{dt} + \frac{d\mathcal{E}^*}{dt} = - \int_{S_1} p_n \mathbf{u} \cdot d\mathbf{S} \quad (19)$$

where

$$\mathbf{K} = \int \frac{1}{2} \varrho u^2 dV, \mathcal{E}^* = \int \varrho (e + \phi) dV. \quad (20)$$

From (13), observing that the normal to S_1 is oppositely oriented from the magnetic and fluid sides, we obtain the energy integral

$$K + \mathcal{E}^* + M^* = \text{constant}. \quad (21)$$

In the incompressible case we have

$$K + M^* = \text{constant}, \quad (22)$$

and as an additional integral, the total volume of the fluid,

$$\int dV = V = \text{constant}. \quad (23)$$

2. EQUILIBRIUM SEPARATION OF THE POTENTIALS

In a non-dissipative fluid, equilibrium is characterized by the pressure balance

$$\varrho^{-1} \nabla p + \nabla \phi = 0. \quad (24)$$

This implies that p and ϱ (and hence all other thermodynamic quantities) are constant on the surfaces of constant ϕ . If the fluid is in thermodynamic equilibrium, the temperature is constant as well. Employing the Gibbs free energy per unit mass, $F = e - T\eta + p/\varrho$, we obtain $dF = -\eta dT + \varrho^{-1} dp$ (η is the entropy per unit mass), and from (24),

$$F + \phi = \text{constant}. \quad (25)$$

For an incompressible fluid, we have

$$p + \varrho\phi = \text{constant}. \quad (26)$$

In equilibrium (scalar pressure), the boundary condition (18) takes the form

$$p = B^2/2\mu. \quad (27)$$

For an arbitrarily given interface there exists a unique equilibrium fluid configuration on one side and a unique magnetic field on the other. The boundary condition (27) singles out a special class of equilibrium shapes. In the simplest case, $\phi = 0$, we have $p = \text{constant}$ in the fluid and $B^2 = \text{constant}$ at the interface. The problem of determining such equilibria is identical to the free-boundary problem for irrotational incompressible fluid flow (B corresponds to fluid velocity) where the condition is that the absolute velocity is constant at the unknown surface.

We now define the potentials P in a way which separates magnetic and fluid effects. In a specific equilibrium state, the fluid pressure is a function of position x defined by the implicit relation (25): $F(p(x), T) + \phi(x) = \text{constant}$. We suppose $\phi(x)$ to be defined in all space. The definition of $p(x)$ may then be extended into the vacuum domain using (25); the extended function is called $p_0(x)$. Referring to (21) we put

$$P_f = \mathcal{E}^* + \int_{V_f} p_0 dV = \mathcal{E}^* + \int_{V_f} (\varrho e + \varrho\phi + p_0) dV \quad (28)$$

$$P_m = M^* + \int_{V_m} p_0 dV = \int_{V_m} \left(\frac{1}{2} \mu^{-1} B^2 + p_0 \right) - \sum_{r=1}^m I_r \Phi_r \quad (29)$$

where V_f and V_m denote the fluid and vacuum domains, respectively. The integral of p_0 has been added to both the fluid and vacuum potentials. Since $V_f + V_m$ is constant during the motion, we have merely changed the value of the constant in

$$K + P_f + P_m = \text{constant}. \quad (30)$$

3. VARIATION OF THE FLUID POTENTIAL

The stability of the fluid is treated by purely thermodynamic arguments. We show for a fluid satisfying any conventional equation, dissipative or not, that motion proceeding from an initial thermodynamic equilibrium state can only increase the potential P_f . Also, for a non-dissipative fluid with no external potential ϕ , a motion proceeding from a state of dynamic equilibrium (p is constant but T is not necessarily constant), can only increase P_f .

We recall that the total entropy $S(\mathcal{E}, V, m)$ is a homogeneous first order function of the three arguments, total energy, volume and mass. We denote the conventional densities (per unit mass) of entropy and internal energy by η, e , respectively, and introduce the densities per unit volume of entropy $\hat{\eta} = \rho\eta = \eta/\tau$, and energy $\hat{e} = \rho e = e/\tau$, where $\tau = 1/\rho$ is the specific volume. The second law of thermodynamics is equivalent to the statement that the functions $\hat{e}(\hat{\eta}, \rho)$ and $e(\eta, \tau)$ are convex functions of their arguments and satisfy

$$T d\eta = de + p d\tau; \quad T d\hat{\eta} = d\hat{e} - F d\rho. \quad (31)$$

At a given point $\mathbf{x}$, consider the function

$$\alpha(\hat{\eta}, \rho) = \hat{e}(\hat{\eta}, \rho) + \phi\rho - T_0\hat{\eta}, \quad (32)$$

where T_0 is the equilibrium value of T . From

$$\begin{aligned} \frac{\partial \alpha}{\partial \hat{\eta}} &= \frac{\partial \hat{e}}{\partial \hat{\eta}} - T_0 = T - T_0 \\ \frac{\partial \alpha}{\partial \rho} &= \frac{\partial \hat{e}}{\partial \rho} + \phi = F + \phi \end{aligned} \quad (33)$$

we see that α is stationary at $T = T_0$, $F + \phi = 0$. The function α is convex in $\hat{\eta}$ and ρ since it differs from $\hat{e}$ by a linear function. We conclude that α has a unique minimum at values of $\hat{\eta}$ and ρ corresponding to $T = T_0$ and $F + \phi = 0$. The minimum property is not altered by adding the constant p_0 to α , and we have

$$\begin{aligned} \hat{e} + \phi\rho - T_0\eta + p_0 &> \hat{e}_0 + \phi\rho_0 - T_0\eta_0 + p_0 \\ &= \varrho_0(F_0 + \phi) = 0. \end{aligned} \quad (34)$$

Integrating over the fluid volume we obtain

$$\mathcal{E}^* - T_0 S + \int_V p_0 dV > 0 = \mathcal{E}_0^* - T_0 S_0 + \int_{V_0} p_0 dV \quad (35)$$

where $\mathcal{E}^*$ is defined by (20). Since the total entropy S cannot decrease in any motion, $S \geq S_0$, we conclude that motion proceeding from an initial equilibrium can only increase the fluid potential P_f .

In the special case $\phi = 0$ it is possible to relax the condition $T = \text{constant}$ of thermodynamic equilibrium and by integrating with respect to mass in the inequality

$$e + p_0\tau - T_0\eta \geq e_0 + p_0\tau_0 - T_0\eta_0 = F_0 \quad (36)$$

we obtain the same result for adiabatic variations at dynamical equilibrium (constant pressure).

If the fluid equations of motion are replaced by the Boltzmann equation, we use the Boltzmann H -function instead of S ,

$$H = \int f \ln f d\xi, \quad (37)$$

where the molecular distribution function $f(\xi, \mathbf{x}, t)$ is a function of velocity ξ as well as $\mathbf{x}$ and t . In equilibrium, f is Maxwellian and

$$H_0 = -R^{-1}\hat{\eta}_0 \quad (38)$$

where R is the gas constant. For any other function f we have

$$H > -R^{-1}\eta \quad (39)$$

where $\hat{\eta}(\hat{e}, \varrho)$ is the usual thermodynamic function of the energy and mass density associated with f ,

$$\varrho = \int f d\xi, \quad \hat{e} = \int \frac{1}{2} \xi^2 f d\xi \quad (40)$$

From the Boltzmann equation, with appropriate boundary conditions at the interface, we obtain

$$\int H(\mathbf{x}, t) dV < \int H(\mathbf{x}, 0) dV. \quad (41)$$

For an initial state of equilibrium, we have from (38), (39),

$$\begin{aligned} \int \hat{\eta}(\mathbf{x}, t) dV &> - \int R H(\mathbf{x}, t) dV \\ &> - \int R H(\mathbf{x}, 0) dV = \int \hat{\eta}_0 dV. \end{aligned} \quad (42)$$

From the Boltzmann equation we obtain the same constraint on η used in proving the previous inequality. Only the primitive variables ϱ and $\hat{e}$ are obtained directly from f ; all other thermodynamic quantities are defined as the conventional functions of ϱ and $\hat{e}$. The fact that these definitions are arbitrary and generally without significance for large deviations from equilibrium does not affect the validity of the final inequality containing the unambiguous quantities $\mathcal{E}$ and V . The parameter p_0 is also unambiguous as the pressure in a state of equilibrium.

We have shown that the fluid potential, P_f , is an absolute minimum at equilibrium. For completeness, we give the second variation of P_f . If the variation is adiabatic, computation yields

$$\delta^2 P_f = \int [\nabla \cdot \delta \mathbf{x}]^2 c^2 dV, \quad (43)$$

where $c^2 = \partial p(\eta, \varrho) / \partial \varrho$ and the result is formally independent of ϕ . The second variation vanishes for any incompressible variation $\nabla \cdot (\delta \mathbf{x}) = 0$. When $\phi = 0$, it is easy to see that P_f is unaltered by incompressible variations even if finite.

4. VARIATION OF THE MAGNETIC POTENTIAL

In variational analysis it is conventional to define the variation δB of an independent function $B(\mathbf{x})$ as

$$\delta B = \left. \frac{\partial}{\partial \varepsilon} B(\mathbf{x}, \varepsilon) \right|_{\varepsilon=0},$$

where $B(\mathbf{x})$ is imbedded in the family of functions $B(\mathbf{x}, \varepsilon)$ so that $B(\mathbf{x}) = B(\mathbf{x}, 0)$. We extend this notation in not insisting that $\varepsilon = 0$ in (5.1). For the variation $\delta \mathbf{x}$ of the interface, we set

$$\delta \mathbf{x} = \frac{\partial}{\partial \varepsilon} \mathbf{x}(S, \varepsilon),$$

where S denotes a coordinate fixed to the surface as it varies.

Since $\nabla \cdot \delta \mathbf{B} = 0$, a vector potential exists with $\delta \mathbf{B} = \nabla \times \mathbf{A}$. The condition $B_n = 0$ further restricts δB , but the restriction on δB (which is complicated if the boundary is varied) is most easily written in terms of the potential: the condition $(\mathbf{A} + \mathbf{B} \times \delta \mathbf{x})_t = 0$, for example, will guarantee that $B_n = 0$. In the light of these remarks we see that the pre-Maxwell equations (2) may be interpreted as variational equations for a harmonic (i.e., irrotational and incompressible) vector field where t corresponds to ε , $\delta \mathbf{B} = \partial \mathbf{B} / \partial t$ to $\delta \mathbf{B}$, $\mathbf{u}$ to $\delta \mathbf{x}$, $\mathbf{E}$ to $-\mathbf{A}$, etc. We shall find it convenient to use time derivatives interchangeably with "variations." In particular, we shall use the "variational" formulas

$$\frac{d}{dt} \int_{V(t)} \psi(\mathbf{x}, t) dV = \int_{V(t)} \frac{\partial \psi}{\partial t} dV + \int_{S(t)} \psi \mathbf{u} \cdot d\mathbf{S} \quad (44)$$

$$\begin{aligned} \frac{d}{dt} \int_{S(t)} \mathbf{A}(\mathbf{x}, t) \cdot d\mathbf{S} &= \int_{S(t)} \left\{ \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \nabla \cdot \mathbf{A} \right\} \cdot d\mathbf{S} \\ &+ \oint_{C(t)} \mathbf{A} \times \mathbf{u} \cdot d\mathbf{x}; \end{aligned} \quad (45)$$

in (44) $S(t)$ is the boundary of $V(t)$, and in (45) $C(t)$ is the boundary of $S(t)$.

First let us consider the magnetic energy M of (6) as a domain functional. Applying (44), we have

$$\delta M = \int_{V_m} \mu^{-1} (\mathbf{B} \cdot \delta \mathbf{B}) dV + \int_{S_1} \frac{1}{2} \mu^{-1} B^2 \mathbf{u} \cdot d\mathbf{S} \quad (46)$$

The relevant potential is M^* defined by (12), for which we obtain after some manipulation the first variation

$$\delta M^* = - \oint_{S_1} \frac{1}{2} \mu^{-1} B^2 \mathbf{u} \cdot d\mathbf{S}. \quad (47)$$

The same formula is obtained for all variations of current, fluxes, and interface subject to the constraints (9). To complete the first variation of the vacuum potential (28), we obtain

$$\delta \int_{V_m} p_0 dV = \oint_{S_1} p_0 \mathbf{u} \cdot d\mathbf{S}. \quad (48)$$

In equilibrium, $B^2/2\mu = p_0$ on the interface, cf. (27), and on adding (47), (48) we conclude that $\delta P_m = 0$ and the potential is stationary. To calculate the second variation, we apply (45) to obtain

$$\begin{aligned} \delta^2 P_m = & - \oint \mu^{-1} \{ \mathbf{B} \cdot \dot{\mathbf{B}} + (\mathbf{u} \cdot \nabla) (\tfrac{1}{2} B^2) \} \mathbf{u} \cdot d\mathbf{S} \\ & - \oint \tfrac{1}{2} \mu^{-1} B^2 (\dot{\mathbf{u}} + \mathbf{u} \nabla \cdot \mathbf{u}) \cdot d\mathbf{S} \\ & + \oint \{ p_0 \dot{\mathbf{u}} + \mathbf{u} \nabla \cdot p_0 \mathbf{u} \} \cdot d\mathbf{S}. \end{aligned} \quad (49)$$

Under the constraint conditions (9) we can show that

$$\oint_{V_m} (\mathbf{B} \cdot \dot{\mathbf{B}}) \mathbf{u} \cdot d\mathbf{S} = - \int_{V_m} \dot{B}^2 dV. \quad (50)$$

Taking the unperturbed boundary ($\epsilon = 0$) in (49) we obtain

$$\delta^2 P_m = \int_{V_m} \mu^{-1} B^2 dV + \oint_{S_1} \frac{\partial}{\partial n} (p_0 - \tfrac{1}{2} \mu^{-1} B^2) u_n^2 dS \quad (51)$$

where we have interpreted the operator $\mathbf{u} \cdot \nabla$ on $p_0 - B^2/2\mu$ as a normal derivative since $p_0 - B^2/2\mu$ is constant on the equilibrium interface. The constraints are not visible in (51) but are implicit in the class of vector fields $\dot{\mathbf{B}}$ which are to be inserted in the formula.

The first term in $\delta^2 P_m$ is always positive, hence stabilizing. The second term is stabilizing if $(B^2/2\mu) - p_0$ increases going into the vacuum domain (the positive normal in (51) is directed outwards). This result may also be stated in terms of ϕ ; we set

$$\begin{aligned} Q(S) &= \frac{\partial}{\partial n} (p_0 - \tfrac{1}{2} \mu^{-1} B^2) \\ &= - \frac{\partial}{\partial n} (\tfrac{1}{2} \mu^{-1} B^2) - \varrho \frac{\partial \phi}{\partial n}. \end{aligned} \quad (52)$$

Since the fluid variation $\delta^2 P_f$ is non-negative, we have stability if Q is positive on the whole interface. If Q is anywhere negative, the question seems open to doubt because the other terms compete. However, the vitally significant fact about the second variation is that the sign of $Q(S)$ alone determines the question of stability. This follows from the fact that one can always find a *localized* special variation, i.e., a special deformation of the interface, $u_n(S)$, confined to an arbitrarily small part of the surface and zero elsewhere, which assigns a value to the first term of (51) smaller than the absolute magnitude of the second term. Consequently, if $Q(S)$ is *anywhere* negative a variation exists which makes $\delta^2 P_m$ negative. Furthermore, the special variation can be chosen to make $\int u_n dS = 0$, so that an incompressible fluid variation can be fitted to it and, hence, $\delta^2 P_f = 0$. Summarizing, we have

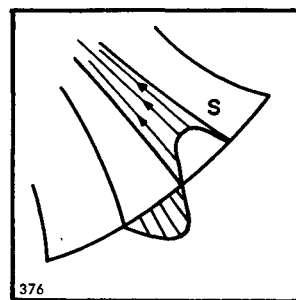


Figure 3

Theorem 1: A sufficient condition for stability, by the criterion of the second variation, is that $Q(S) > 0$ for all S . A necessary condition is $Q(S) \geq 0$ for all S ; i.e., a sufficient condition for instability is $Q(S) < 0$ anywhere on S .

The explicit construction of the special variation mentioned above is too lengthy to be given here. However, it is easily described intuitively. In some instances there exists a special variation in which $\mathbf{B}$ is unchanged at every point and hence $\int \dot{B}^2 dV = 0$. This can be accomplished by a perturbation of the interface which follows a flux tube of the original magnetic field (hence leaves $\dot{\mathbf{B}} = 0$) and leaves the volume unchanged (see Fig. 3). This possibility makes use of the fact that the interface is known to be analytic and $\mathbf{B}$ can be continued across the surface. In such a perturbation all terms except $Q(S)$ are zero. Such a variation does not always exist. The perturbation must follow a flux tube and to specify it at one point of a magnetic line determines it for the entire line. In an axially symmetric geometry where the lines extend infinitely in both directions the construction is possible. On a closed surface such as a torus the variation cannot be extended indefinitely along a magnetic line unless the geometry is so special that the lines close. We may take such a variation and "taper" it gradually to zero along the length of the line. This will not make the first term of $\delta^2 P_m$ zero, but will make it arbitrarily small compared to the second term.

The local nature of this stability criterion is a property of the free-boundary equilibrium alone (see Section 10).

5. STABLE AND UNSTABLE CONFIGURATIONS

The second variation condition for stability has a simple geometrical interpretation. For a vacuum magnetic field, $\nabla(\tfrac{1}{2} B^2) = (\mathbf{B} \cdot \nabla) \mathbf{B}$. Consequently, at a point on the interface,

$$\frac{\partial}{\partial n} (\tfrac{1}{2} B^2) = \mathbf{n} \cdot (\mathbf{B} \cdot \nabla) \mathbf{B} = \kappa B^2$$

where κ is the curvature of the magnetic line in the interface which passes through the point in question. In the case $\phi = 0$, we note that the principal normal to a magnetic line on the interface is normal to the surface, hence the magnetic line is a geodesic. The

sign of κ is positive if the center of curvature of the magnetic line is toward the fluid, and negative otherwise. When $\phi = 0$ we can restate Theorem 1 as

Theorem 2: When $\phi = 0$, a sufficient condition for stability (positive second variation) is that the center of curvature of the magnetic lines on the interface lies always toward the vacuum; sufficient for instability is a reverse curvature anywhere.

An immediate consequence of Theorem 2 is

Theorem 3: When $\phi = 0$, there exists no bounded stable fluid configuration separated from a magnetic field by an everywhere smooth interface.

The proof is simple. Consider a "support plane" which is tangential to the boundary and leaves the entire fluid to one side. The magnetic line which passes through the point of tangency has its center of curvature on the fluid side assuring instability.

This theorem is relevant only for a fluid domain bounded by a topological torus. A non-singular tangential direction field can exist on no other kind of surface, and a singularity of the tangential magnetic field would violate the boundary condition $B^2 = 2\mu p_0 = \text{constant}$.

The term $\rho \partial \phi / \partial n$ in (52) is the normal component of the external force (per unit volume) at the interface. If we allow an external force field to act under the conditions of Theorem 3, then stability can be obtained only if this force is directed inward at every point where the tangent plane is a plane of support. However, if one could find such an inward directed force, there would be no need for the magnetic field as a containing mechanism.

The curvature criterion shows the inherent instability of the pinch geometry, in which a plasma cylinder and surrounding field are enveloped in a concentric cylindrical conductor (Fig. 4a).

The curvature of the magnetic lines is clearly in the unstable direction for any helical or azimuthal field. The only exception is the limiting case of a purely axial field (Fig. 4b) for which the curvature is zero. The axial configuration is neutral since there do exist perturbations not affecting the magnetic potential. Such a situation is unstable by the criterion that large deviations from equilibrium can arise in the course of time. However, such neutral perturbations must be cylindrical, i.e., infinite in extent with straight magnetic lines. For any finite perturbation, the magnetic potential can be shown to increase. We therefore consider this configuration to be stable.

The inversion of the pinch configuration consisting of a cylindrical tunnel in the fluid created by a current-

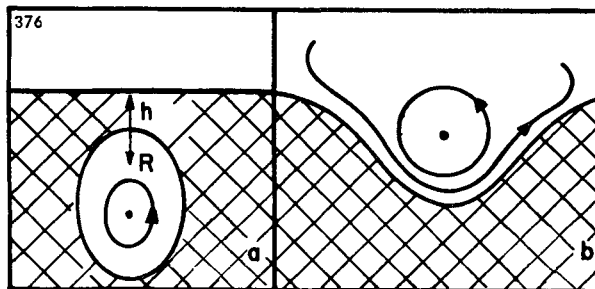


Figure 5

carrying wire is stable (Fig. 4c). Evidently the hole created within a conducting fluid by a wire of almost any shape will be stable. In the presence of a gravity potential ϕ the equilibrium shape of the hole will be altered. The stability criterion is now that the radius of curvature R of the magnetic line at each point of the interspace be less than the depth h below the surface. If the wire is brought too close to the surface, the top of the hole becomes unstable and the configuration presumably shifts to the one of Fig. 5b.

If accelerations are present, we should expect the interpretation of the local and instantaneous acceleration as an external force field to give qualitatively correct insights. For example, in a radially oscillating pinched discharge one would expect an inward acceleration to be destabilizing, an outward one stabilizing. Roughly, the effect would be destabilizing while the plasma is largest and most stabilizing when the compression is greatest.

Interesting configurations arise if more than one wire is inserted in the fluid. Two neighboring parallel wires carrying opposite currents are depicted in Fig. 6a. A very interesting configuration arises from four such wires, alternating in current direction (Fig. 6b). A separated *cusped* configuration appears in the center. By removing the surrounding unconnected fluid we obtain a stable configuration of finite extent (two-dimensionally). This does not contradict Theorem 3 because the boundary is not smooth; no tangent plane is a plane of support. From the direction of the magnetic field it is clear that the "destabilizing curvature" is not merely concentrated at the cusps. This and similar two-dimensional configurations can be found by conformal mapping; a special case is the familiar hypocycloid $x^{2/3} + y^{2/3} = 1$. A diverse family of related two- and three-dimensional stable cusped geometries is treated in an accompanying paper.⁵

6. MAGNETIC VARIATION IN THE LARGE

From Theorem 1, one would venture to guess that the total potential $P = P_m + P_f$ is a minimum in the large if the quantity $B^2/2\mu - p_0$ is positive in the entire vacuum region; the quantity is zero on the interface and the second variation condition is that it increase toward the vacuum. We shall now present a precise formulation of this condition. Here, for simplicity, we treat only the case $B_n = 0$.

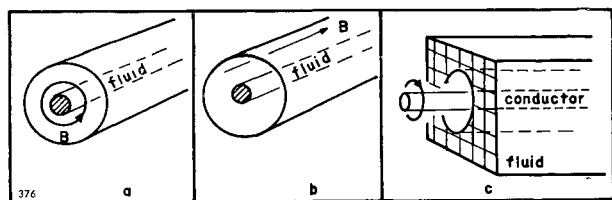


Figure 4

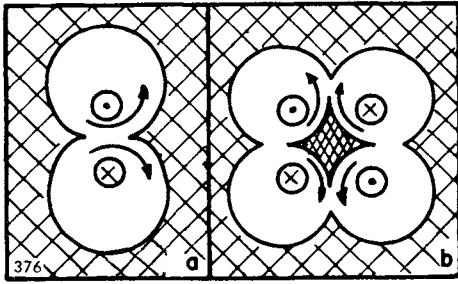


Figure 6

Let $V = V_m + V_f$ denote the entire fixed domain. An *admissible configuration* is any domain V_f given with the entire fluid state $\rho(x)$, $p(x)$, etc., where the fluid state is arbitrary except that it is assumed to be *compatible* with the original equilibrium: the same total mass, a total entropy which is no smaller, or, for an incompressible fluid, the same volume. The magnetic field corresponding to this configuration is uniquely defined by the domain V_m and the magnetic constraints $B_n = 0$ and (9). We further restrict admissible configurations to deformations which do not change the topology of the equilibrium configuration. Within this class we further restrict our attention to the accessible configurations defined below. Remembering that p_0 is extended into the vacuum domain by (25), we state

Theorem 4: For an equilibrium configuration with the property that $B^2/2\mu - p_0$ is positive in some domain $\bar{V}$ which encloses the equilibrium fluid domain V_f^0 , the equilibrium potential $P^0 = P_m^0 + P_f^0$ is a minimum compared to the potential of any accessible configuration for which the fluid domain V_f' is contained within $\bar{V}$. If, on the contrary, there exists a domain contiguous to V_f^0 for which $B^2/2\mu - p_0$ is negative, then P^0 is not a minimum.

The intuitive significance of the Theorem is exemplified by the cusp geometry (Fig. 6b). The value of $B^2/2\mu$ rises from the value p_0 on the central fluid domain, reaches a maximum, and drops back to p_0 at the outer fluid boundary. The question of stability with respect to finite displacements is very similar to that for the container of water depicted in Fig. 2b.

We shall only outline the proof. The second part of the Theorem (necessity) is almost identical to that for the like part of Theorem 2. The first part (sufficiency) is inherently more difficult, since, as is clear in the cusp example, there may be states neighboring the equilibrium for which $\partial(B^2/2\mu - p_0)/\partial n < 0$.

We effect the comparison between V_f^0 and V_f' by introducing intermediate stages. Let V_+ denote that part of V_f^0 not covered by V_f' , and V_- that part of V_f' not covered by V_f^0 :

$$V_+ = V_m' \cap V_f^0; \quad V_- = V_m^0 \cap V_f'. \quad (53)$$

Let S_+ denote that part of the equilibrium interface in common with V_+ . On S_+ , $\mathbf{B}^0 = \nabla \psi$ is a tangential direction field. We assume and later verify that

ψ can be extended continuously and differentially into V_+ in such a way that, setting $\mathbf{X} = \nabla \psi$, we have

$$X^2/2\mu - p_0 \leq 0, \text{ in } V_+. \quad (54)$$

The definition of $\mathbf{X}$ is extended into V_m^0 by setting $\mathbf{X} = \mathbf{B}_0$.

Since

$$\int_{V_+} [X^2/2\mu - p_0] dV \leq 0$$

and

$$\int_{V_-} [(B^0)^2/2\mu - p_0] dV \geq 0,$$

we have

$$\int_{V_m^0} [\frac{1}{2}\mu^{-1}(B^0)^2 - p_0] dV \geq \int_{V_m'} [\frac{1}{2}\mu^{-1}X^2 - p_0] dV. \quad (55)$$

If this construction of $\mathbf{X}$ is possible, the configuration $V_f' + V_m'$ is said to be *accessible*. This inequality is valid for accessible configurations. It is conjectured that in three dimensions all admissible configurations with V_f in $\bar{V}$ are accessible. The conjecture has been completely proved only for two-dimensional or axially symmetric equilibrium configurations. In three dimensions, a certain domain between V_f^0 and $\bar{V}$ has been shown to be accessible.

Let α denote an index of range 1, ..., m ; β an index $m+1$, ..., n . The constant constraints (9) are the currents I_α and fluxes Φ_β . From (6), (12), (29) we obtain (55) in the form

$$\int_{V_m'} \frac{1}{2}\mu^{-1}X^2 dV - \sum I_\beta^0 \Phi_\beta^0 \leq \int_{V_m'} p_0 dV - P_m^0. \quad (56)$$

The magnetic fields $\mathbf{B}^0$ and $\mathbf{B}'$ satisfy the same constraints $I_\alpha^0 = I_\beta'$, $\Phi_\beta^0 = \Phi_\beta'$. By its construction, $\mathbf{X}$ has the same constant current constraints I_α^0 as $\mathbf{B}^0$; fluxes are not defined for $\mathbf{X}$ since it satisfies $\nabla \times \mathbf{X} = 0$ but not $\nabla \cdot \mathbf{X} = 0$. Consider the functional

$$\int_{V_m'} \frac{1}{2}\mu^{-1}\mathbf{X}^2 dV - \Phi_\beta^0 I_\beta[\mathbf{X}], \quad (57)$$

where

$$I_\beta[\mathbf{X}] = \mu^{-1} \oint_{\Gamma_\beta} \mathbf{X} \cdot d\mathbf{x}, \quad (58)$$

to be defined on the class of all irrotational vectors $\mathbf{X}$ on V_m' which have the given currents I_α^0 ; it is minimized on this class by the vector $\mathbf{B}'$.⁴ Noting that $\Phi_\beta^0 = \Phi_\beta'$, and that $M^* = \sum \frac{1}{2} I_\beta \Phi_\beta - \sum \frac{1}{2} I_\alpha \Phi_\alpha$, we conclude for the particular $\mathbf{X}$ constructed above,

$$\begin{aligned} & \int_{V_m'} \frac{1}{2}\mu^{-1}\mathbf{X}^2 dV - \sum \Phi_\beta^0 I_\beta^0 \\ & \geq \int_{V_m'} \frac{1}{2}\mu^{-1}B'^2 dV - \Phi_\beta' I_\beta' = -M^*[\mathbf{B}]. \end{aligned} \quad (59)$$

Comparison with (56) yields the desired result,

$$M^*[\mathbf{B}'] + \int_{V_m} p_0 dV = P_m[\mathbf{B}'] \geq P_m[\mathbf{B}^0]. \quad (60)$$

7. VARIATION IN TERMS OF INDUCTANCE

There is one class of equilibrium configurations for which stability analysis in the large is particularly easy and even reducible to finite dimensional terms. A first restriction is that $B_n = 0$, $\phi = 0$. For simplicity, we assume also that the fluid is incompressible. The potential M^* is an explicit function of fluxes, currents and inductance; the dependence on the domain is reduced to a dependence on the finite number of variables L_{rs} . Varying (7) and (6), we obtain

$$\dot{\Phi}_r = \sum (\dot{L}_{rs} I_s + L_{rs} \dot{I}_s), \quad \dot{I}_r = \sum (\dot{\lambda}_{rs} \Phi_s + \lambda_{rs} \dot{\Phi}_s) \quad (61)$$

and

$$\begin{aligned} \dot{M} &= \sum_{r,s} (I_r L_{rs} \dot{I}_s + \frac{1}{2} I_r \dot{L}_{rs} I_s) \\ &= \sum_{r,s} (\Phi_r \dot{\lambda}_{rs} \dot{\Phi}_s + \frac{1}{2} \dot{\Phi}_r \dot{\lambda}_{rs} \Phi_s). \end{aligned} \quad (62)$$

Taking $\sum \dot{I}_r \Phi_r \pm \sum I_r \dot{\Phi}_r$ from (61) and combining with (62), we obtain

$$\begin{aligned} - \sum_{r,s} \frac{1}{2} I_r \dot{L}_{rs} I_s &= \sum_{r,s} \frac{1}{2} \dot{\Phi}_r \dot{\lambda}_{rs} \Phi_s \\ &= \sum \frac{1}{2} (\dot{I}_r \Phi_r - I_r \dot{\Phi}_r) = - \int \frac{1}{2} \mu^{-1} B^2 \mathbf{u} \cdot d\mathbf{S}, \end{aligned} \quad (63)$$

where the last equality derives from (9), (13), (14). All these expressions must then equal the first variation of M^* under the constraints (9). If a given configuration has the property that each component L_{rs} of the inductance matrix is stationary for an arbitrary incompressible variation of the domain, it follows that M^* is stationary and the configuration is in equilibrium. This is a special type of equilibrium configuration which remains so independent of the values of the fluxes and currents; the boundary condition, $B^2 = \text{constant}$, is satisfied in every case. One such vacuum domain is the region between two concentric cylinders. For a purely axial field, an azimuthal field, or any helical combination of the two, B^2 is constant on each boundary component. Each of the entries of the inductance matrix is stationary for incompressible variation of either boundary. Equilibrium for such domains is a purely geometrical property and we shall find that stability also is purely geometrical. We call such equilibria *universal*.

The second variation of M^* for a universal equilibrium is simply calculated,

$$\delta^2 M^* = - \sum \frac{1}{2} I_r \ddot{L}_{rs} I_s. \quad (64)$$

If $\ddot{L}_{rs}$ is negative definite, then not only is the equilibrium stable, but it is stable no matter how the magnetic field is inserted into the domain. For example, take the domain between two concentric cylinders with the inner cylinder a fixed conductor and the outer cylinder the fluid interface (Fig. 4c).

This configuration is stable for any pitch of the helical magnetic field.

For the analysis of stability in the large, we introduce a partial ordering in the space of symmetric matrices. If the matrix $L'_{rs} - L_{rs}$ is positive definite we write $L_{rs} < L'_{rs}$. In this context the meaning of a maximum on a given set of matrices is clear. Associated with the class of admissible domains V_m' there is a class of inductance matrices L_{rs}' which is represented by a point set in the $n(n+1)/2$ -dimensional space of matrix elements. We prove two theorems:

Theorem 5: If a universal equilibrium is stable for one species of constraints, it is stable for all.

Theorem 6: A necessary and sufficient condition that a universal equilibrium be stable is that the inductance be a maximum.

The first theorem is an instance of the results of Section 7, but a direct proof in this special case is much simpler. Suppose, for example, that stability holds for constant flux constraints; i.e., for all Φ .

$$M'(\Phi) = \sum_{r,s} \frac{1}{2} \Phi_r \lambda'_{rs} \Phi_s > M^0(\Phi) = \sum_{r,s} \frac{1}{2} \Phi_r \lambda^0_{rs} \Phi_s. \quad (65)$$

We wish to prove stability under other constraints, i.e.,

$$\begin{aligned} M_*' &= \sum (\frac{1}{2} I_\beta' \Phi_\beta^0 - \frac{1}{2} I_\alpha^0 \Phi_\alpha') > M_*^0 \\ &= \sum (\frac{1}{2} I_\beta^0 \Phi_\beta^0 - I_\alpha^0 \Phi_\alpha^0). \end{aligned}$$

From (65), we have

$$\begin{aligned} M_*'(\Phi', I') &= M'(\Phi') - I_\alpha' \Phi_\alpha' > M^0(\Phi') \\ &- \sum I_\alpha' \Phi_\alpha' = M^0(\Phi_\alpha', \Phi_\beta^0) - \sum I_\alpha^0 \Phi_\alpha'. \end{aligned} \quad (66)$$

With λ_{rs}^0 fixed, $M^0(\Phi)$ is convex in the variables Φ_r . Convexity follows also for $M^0(\Phi_\alpha', \Phi_\beta^0) - \sum I_\alpha^0 \Phi_\alpha'$, where the equilibrium values I_α^0, Φ_β^0 are fixed; this implies a unique minimum at $\Phi_\alpha = \Phi_\alpha^0$; i.e.

$$\begin{aligned} M^0(\Phi_\alpha', \Phi_\beta^0) - \sum I_\alpha^0 \Phi_\alpha' &> M^0(\Phi^0) \\ &- \sum I_\alpha^0 \Phi_\alpha^0 = M_*^0(\Phi^0, I^0). \end{aligned}$$

With (66), this completes the demonstration. The converse is easily demonstrated by constructing the appropriate inequality for stability under the mixed constraints condition and proving stability for constant fluxes.

In Theorem 6, a maximum inductance yields stability for constant current constraints directly from $M^* = -M = - \sum_{r,s} \frac{1}{2} I_r L_{rs} I_s$ and, hence, for all other constraints by Theorem 5. Conversely, stability for any species of constraints implies the same for constant currents and, from the form of M^* , that L_{rs} is a maximum.

8. VIBRATIONAL STABILITY

Here we consider only an irrotational incompressible fluid flow and stipulate that all the circulations

(which are constants of the motion) are zero. No external potential is superposed. This problem is mathematically elegant because both the fluid velocity and magnetic field are harmonic vectors and the analysis involves only the interface. The fluid equations are given in terms of a velocity potential ϕ ,

$$\Delta \phi = 0, \quad v = \nabla \phi, \quad \frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + p/\rho = 0.$$

At the boundary, setting $p = B^2/2\mu$, we have

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{B_0^2}{2\mu\rho} = 0. \quad (67)$$

In (67) we take the Lagrangian derivative $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ following the fluid and linearize conventionally to obtain

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{\mu\rho} \mathbf{B}_0 \cdot \frac{\partial \mathbf{B}}{\partial t} + v_n \frac{\partial}{\partial n} \left(\frac{B_0^2}{2\mu\rho} \right) = 0 \quad (68)$$

where $\mathbf{B}_0$ is the magnetic field at the unperturbed interface and the condition (68) is assumed to hold on the unperturbed boundary. Separating time in the form $f(x,t) = e^{i\omega t} \bar{f}(x)$ we then drop the bar and obtain in (9.2),

$$-\omega^2 \phi + \mu\rho^{-1} \mathbf{B}_0 \cdot \frac{\partial \mathbf{B}}{\partial t} + v_n \frac{\partial}{\partial n} \left(\frac{B_0^2}{2\mu\rho} \right) = 0. \quad (69)$$

We choose not to write $\partial B/\partial t$ as $i\omega B$ since it is related to v_n by a time-independent boundary condition. The boundary condition (69) expressed in terms of ϕ and $\partial\phi/\partial n = v_n$ presumably serves to determine the harmonic function $\phi(x)$.

For the variational formulation, we write, integrating by parts

$$K = \int_{V_f} \frac{1}{2}\rho v^2 dV = \int_{V_f} \frac{1}{2}\rho (\nabla \phi)^2 dV = \frac{1}{2}\rho \int_{S_f} \phi v_n dS.$$

There exists a symmetric positive definite integral operator relating ϕ and $v_n = \partial\phi/\partial n$ on the surface,

$$\phi(S) = \int L_f(S, S') v_n(S') dS'$$

and, hence,

$$K = \frac{1}{2}\rho \int v_n(S) L_f(S, S') v_n(S') dS dS'. \quad (70)$$

From (50) and (51) the second variation of M^* is

$$U^* = \delta^2 M^* = \int_{S_f} \mu^{-1} \mathbf{B}_0 \cdot \dot{\mathbf{B}} u_n dS + \int_{S_f} \frac{\partial}{\partial n} \left(\frac{B_0^2}{2\mu} \right) u_n^2 dS, \quad (71)$$

where the fluid normal is taken positive and the variation u_n may be identified with the velocity v_n . In (71) we introduce the linear operator,

$$\mathbf{B}_0 \cdot \dot{\mathbf{B}} = \int L_m(S, S') u_n(S') dS.$$

Again, it can be shown that L_m is symmetric positive definite. We consider K and U^* in (70) and (71) as quadratic functionals of the surface function $u_n(S)$ and vary the ratio U^*/K or, equivalently, introduce a Lagrangian multiplier and take the variation

$$\delta(U^* - \lambda K) = \int \left[\mu^{-2} \mathbf{B}_0 \cdot \dot{\mathbf{B}} + \mu^{-1} u_n \frac{\partial B_0^2}{\partial n} - \lambda \rho \phi \right] \delta u_n dS.$$

Since $\int \delta u_n dS = 0$, it follows that $U^* - \lambda K$ is stationary if the bracket is constant. This constant can be absorbed in ϕ as an additive time function to make the bracket vanish. This is exactly the boundary condition (69) provided that

$$\omega^2 = \frac{1}{2}\lambda.$$

In that event, $U^* = \lambda K$. The eigenvalues of U^* , with respect to K as unit form, are essentially the vibration frequencies. If U^* is positive then the lowest frequency is given by the minimum of U^*/K . If U^* can be made negative at all, then it can be shown that U^*/K can be made arbitrarily large negative and no minimum exists. This formal analysis, of course, needs to be supplemented by an investigation of the spectrum of the operator.

Some analysis of this kind has been made in the compressible case.³

9. A STABILITY CRITERION WITH VOLUME CURRENTS

One of the main features of equilibrium configurations in which a sheet current separates a field-free fluid from a vacuum magnetic field is the entirely local character of the stability criterion (Theorem 1). The violation anywhere of a certain inequality makes the whole system unstable.

The situation is quite different for a configuration of mixed fluid and magnetic field with distributed volume current. In this case, we shall show that a sufficiently localized disturbance is always stable. This means that instabilities can develop only because of cooperative behavior involving an appreciable part of the system.

The criterion is formulated in terms of the quantities

$$L = \int_{\mathbf{B} \text{ line}} \frac{d\mathbf{s}}{|\mathbf{B}|} \text{ and } \frac{dB}{dx} (\geq 0) \text{ where } \left(\frac{dB}{dx} \right)^2 \text{ is the}$$

largest eigenvalue of the matrix $\left(\frac{\partial \mathbf{B}}{\partial x_i} \cdot \frac{\partial \mathbf{B}}{\partial x_j} \right)$. The pro-

duct of L with a representative value of B represents a weighted length of the $\mathbf{B}$ line where the weighting factor may be thought of as the volume per unit length of a flux tube following the line. Both the quantity dB/dx and the current density $\mathbf{j}$ contribute to the rate of change of $\mathbf{B}$. We prove

Theorem 7: For an incompressible fluid, an equilibrium configuration is stable against all disturbances confined to a region R which contains no closed $\mathbf{B}$ line, if J does not vanish identically in any subregion of positive volume, and if along each $\mathbf{B}$ line within R the inequality

$$L \max \{ \mu J + dB/dx \} < \pi \quad (72)$$

is satisfied.

The criterion is sufficient for stability but of course stability may well occur when the criterion is violated. A similar theorem holds for compressible fluids. The inequality (72) can be interpreted as a requirement that the weighted length of each $\mathbf{B}$ line in the disturbance region is to be less than the distance in which the fractional change in $\mathbf{B}$ is appreciable.

We establish Theorem 7 by deriving positive definiteness of the second variation from (72); namely, that

$$\int_R \dot{\mathbf{B}}^2 dV + \mu \int_R ([\dot{\mathbf{B}} \times \mathbf{J}] \cdot \mathbf{u}) dV > 0 \quad (73)$$

for arbitrary continuous perturbations $\mathbf{u}$ which vanish identically outside R and which differ from zero in at least one point of the region.

Using the electromagnetic equations (2) (omitting the restriction $\mathbf{J} = 0$) and the relations

$$\begin{aligned} 2\mu \left| \int_R [\dot{\mathbf{B}} \times \mathbf{J}] \cdot \mathbf{u} dV \right| &\leq \int_R \dot{B}^2 dV \\ &\quad + \mu^2 \int_R (\mathbf{u} \times \mathbf{J})^2 dV, \\ [\mathbf{u} \cdot \nabla \mathbf{B}]^2 &= \sum_{i,j=1}^3 \left(\frac{\partial \mathbf{B}}{\partial x_i} \cdot \frac{\partial \mathbf{B}}{\partial x_j} \right) u_i u_j, \\ \int_a^b u_s^2 ds &\geq \frac{\pi^2}{b^2 - a^2} \int_a^b \mathbf{u}^2 ds, \end{aligned} \quad (74)$$

we obtain the general inequality which yields the theorem; namely,

$$\begin{aligned} &\int_R \dot{B}^2 dV + \mu \int_R [\dot{\mathbf{B}} \times \mathbf{J}] \cdot \mathbf{u} dV \\ &\geq \frac{\pi^2}{2} \int_R \frac{\mu J}{\mu J + \frac{dB}{dx}} \left\{ \frac{1}{[L(x, y, z)]^2} \right. \\ &\quad \left. - \frac{1}{\pi^2} \left(\mu J + \frac{dB}{dx} \right)^2 \right\} u^2 dV. \end{aligned} \quad (75)$$

The stability criterion of Theorem 7 can be applied to any sort of device in which all $\mathbf{B}$ lines begin and end on conducting plates. If inequality (72) is satisfied, then the device is stable.

As an application we treat the stability of magnetic mirror configurations with conducting end-plates, that is, such axially symmetric configurations as have

flux tubes like those depicted in Fig. 7. A class of equilibrium configurations having this form is given by the equations

$$\begin{aligned} B_x &= B_0 (1 + \lambda r^2) - k \cos \frac{\pi z}{a} J_0 \left(\frac{i\pi r}{a} \right), \\ B_r &= i k \sin \frac{\pi z}{a} J_1 \left(\frac{i\pi r}{a} \right) \\ p &= \frac{1}{2} B_0^2 \lambda [R^2 (2 + \lambda R^2) - r^2 (2 + \lambda r^2)] \\ &\quad - \frac{2\lambda k a B_0}{i\pi} \left[R J_1 \left(\frac{i\pi R}{a} \right) - r J_1 \left(\frac{i\pi r}{a} \right) \cos \frac{\pi z}{a} \right], \end{aligned} \quad (76)$$

containing the parameters B_0, λ, k, a, R .

Three physical parameters may be varied: β (the ratio of plasma energy to magnetic energy), the average mirror ratio (the ratio of maximum to minimum of B on a $\mathbf{B}$ line averaged over the machine) and the aspect ratio (the ratio of the midplane diameter to axial length). We have prepared a table of stable configurations on the IBM 704 at the Institute of Mathematical Sciences according to the criterion (72) applied to the equilibria (76). The table shows that, for a fixed aspect ratio, an increase in the average mirror ratio will decrease the maximum β for which the stability criterion is satisfied. However, if the mirror ratio is held fixed, then there is an optimum aspect ratio for achieving the highest β compatible with the stability criterion; these values are plotted in Fig. 8 where the maximum values of β are indicated.

10. STABILIZATION WITH ALTERNATING FIELDS

We consider the problem of stabilizing a pinched plasma by applying alternating external magnetic fields. We suppose that the external circuits are so arranged that at each point of the plasma boundary surface, the magnetic field is of constant magnitude and rotating with constant angular velocity. The specific case we treat is the simplest configuration which displays significant features of the problem and yet is susceptible to analysis. It is the case of a perfectly conducting fluid supported against gravity at a horizontal plane interface by a vacuum magnetic field parallel to the interface. If the field is constant, this equilibrium position is unstable.² When the magnetic field is uniform in space and has constant

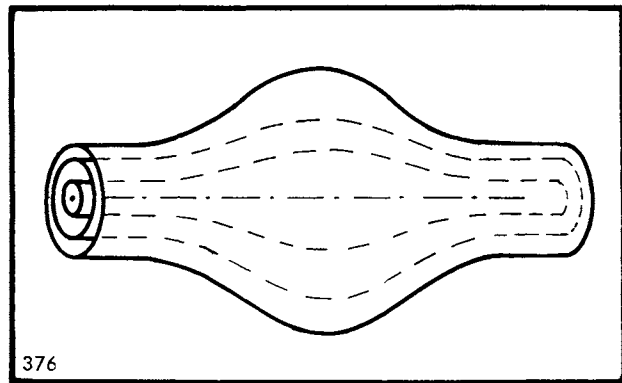


Figure 7

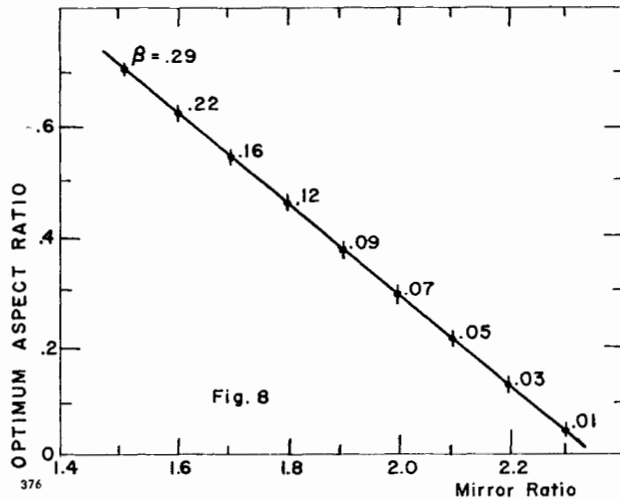


Figure 8

magnitude, we shall see that rotation about a vertical axis with constant angular velocity has a stabilizing effect.

We introduce rectangular coordinates with the y axis vertical and the horizontal x, z plane representing the plane of the interface. In equilibrium, the fluid is at rest in the half-space $y > 0$, the hydrostatic pressure due to the gravity potential gy is counterbalanced at the interface by the magnetic pressure from below. For the sake of mathematical simplicity, we assume the fluid is incompressible.

We make the assumptions customary in this kind of infinitesimal stability analysis. We imagine the perturbed interface $y = f(x, z, t)$ given at time $t = 0$; the problem is to determine the subsequent motion.

In the fluid, the velocity u is derivable from a velocity potential $\phi(x, y, z, t)$ by the relation $u = \nabla\phi$. At $y = +\infty$ we assume that both u and the perturbation in the pressure p vanish. The pressure p is determined from Bernoulli's law

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} + gy = C(t). \quad (76)$$

At $y = -\infty$ we assume the magnetic field is equal to the unperturbed field $B_0(t)$. The direction of $B_0(t)$ as a function of time will be prescribed later. At the interface, we have $B_n = 0$ and $B^2/2\mu = p$. The determination of B can be reduced to a scalar potential problem by observing that B can be written as $B = B_0(t) + \nabla\psi$, where $\Delta\psi = 0$.

The problem is linearized by neglecting the u^2 term in Bernoulli's law and the boundary conditions are imposed not at the actual interface but at the unperturbed surface $y = 0$. Recall that the interface $y = f(x, z, t)$ should not be prescribed, for that would fix the potentials ϕ and ψ at once.

Consider a component of the Fourier expansion of $y = f(x, z, t)$, say $y = r(t) \sin kx$. For this component we may use the method of separation of variables to find explicit formulas for ϕ and ψ ; these, when put

into the linearized pressure equation at the mean interface $y = 0$, yield a differential equation for $r(t)$

$$\ddot{r} + \left(\frac{B_0^2 k^2}{2\rho\mu} \cos^2 \theta(t) - gk \right) r = 0, \quad (77)$$

where $\theta(t)$ is the angle $B_0(t)$ makes with the x axis. This includes the static case where $\theta(t)$ is constant. For $\theta(t) = \pi/2$ we have the well-known result that the growth constant of these waves is $(gk)^{1/2}$.

In the dynamic case, we take constant angular velocity $\theta = \omega t$. Equation (77) then takes the form of Mathieu's equation

$$d^2 r / d\theta^2 + (a + b \cos 2\theta) r = 0, \quad (78)$$

where

$$a = \frac{1}{2\omega} \left(\frac{B_0^2 k^2}{2\rho\mu} - gk \right), \quad b = \frac{1}{2\omega} \frac{B_0^2 k^2}{2\rho\mu}. \quad (79)$$

To investigate stability for a particular value of ω , one must study all values of k ; i.e., all points of the curve (79) in the (a, b) plane. The values of the parameters (a, b) which yield stability have been charted in part. Meixner and Schäfke⁶ give a stability chart for $|a|, |b| < 50$; for large values of a and b the solutions of (78) have been studied in very complete form by Langer.⁷ By using their results and methods we find that the application of periodic fields does not produce stability. However, the rate at which instabilities grow is controlled in the sense that the growth constant has an upper bound inde-

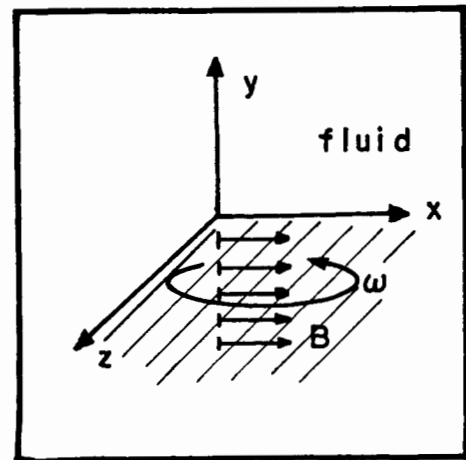


Figure 9

pendent of k . In contrast to the static case, where for large k the growth constant is proportional to $k^{1/2}$ and may be arbitrarily large, in the dynamic case for a small angular velocity ω , the growth constant for large k is at most $g(\rho\mu)^{1/2}/2B_0$. This may yet be too large for some practical purposes, but the application of alternating fields does produce a stabilizing tendency in the presence of acceleration.

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Hydromagnetic Equilibria and Force-Free Fields

By H. Grad and H. Rubin*

INTRODUCTION — ELEMENTARY PROPERTIES

The equations governing the equilibrium of a perfectly conducting fluid in the presence of a magnetic field are

$$\begin{aligned}\nabla p &= \mathbf{J} \times \mathbf{B} \\ \nabla \times \mathbf{B} &= \mu \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \\ \text{or } \nabla(p + B^2/2\mu) &= (\mathbf{B} \cdot \nabla) \mathbf{B} / \mu \\ \nabla \cdot \mathbf{B} &= 0,\end{aligned}\quad (1)$$

where p is the fluid pressure, $\mathbf{B}$ the magnetic field, and $\mathbf{J}$ the current density. These equations admit a large variety of solutions, i.e., of equilibrium configurations in which a conducting fluid is balanced by a magnetic field. It is our purpose to survey these possibilities with the expectation that, if a solution has been found to exist mathematically (and is, in addition, stable) it can actually be constructed by sufficient exercise of experimental ingenuity. In this context, it is extremely important to discover exactly what data should be specified in order to determine a solution uniquely.

In addition to certain general properties of these equations, we shall consider their solution in terms of arbitrary functions, the solution of well-posed boundary value problems, and several alternative formulations of the problem in terms of the calculus of variations. In principle, either the differential equations or the variational characterization can be taken as the definition of equilibrium; the two are only approximately equivalent. Moreover, a certain type of variational formulation (slightly different from that treated in this paper) can be instrumental in a stability analysis, which is essential to give physical meaning to an equilibrium configuration.¹

By inspection of (1), since $\mathbf{B}$ and $\mathbf{J}$ are perpendicular to ∇p , we see that p is constant on $\mathbf{B}$ lines and on $\mathbf{J}$ lines; equivalently, the $\mathbf{B}$ lines and $\mathbf{J}$ lines lie on constant p surfaces.

In the case of a unidirectional magnetic field (e.g., $B_z(x, y)$) the general solution is $p + B^2/2\mu = \text{constant}$. The fluid pressure is balanced by the

magnetic pressure; either p or B can be given arbitrarily and the other one "filled in".

The integral form of (1) is

$$\oint_S \{ (p + B^2/2\mu) \mathbf{n} - \frac{1}{\mu} \mathbf{B} B_n \} dS = 0, \quad \oint_S B_n dS = 0. \quad (2)$$

These equations are more fundamental than the differential equations (1). They are equivalent to (1) when the functions are smooth. In addition, at a discontinuity surface, the integral relations (2) imply that B_n vanishes and that $p + B^2/2\mu$ is continuous. A discontinuity surface must be a flux surface; otherwise the Maxwell magnetic stress tensor would not be compatible with a scalar pressure.

A FLUID-DYNAMICAL ANALOGUE

Using an appropriate identification of variables, equations (1) become identical to the equations of steady incompressible inviscid flow. Setting the fluid density equal to unity, these equations are

$$\begin{cases} (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p^* = 0 \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \quad (3)$$

where p^* denotes the fluid pressure. The identifications are $\mathbf{u} \sim \mathbf{B}/\sqrt{\mu}$ and $-p^* \sim p + B^2/2\mu$; the negative of the pressure p in the magnetic case then corresponds to the Bernoulli constant $p^* + u^2/2$. The interesting case is *rotational* flow, for which the vorticity $\nabla \times \mathbf{u} \neq 0$ corresponding to $\mathbf{J} \neq 0$.

It is interesting to consider the analogue of the fluid-dynamical *free boundary* or cavitation problem in which an irrotational flow is separated at a discontinuity surface (vortex sheet) from stagnant fluid or a cavity. The separation surface is determined by the extra boundary condition $|\mathbf{u}| = \text{constant}$. This is mathematically equivalent to a vacuum magnetic field ($\mathbf{J} = 0$) separated at an interface (current sheet) from a field-free conducting fluid. Since p is constant in the conducting fluid, we obtain the free boundary condition $|\mathbf{B}| = \text{constant}$.

Although the two problems are mathematically identical, it is perfectly possible for a significant fluid-dynamical problem to be uninteresting in the magnetic analogue (e.g., a jet from an orifice, Fig. 1a) and vice versa (e.g., the cusped equilibrium,² Fig. 1b).

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SOLUTION OF BOUNDARY VALUE PROBLEMS

The characteristics of a system of partial differential equations³ give immediate qualitative and possibly quantitative information concerning relevant initial value or boundary value problems. An elementary calculation yields four characteristics for the system (1). Two of the characteristics are purely imaginary as for the potential equation. Corresponding to these two characteristics, one would expect to prescribe a single scalar boundary value on the entire boundary of the domain, e.g., the normal component B_n of the magnetic field.

The remaining two characteristics are real; *viz.*, the $\mathbf{B}$ lines counted twice. Corresponding to each real characteristic, one would expect to be able to specify a single scalar quantity at one end of each $\mathbf{B}$ line. In a geometry as in Fig. 2 in which every $\mathbf{B}$ line

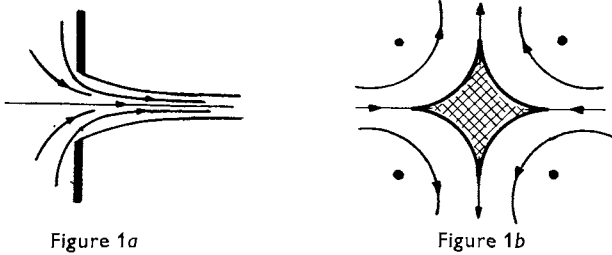


Figure 1a

Figure 1b

intersects each end of the tube, one would expect to give both additional quantities at one end or one at each end. For example, one may conjecture that the following specification of boundary values (in addition to B_n on the whole surface) would be appropriate where t, n designate tangential and normal components:

Problem A_1 : ϕ given on S_1 , ϕ given on S_2 (compatibly);

Problem A_2 : ϕ given on S_1 , J_n given on S_1 ;

Problem A_3 : ϕ given on S_1 , J_n given on S_2 ;

Problem A_4 : B_t given on S_1 (as well as B_n : the vector $\mathbf{B}$ is thus given).

In these problems, we prescribe ϕ on boundary surfaces in such a way that the ϕ lines are "simple" (not closed) as shown in Fig. 3. Moreover, in Problem A_1 , since ϕ is constant on $\mathbf{B}$ lines, matching magnetic flux on both ends imposes a simple compatibility requirement on ϕ . For example, if ϕ is given on S_1 and the lines of constant ϕ are specified on S_2 , this condition fixes the values of ϕ on these given lines.

An iteration scheme for Problems A_2 and A_3 offering likelihood of convergence is as follows:

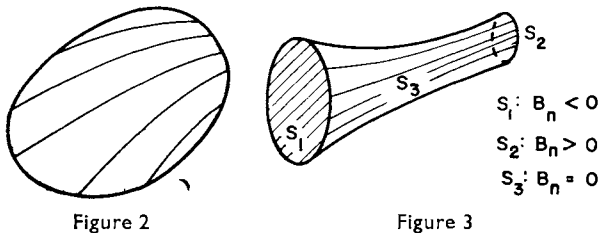


Figure 2

Figure 3

$$\begin{aligned} S_1: B_n &< 0 \\ S_2: B_n &> 0 \\ S_3: B_n &= 0 \end{aligned}$$

suppose that, at a certain stage of the iterations, we have a vector field $\mathbf{B}$ satisfying $\nabla \cdot \mathbf{B} = 0$ and the boundary condition for B_n . We find ϕ everywhere in the domain by carrying the boundary values of ϕ along these $\mathbf{B}$ lines. The component $\mathbf{J}_\perp$ perpendicular to $\mathbf{B}$ is then obtained in the domain from the equation $\nabla \cdot \phi = \mathbf{J} \times \mathbf{B}$. We write $\mathbf{J}_\parallel = \sigma \mathbf{B}$ for the parallel component and employ the requirement $\nabla \cdot \mathbf{J} = 0$ to obtain along each $\mathbf{B}$ line the ordinary differential equation

$$\mathbf{B} \cdot \nabla \sigma + \nabla \cdot \mathbf{J}_\perp = 0 \quad (4)$$

for σ . The given "initial" condition for J_n at one end allows this equation to be solved uniquely on each line so that $\mathbf{J}$ is determined everywhere. We now solve for a new $\mathbf{B}$ from $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{B} = \mathbf{J}$ and the boundary condition for B_n , and then we continue the iterations.

We next consider two-dimensional problems; *i.e.*, problems in which no quantity depends on z . There are several possibilities: (1) $\mathbf{B}$ can have the single component B_z and $\mathbf{J}$ the two components J_x and J_y ; (2) $\mathbf{B}$ can have the two components B_x and B_y and $\mathbf{J}$ the single component J_z ; (3) both $\mathbf{B}$ and $\mathbf{J}$ can be general (three-component) vectors depending on x and y alone.

The general solution in the first case has already been given explicitly, $\phi + B^2/2\mu = \text{constant}$. In the second case, the number of characteristics is three rather than four; the $\mathbf{B}$ lines are counted only once. The two-dimensional analogue to Problem A_1 will be considered later. Corresponding to Problems A_2 , A_3 and A_4 we have (see Fig. 4)

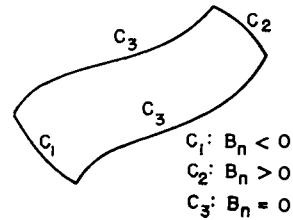


Figure 4

Problem B_1 : ϕ given on C_1 or C_2 ;

Problem B_2 : $\mathbf{B}$ given on C_1 or C_2 .

In the third (general two-dimensional) case, the characteristics are counted as in the full three-dimensional case. Correspondingly we list:

Problem C_2 : ϕ given on Γ_1 , $\mathbf{J}_n$ given on Γ_1 ;

Problem C_3 : ϕ given on Γ_1 , $\mathbf{J}_n$ given on Γ_2 ;

Problem C_4 : $\mathbf{B}$ given on Γ_1 (as above, the analogue to Problem A_1 is left to later).

The two-dimensional Problems C_2 and C_3 are equivalent.

With axial symmetry, the θ coordinate corresponds to the z coordinate of the two-dimensional case. We have the same three subdivisions: (1) B_θ, J_r, J_z ; (2) B_r, B_z, J_θ ; (3) $\mathbf{B}$ and $\mathbf{J}$ general (three-component) vectors depending on r and z alone. In the first case, B_θ and ϕ must be functions of r alone such that

$$\partial/\partial r(\phi + \frac{1}{2}\mu B_\theta^2) = -B_\theta^2/\mu r. \quad (5)$$

In the second and third cases, we identify Problems D_1 , D_2 and E_2 , E_3 , E_4 in correspondence with B_1 , B_2 , and C_2 , C_3 , C_4 , respectively.

Additional justification of the above conjectures will be given later; here they are suggested merely by the counting of characteristics.

The problem of force-free fields, *viz.*, solution of the system

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \quad \nabla \cdot \mathbf{B} = 0 \quad (6)$$

is a special case of the equilibrium problem obtained by specializing boundary values of pressure to $\phi = \text{constant}$. We identify Problem F_1 as the special case of A_2 or A_3 , F_2 as the special case of C_2 or C_3 , and F_3 as the special case of E_2 or E_3 . Problems F_2 and F_3 are the two-dimensional and axially symmetric versions of F_1 .

CHARACTERIZATION OF THE MAGNETIC FIELD IN A PRESSURE SURFACE

We first introduce the concept of a *surface harmonic* vector field on a surface S (e.g., a constant ϕ surface). A tangential vector field $\mathbf{X}^{(2)} = \nabla^{(2)}\phi$ is said to be a *surface gradient* (or irrotational vector field) if it is the projection of a three-dimensional gradient or if

$$\oint_C \mathbf{X}^{(2)} \cdot d\mathbf{x} = 0$$

for every closed curve C which bounds a portion of S . The *conjugate* vector field $\mathbf{n} \times \mathbf{X}^{(2)} = \mathbf{Y}^{(2)}$ (where $\mathbf{n}$ is the normal to S) is said to be a *surface curl* (or solenoidal vector field); we have

$$\oint_C Y_\nu^{(2)} ds = 0$$

where ν denotes the normal in S to the curve C .

If $\mathbf{X}^{(2)} = \nabla^{(2)}\phi = \mathbf{n} \times \nabla^{(2)}\psi$ (that is, if $\mathbf{X}^{(2)}$ is both irrotational and solenoidal), we call it *harmonic* and say that ϕ and ψ are conjugate surface harmonics. In the special case where S is a plane, ϕ and ψ satisfy the Cauchy-Riemann equations.

In a simply-connected plane domain, a harmonic vector $\mathbf{X}^{(2)}$ is uniquely determined by the boundary values of the normal or tangential components of $\mathbf{X}^{(2)}$. Specifying either at the boundary is equivalent (except for a trivial additive constant) to specifying ψ or ϕ at the boundary. Exactly the same facts hold on an arbitrary surface S .

In a multiply-connected domain, the solution to such a boundary-value problem is uniquely determined only when certain additional data called *periods* are prescribed. These may be specified values of the circulations $\oint \mathbf{X}^{(2)} \cdot d\mathbf{x}$ on each independent closed circuit or of the fluxes $\int X_\nu^{(2)} ds$ on arcs which cut these circuits (see Fig. 5). If these periods are non-zero, the functions ϕ and ψ are multiple-valued.

The above theory can be generalized to include *weighted* surface harmonics which satisfy the equation

$$\mathbf{X}^{(2)} = \nabla^{(2)}\phi = \sigma \mathbf{n} \times \nabla^{(2)}\psi; \quad (7)$$

here ϕ and ψ are conjugate harmonic with respect to a positive weight function σ .

Since $\mathbf{J}$ has no component normal to a pressure surface S_p , we conclude that $\mathbf{B}$ is a surface gradient on S_p ; i.e.,

$$\mathbf{B} = \nabla^{(2)}\phi = \nabla\phi - \mathbf{n}(\partial\phi/\partial n). \quad (8)$$

In Appendix I, it is shown that, since $\nabla \cdot \mathbf{B} = 0$ and ϕ is constant on $\mathbf{B}$ lines, there exists a function ω such that

$$\mathbf{B} = \nabla\phi \times \nabla\omega = |\nabla\phi| \mathbf{n} \times \nabla^{(2)}\omega. \quad (9)$$

We see that $\mathbf{B}$ is a weighted surface harmonic with weight $|\nabla\phi|$ on any S_p . The weight $|\nabla\phi|$ arises in converting from an actual three-dimensional solenoidal vector with $\nabla \cdot \mathbf{B} = 0$ to a two-dimensional area-weighted solenoidal vector. We note that $\mathbf{B}$ will be an actual three-dimensional gradient instead of a surface gradient only if $\mathbf{J} = 0$ and, hence, ϕ is constant.

These remarks allow us to extend the previously conjectured existence theorems to cases in which the ϕ lines on the boundary are closed curves (see Fig. 6). On a surface S_p , the magnetic field is determined only when, in addition to the normal component of B at each end, we give a period, e.g., the mmf

$$\tau(\phi) = \oint_C \mathbf{B} \cdot d\mathbf{x} \quad (10)$$

on a curve C circling S_p . This argument suggests the following modification of Problem A_1 :

Problem G_1 : ϕ given on S_1 , ϕ given on S_1 , τ given for each ϕ .

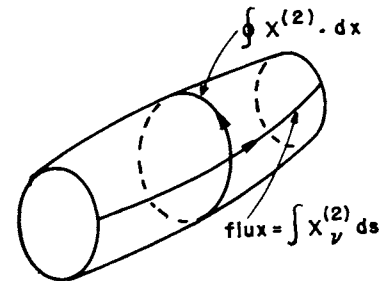


Figure 5

We are now also able to insert the two-dimensional (G_2) and axially symmetric (G_3) analogues of A_1 :

Problem G_2 : ϕ given on Γ_1 , τ given for each ϕ ;

Problem G_3 : ϕ given on S_1 , τ given for each ϕ .

In Problem G_2 , τ is defined as the line integral of $\mathbf{B}$ over a finite distance z , interpreting the figure to be periodic in the z direction. In Problem G_3 , $\tau = 2\pi B_\theta r$. In general, the value of $\tau(\phi)$ can be interpreted as the degree of "twist" of the magnetic lines on each tubular ϕ surface. It should be noted that J_n is an alternative way of specifying this twist.

SOLUTION IN TERMS OF ARBITRARY FUNCTIONS

First we consider the general two-dimensional case in which B_x , B_y , B_z depend only on x , y . Since

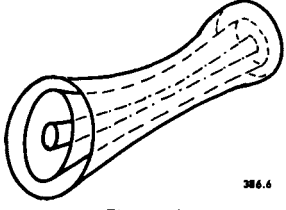


Figure 6

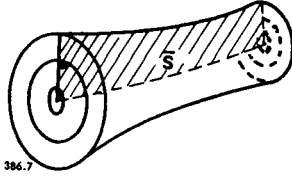


Figure 7

$\nabla \cdot \mathbf{B} = 0$, we can introduce a stream function $\psi(x, y)$ for the B_x, B_y components of the field; we then have

$$B_x = -\frac{\partial \psi}{\partial y}, \quad B_y = \frac{\partial \psi}{\partial x} \quad (11)$$

so that

$$\mathbf{B} = \mathbf{n} \times \nabla \psi + \mathbf{n} B_z \quad (12)$$

where $\mathbf{n} = (0, 0, 1)$. It can be seen that p and B_z are constant on the curves of constant ψ in the x, y plane. However, p and B_z may be multiple-valued functions of ψ . The stream function $\psi(x, y)$ satisfies the non-linear potential equation

$$\Delta \psi + \mu p'(\psi) + \mu f'(\psi) = 0 \quad (13)$$

where

$$f(\psi) = \frac{1}{2\mu} B_z^2; \quad (14)$$

p' and f' refer to derivatives with respect to ψ . In (13), $p(\psi)$ and $f(\psi)$ are arbitrary functions of ψ . Hence, our system of four first-order equations has been reduced to a single second-order equation containing two arbitrary functions. One can expect a solution to be determined by specifying the arbitrary functions and boundary values for ψ as for the standard two-dimensional potential equation. For solutions depending on x only, the equation

$$\frac{d^2 \psi}{dx^2} + \mu p'(\psi) + \mu f'(\psi) = 0 \quad (15)$$

can be integrated explicitly giving many interesting equilibrium configurations.

In the case of axial symmetry in which B_r, B_θ, B_z depend only on r, z , we introduce a stream function $\psi(r, z)$ such that

$$B_r = \frac{1}{r} \frac{\partial \psi}{\partial z}, \quad B_z = -\frac{1}{r} \frac{\partial \psi}{\partial r} \quad (16)$$

and, hence

$$\mathbf{B} = \frac{1}{r} \mathbf{n} \times \nabla \psi + \mathbf{n} B_\theta \quad (17)$$

where $\mathbf{n} = (n_r, n_\theta, n_z) = (0, 1, 0)$. Here $\psi(r, z)$ satisfies the equation

$$\Delta^* \psi + \mu r^2 p'(\psi) + \mu f'(\psi) = 0 \quad (18)$$

where

$$\Delta^* = \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad (19)$$

and

$$f(\psi) = \frac{1}{2\mu} r^2 B_\theta^2. \quad (20)$$

In the case of cylindrical symmetry, in which there is dependence on r alone, we may give any two of p, ψ and B_θ as functions of r and immediately compute the other.

Interesting special solutions of Eq. (15) and Eq. (18) can be found by separation of variables after taking f' and p' to be linear in ψ . In some cases eigenvalue problems arise. It is thus clear that uniqueness cannot be expected in general. However, it is possible to prove uniqueness as well as existence even with quite arbitrary $f(\psi)$ and $p(\psi)$ for domains which are not too large.

The force-free special cases are obtained by the simple expedient of setting $p'(\psi) = 0$.⁴

The reduction in terms of arbitrary functions offers support for the conjectured existence of solutions to Problems B₁, C₂ or C₃, D₁, E₂ or E₃, F₂, and F₃. In each case, the boundary data over and above B_n serve to determine the functions $p(\psi)$ and $f(\psi)$.

VARIATIONAL ANALYSIS.

ADDITIONAL BOUNDARY VALUE PROBLEMS

It is well known that solutions of the fluid free-surface problem can be described as those vector functions $\mathbf{u}(x)$ which make stationary the functional $\int \frac{1}{2} u^2 dV$ when the fluid volume is held fixed. Analogously, in the magnetic case we vary $\int B^2 / 2\mu dV$ holding the volume of the magnetic domain fixed. Or we can drop the restriction to constant volume and, as suggested by the Lagrange multiplier rule, vary

$$\begin{aligned} \int_{V_m} \frac{1}{2\mu} B^2 dV + p_0 \int_{V_m} dV &= \int_{V_m} \frac{1}{2\mu} B^2 dV \\ &\quad - \int_{V_f} p_0 dV + \text{constant} \\ &= \int_{V_f + V_m} \left(\frac{1}{2\mu} B^2 - p \right) dV + \text{constant}. \end{aligned} \quad (21)$$

Here V_m represents the vacuum (magnetic) domain and V_f the conducting fluid domain; the sum $V_m + V_f$ is fixed. The pressure p takes the value zero in the vacuum and the constant value p_0 in the fluid. We shall see that, if interpreted properly, the same functional,

$$Q = \int \left(\frac{1}{2\mu} B^2 - p \right) dV \quad (22)$$

serves for the general equilibrium problem with fluid and magnetic field mixed; here p as well as $\mathbf{B}$ will be a function of x, y, z . The class of admissible pairs $(\mathbf{B}, p)$ allowed to compete in the variation of the given functional is restricted by the conditions that $\mathbf{B}$ be solenoidal and p be constant on the $\mathbf{B}$ lines of the associated $\mathbf{B}$,

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0 \\ \mathbf{B} \cdot \nabla p &= 0,\end{aligned}\quad (23)$$

as well as by boundary conditions that will vary from problem to problem.

A simple way of incorporating both constraints (23) is to represent $\mathbf{B}$ in the form

$$\mathbf{B} = \nabla p \times \nabla \omega \quad (24)$$

(see Appendix I). The function ω may be multiple-valued if the p surfaces are not simple.

We perform the variation, obtaining

$$\begin{aligned}\delta \int_V (\tfrac{1}{2} B^2 - p) dV &= \int_V (\mathbf{B} \cdot \delta \mathbf{B} - \delta p) dV \\ &= \int_V \{ \mathbf{B} \cdot (\nabla \delta p \times \nabla \omega + \nabla p \times \nabla \delta \omega) - \delta p \} dV \\ &= \int_V \{ \delta p (\mathbf{J} \cdot \nabla \omega - 1) - \delta \omega (\mathbf{J} \cdot \nabla p) \} dV \\ &\quad - \oint_S \{ \delta p (\mathbf{B} \times \nabla \omega) - \delta \omega (\mathbf{B} \times \nabla p) \} \cdot \mathbf{n} dS = 0. \quad (25)\end{aligned}$$

For arbitrary variations of δp and $\delta \omega$ in the volume integral, we conclude

$$\begin{aligned}\mathbf{J} \cdot \nabla \omega &= 1 \\ \mathbf{J} \cdot \nabla p &= 0\end{aligned}\quad (26)$$

from which we easily obtain $\nabla p = \mathbf{J} \times \mathbf{B}$ as the Euler equation.

Next we turn to the boundary variation and first consider the case of simple p surfaces (Fig. 3), with p given at both ends S_1 and S_2 . It can easily be verified that there is no contribution to the variation (25) from S_3 on which $B_n = 0$. On S_1 and S_2 , from $\delta B_n = 0$ we conclude that $\delta \omega$ is a function of p ; however, we are certainly free to fix the value of ω at one end of each p line on S_1 , and this makes $\omega \delta = 0$ everywhere. We then have:

Theorem 1: In the tubular geometry of Fig. 2, if p is given (compatibly) at the ends S_1 and S_2 in such a way that the p lines are simple, then Q is made stationary by any solution of the system (1) which satisfies these boundary conditions. This theorem corresponds to the conditions of Problem A₁.

Next consider tubular p surfaces (Fig. 6). Since ω and ϕ (see Eq. (8)) may be multiple-valued, we cut the domain on a surface S which extends from the axis of the nested surfaces S_p to the outer boundary S_3 (Fig. 7). The boundary integral in (25) must now be taken over both sides of $\bar{S}$ as well as S_1 , S_2 and S_3 . As in the previous case, the contribution from S_3 vanishes. A little manipulation shows that the contribution from the cut $\bar{S}$ also vanishes. On S_1 or S_2 , since $\delta B_n = 0$, we must have $\sigma \omega = f(p)$. We obtain for this part of the variation,

$$\begin{aligned}\int_{S_1+S_2} f(p) (\mathbf{B} \times \nabla p) \cdot d\mathbf{S} &= \int_{S_2-S_1} f(p) \tau(p) dp \\ &= \int_{S_2} (f_2(p) - f_1(p)) \tau(p) dp\end{aligned}\quad (27)$$

using Eq. (II.11) of Appendix II. The difference $\delta \omega_2 - \delta \omega_1$ represents the variation of the twist from

S_1 to S_2 on a given pressure surface. In order to have Q stationary we must either fix this twist beforehand so that $\delta \omega_2 - \delta \omega_1 = 0$ or else specify that $\tau(p) = 0$; the latter is a natural boundary condition in this problem. To specify the twist for a class of admissible vector fields $\mathbf{B}$, we require the magnetic lines which originate on a ray C' on S_1 to end on another given ray C'' on S_2 (Fig. 8). We have:

Theorem 2: With tubular p surfaces, p given compatibly at the ends S_1 and S_2 and two given rays C' and C'' identified, Q is stationary if $\mathbf{B}$ and p satisfy (1). If the twist is not specified, Q is stationary for a solution of (1) which satisfies $\tau(p) = 0$. There are really two ways of specifying twist, since the specification of $\tau(p)$ or the identification of two rays C' and C'' are, in an intuitive sense, equivalent.

It is easy to alter the problem by the use of the Lagrange multiplier rule so as to be able to specify τ instead of the twist. This variational formulation corresponds to Problems G₁, G₂ and G₃.

Some modifications of these variational problems are necessary to obtain force-free fields. We drop the term p in the variational functional, and obtain

$$M = \int \frac{1}{2\mu} B^2 dV. \quad (28)$$

Instead of the representation $\mathbf{B} = \nabla p \times \nabla \omega$ we now take

$$\mathbf{B} = \nabla \pi \times \nabla \omega \quad (29)$$

where π is no longer identified with the pressure. It is now easy to verify the following theorems:

Theorem 3: For admissible sets $(\mathbf{B}, \pi)$ with simple π surfaces and π given compatibly on S_1 and S_2 , M is stationary when $\mathbf{B}$ is force-free.

Theorem 4: For admissible sets $(\mathbf{B}, \pi)$ with tubular π surfaces and π given compatibly on S_1 and S_2 as well as the twist from S_1 to S_2 , M is stationary when $\mathbf{B}$ is force-free.

If one is willing to place a large burden on the verification of the compatibility of given boundary data, then it is possible to formulate problems in which p (or π) is given at both ends S_1 and S_2 without regard to the simplicity of p lines. We have:

Theorem 5: Consider admissible classes $(\mathbf{B}, p)$ [or $(\mathbf{B}, \pi)$] which satisfy $\nabla \cdot \mathbf{B} = 0$ and p [or π] constant on $\mathbf{B}$ lines, p [or π] given compatibly at both ends, and a fixed (compatible) identification of the ends of each $\mathbf{B}$ line. Then Q [or M] is rendered stationary when $(\mathbf{B}, p)$ satisfies $\nabla p = \mathbf{J} \times \mathbf{B}$ [$\mathbf{J} \times \mathbf{B} = 0$].

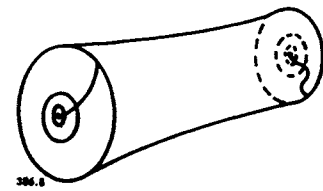


Figure 8

This theorem follows when we note that assignment of B_n and p [or π] at both ends uniquely fixes the correspondence between $\mathbf{B}$ lines if the p surface [π surfaces] are simple, and if they are not simple, the additional specification of twist serves the same purpose. Physically, the boundary condition that the ends of each $\mathbf{B}$ line be fixed corresponds to a perfectly conducting fluid in contact with perfectly conducting end walls S_1 and S_2 . In the force-free case, the reference is to a so-called "pressureless plasma" in which the gas pressure is negligible compared to the magnetic pressure, but which is, nevertheless a good conductor; that is physically realizable since the conductivity of a plasma is independent of its density.

It is interesting to compare the variational problem for the force-free field with the classical Dirichlet's principle which states that M is minimized subject to given B_n (and $\mathbf{V} \cdot \mathbf{B} = 0$) when $\mathbf{V} \times \mathbf{B} = 0$. The additional requirement that the ends of each $\mathbf{B}$ line be fixed prevents this minimum from being attained and yields $(\mathbf{V} \times \mathbf{B}) \times \mathbf{B} = 0$ as the Euler equation instead of $\mathbf{V} \times \mathbf{B} = 0$.

As a final example, we take as our domain the interior of a topological torus and look for solutions which have the outer boundary of the torus as a constant pressure surface. No boundary values can be given since no $\mathbf{B}$ lines are accessible. However, the variational formulation itself suggests what data are required to obtain a well-posed problem. As before, we take admissible sets $(\mathbf{B}, p)$ which satisfy $\mathbf{B} = \nabla p \times \nabla \omega$, where the p surfaces now are nested toruses about some closed curve C_0 as axis. For simplicity we take p to be monotone, $p_0 > p > p_1$, where p_0 is the value taken on the axis C_0 and p_1 is the value taken on the outer boundary. To make ω single valued, it is necessary to introduce two surfaces $\bar{S}_1$ and $\bar{S}_2$ as cuts; $\bar{S}_1$ is a transverse cut across the torus (leaving ends similar to S_1 and S_2 of Fig. 2), and $\bar{S}_2$ extends from the axis C_0 to the outer boundary. The only contribution to the variation (25) is on the cuts; we have

$$\delta Q = \int_{\bar{S}_1 + \bar{S}_2} [\delta p] (\nabla \omega \times \mathbf{B}) \cdot \mathbf{n} dS - \int_{\bar{S}_1 + \bar{S}_2} [\delta \omega] \nabla p \times \mathbf{B} \cdot \mathbf{n} dS. \quad (30)$$

Since p (therefore δp) is single valued, $[\delta p] = 0$. The periods $[\omega]$ are given by

$$\sigma_i(p) = \int_{C_i} \nabla \omega \cdot d\mathbf{x}. \quad (31)$$

where C_i are the independent closed curves on the torus S_p . In Appendix II it is shown that

$$\begin{cases} \sigma_1(p) = \frac{d}{dp} \Phi_2(p) \\ \sigma_2(p) = -\frac{d}{dp} \Phi_1(p) \end{cases} \quad (32)$$

where $\Phi_i(a)$ represents the magnetic flux across that

part of $\bar{S}_i$ for which $p_1 < p < a$. From Eq. (II.11) of Appendix II, we obtain the form

$$\delta Q = \int_{p_0}^{p_1} \{\tau_1 \delta \sigma_2 - \tau_2 \delta \sigma_1\} dp \quad (33)$$

where p_0 and p_1 are the values taken by p on the axis C_0 and on the outer boundary of the torus, respectively. We now state:

Theorem 6: In the class of admissible $(\mathbf{B}, p)$ defined by $\mathbf{V} \cdot \mathbf{B} = 0$, $\mathbf{B} \cdot \nabla p = 0$ and given $\Phi_1(p)$, $\Phi_2(p)$ (or $\sigma_1(p)$, $\sigma_2(p)$) for $p_0 > p > p_1$, Q is stationary when $\nabla p = \mathbf{J} \times \mathbf{B}$.⁵

We can obtain a formulation with given $\tau_i(p)$ rather than $\Phi_i(p)$ by modifying Q according to the conventional Lagrange multiplier rule.

For force-free fields, exactly the same analysis yields the result that Q is stationary if Φ_1 is a given function of Φ_2 and Φ_2 attains the fixed values Φ_2 and 0 on C_0 and on the outer boundary; in parametric form we give $\Phi_1(\pi)$ and $\Phi_2(\pi)$.

As a special case of a toroidal geometry, we can take a problem with cylindrical symmetry (dependence on r alone) and introduce periodicity to provide the toroidal topology. In this case, the above conjectures are trivially proved.

We summarize various sets of data which are believed to define definite equilibrium configurations. In the tubular domain of Fig. 2, we specify B_n as shown and we also specify p and J_n at one end or p at one end and J_n at the other. The justification lies in the counting of characteristics, in a heuristically appealing iteration scheme and in a direct verification by integrating in terms of arbitrary functions in certain special cases. Or we can specify p at one end and the corresponding terminal points of each B line in a manner compatible with the given values of B_n . This is confirmed by counting of characteristics supplemented by the known properties of surface harmonics, by variational analysis and by integration in terms of arbitrary functions in certain special cases. In the toroidal geometry, we conclude that the two fluxes $\Phi_i(p)$ or mmfs $\tau_i(p)$ can be specified arbitrarily; the justification is proved by variational analysis and by explicit solution in special cases.

APPENDIX I

REPRESENTATION OF AN INCOMPRESSIBLE VECTOR FIELD IN TERMS OF TWO STREAM FUNCTIONS

We note, for arbitrary $\phi(x, y, z)$, $\psi(x, y, z)$ and arbitrary $\alpha(\phi, \psi)$ (which means that α is constant on the intersection of $\phi = \text{constant}$ and $\psi = \text{constant}$), that

$$\nabla \cdot (\alpha \nabla \phi \times \nabla \psi) = 0. \quad (\text{I.1})$$

We remark further that, if (I.1) holds for $\alpha(x, y, z)$, $\phi(x, y, z)$, $\psi(x, y, z)$ in a domain where ϕ and ψ are independent functions (i.e., $\nabla \phi \times \nabla \psi \neq 0$), then it follows that $\alpha = \alpha(\phi, \psi)$.

Next consider a solenoidal vector field $\mathbf{B}$ and a small region of space which is simply covered by the $\mathbf{B}$ lines which intersect a transverse surface S . We have:

Theorem 1.1: There exist functions $\phi, \psi, \alpha(\phi, \psi)$ such that $\mathbf{B} = \alpha \nabla \phi \times \nabla \psi$.

To prove Theorem 1.1 we choose any independent ϕ and ψ on S (i.e., $\phi = \text{constant}$ and $\psi = \text{constant}$ form a coordinate system). We then carry the values of ϕ and ψ off S on the $\mathbf{B}$ lines. Since $\nabla \phi \times \nabla \psi$ is parallel to $\mathbf{B}$, we conclude that $\mathbf{B} = \alpha \nabla \phi \times \nabla \psi$ and, hence, $\alpha = \alpha(\phi, \psi)$ by the above remark.

We note that the flux through S ,

$$d\Phi \equiv B_n dS = d\phi d\psi, \quad (\text{I.2})$$

which justifies the terminology "stream functions".

We again take an arbitrary vector field $\mathbf{B}$ with $\nabla \cdot \mathbf{B} = 0$ and a small region and assert:

Theorem 2: There exist ϕ, ψ such that $\mathbf{B} = \nabla \phi \times \nabla \psi$. In fact, if ϕ is any given function which is constant on $\mathbf{B}$ lines, then there exists ψ such that $\mathbf{B} = \nabla \phi \times \nabla \psi$.

Choose ϕ arbitrarily on S so that the ϕ curves are simple. Introduce s as the arc length on a ϕ curve, ν as the normal to the ϕ curve in S . Note that if $\mathbf{B} = \nabla \phi \times \nabla \psi$ then $B_n = (\partial \phi / \partial \nu) / (\partial \phi / \partial S)$. This suggests that we construct ψ as

$$\psi(s) = \int \frac{1}{B_n} \frac{\partial \phi}{\partial \nu} dS, \quad (\text{I.3})$$

integrated along each ϕ line (the value of ψ at one end of each ϕ line can be arbitrarily assigned). As before, we carry the values of ϕ and ψ off S as constants on $\mathbf{B}$ lines. From the previous theorem, $\mathbf{B} = \alpha \nabla \phi \times \nabla \psi$. By the construction of ψ , $\alpha = 1$ on S ; since α is constant on each $\mathbf{B}$ line, $\alpha = 1$ throughout.

If the ϕ lines are taken as closed curves, then the construction yields a ψ which is not single valued, and a cut should be introduced.

APPENDIX II

INTEGRATION FORMULAS

We shall use two basic formulas. First, given two functions $f(x)$ and $g(x)$ defined in a space $x = (x_1, \dots, x_n)$, we have

$$\int_{a < g < b} f(x) dx = \int_a^b \langle f \rangle_g dg \quad (\text{II.1})$$

where

$$\langle f \rangle_g = \int_{S_g} f(dS / |\nabla g|). \quad (\text{II.2})$$

The integral (II.1) over the shell $a < g < b$ is written as an iterated integral first on the surface S_g , then with respect to g .⁶ The second identity is,

$$\int_S \nabla \phi \times \nabla \psi \cdot d\mathbf{S} = \int_S d\phi d\psi = \int_C \phi d\psi = - \int_C \psi d\phi; \quad (\text{II.3})$$

here S is an arbitrary surface in three-space with C as its boundary.⁷ The surface S may have to be cut to make ϕ and ψ single valued. If S is a torus, on which there are the two independent closed curves C_1 and C_2 , this formula reduces to

$$\int_S \nabla \phi \times \nabla \psi \cdot d\mathbf{S} = [\phi]_1 [\psi]_2 - [\phi]_2 [\psi]_1 \quad (\text{II.4})$$

where

$$[\phi]_i = \int_{C_i} d\phi, \quad [\psi]_i = \int_{C_i} d\psi \quad (\text{II.5})$$

are the periods of ϕ and ψ , respectively.

We apply these formulas to obtain the identity

$$\int_{a < p < b} B^2 dV = \int_a^b dp \int_{S_p} d\phi d\omega \quad (\text{II.6})$$

for an equilibrium $\mathbf{B}$, i.e., for $\mathbf{B}$ satisfying

$$\mathbf{B} = \nabla^{(2)} \phi = \nabla p \times \nabla \omega. \quad (\text{II.7})$$

Using (II.2) and then (II.3) we verify

$$\begin{aligned} \langle B^2 \rangle_p &= \int_{S_p} \nabla^{(2)} \phi \cdot (\nabla p \times \nabla \omega) \frac{dS}{|\nabla p|} \\ &= \int_{S_p} \nabla \omega \times \nabla^{(2)} \phi \cdot d\mathbf{S} = \int d\omega d\phi. \end{aligned} \quad (\text{II.8})$$

In the special case of a torus, we have

$$\int_{a < p < b} B^2 dV = \int \{ \tau_1(p) \sigma_2(p) - \tau_2(p) \sigma_1(p) \} dp \quad (\text{II.9})$$

where

$$\begin{cases} \sigma_i(p) = \oint_{C_i} \nabla \omega \cdot d\mathbf{x} \\ \tau_i(p) = \oint_{C_i} \mathbf{B} \cdot d\mathbf{x}. \end{cases} \quad (\text{II.10})$$

On a transverse surface S , we compute

$$\int_{a < p < b} (\mathbf{B} \times \nabla p \cdot \mathbf{n}) dS = - \int_a^b \tau(p) dp. \quad (\text{II.11})$$

We have $\mathbf{n}$ as the normal to S , and introduce ν as the normal in S to a p curve, and S as arc length along a p curve (Fig. 9). The proof is

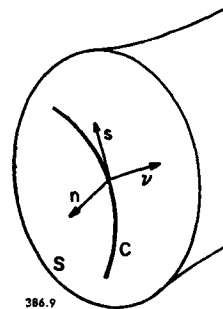


Figure 9

$$\begin{aligned} \langle \mathbf{B} \times \nabla p \cdot \mathbf{n} \rangle_{\mathbf{v}} &= \int_{C_p} (\mathbf{B} \times \nabla p \cdot \mathbf{n}) \frac{ds}{|\nabla^{(2)} p|} \\ &= \int_{C_p} (\mathbf{B} \times \nabla \cdot \mathbf{n}) dS = - \oint_{C_p} \mathbf{B} \cdot d\mathbf{x} = -\tau(p). \end{aligned} \quad (\text{II.12})$$

Again, on a transverse surface S ,

$$\begin{aligned} \langle B_n \rangle_p &= \int_{C_p} (\nabla p \times \nabla \omega \cdot \mathbf{n}) \frac{ds}{|\nabla^{(2)} p|} \\ &= \oint_{C_p} \nabla \cdot \omega d\mathbf{x} = \sigma(p). \end{aligned} \quad (\text{II.13})$$

Consequently,

$$\Phi(b) - \Phi(a) = \int_{a < p < b} B_n dS = \int_a^b \sigma(p) dp \quad (\text{II.14})$$

or

$$\sigma(p) = \Phi'(p). \quad (\text{II.15})$$

Taking signs into account, we can rewrite (II.9) as

$$\int_{a < p < b} B^2 dV = \int_a^b \{ \tau_1 \Phi_1' + \tau_2 \Phi_2' \} dp \quad (\text{II.16})$$

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Some Stable Hydromagnetic Equilibria

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The problem of hydromagnetic instability in connection with the controlled thermonuclear program was first raised in 1951 by Spitzer, who gave arguments that the pinch discharge should be unstable. A detailed analysis by Kruskal and Schwarzschild¹ showed that this configuration should, in fact, be unstable. This problem was raised again in 1954 by Teller² who gave strong intuitive reasons for suspecting all the devices to be unstable.

An energy principle³ for the treatment of hydromagnetic instabilities was developed and from it a criterion for the stability of general axisymmetric equilibria was obtained by the summer of 1955. It was seen by this criterion that the stellarator is unstable with respect to the "interchange" instability. This "interchange" can be visualized by considering an axisymmetric equilibrium in which the plasma and the magnetic field are imbedded in each other, and in which the magnetic lines of force lie entirely in r, z planes (in the usual cylindrical coordinate system). A displacement of the plasma, which interchanges lines of force in such a way that the magnetic field and its magnetic energy are unchanged, can be carried out by first specifying, on a cross section with $z = \text{constant}$, a mapping of the magnetic lines of force into themselves, and then extending this displacement throughout the volume by making each line of force continue to go into the same line as that assigned on this cross section. Since the magnetic field strength is given by the density of lines, the magnetic energy is clearly unchanged. However the material energy may increase or decrease. Consider a displacement in which two flux shells with equal amounts of flux $d\psi$ are interchanged. If the volumes in these flux shells are not equal the plasma in one shell will expand and that in the other will compress. Thus the resulting change in the total compressional energy of the plasma will depend on the pressure profile and is given,³ up to a positive constant factor, by $(V''/V') [V''/V' + p'/\gamma p]$, where primes denote differentiation with respect to ψ , $V(\psi)$ is the volume in a flux tube with flux ψ and γ is the ratio of the specific heats of the gas. If the magnetic field lines are not straight

and if the material pressure is sufficiently small, then V'' is positive and non-zero while the factor in parenthesis goes to negative infinity at the boundary of the confined plasma so that the system is unstable. If the magnetic lines of force are perfectly straight as in an infinite cylinder and p' is negative, V'' is negative and the system is stable with respect to this interchange. However, displacements can be found with respect to which the infinite cylinder is neutral. Since in the stellarator there will be small ripples in the magnetic lines of force due to the discreteness of the coils which produce the field, we expect the stellarator to be unstable.

This argument (for instability) applies in general to more complicated situations such as stellarators, since nothing would prevent us from carrying out the "interchange" displacement unless the system is such that the magnetic lines of force return on themselves. In the latter case, our original assignment of the mapping of lines into lines would not match when we bring the lines around on to themselves. Nevertheless, one might ask whether even in re-entrant systems, interchanges can be constructed which lower the energy of the plasma and leave the magnetic field unchanged. The answer can be given in terms of the rotational transform angle ι ,⁴ which can be defined on each magnetic surface as 2π times the average number of turns about the magnetic axis which a line of force makes as it is followed once around the system. (The magnetic axis is that line of force which closes on itself.)

We consider first a system in which ι is constant over the cross section; i.e., independent of the flux ψ within each magnetic surface. If in some such system every magnetic line of force returns on itself after n times around the system ($\iota = 2\pi m/n$ for every ψ), an interchange can be selected so localized that it need not match until the n th time around, when it matches perfectly. If ι is not a rational multiple of 2π the matching can never be achieved exactly, but it can be so closely approximated by a rational multiple of 2π that the matching comes arbitrarily close. One can choose the displacement sufficiently close to an interchange to make the change in the magnetic energy negligible. Thus for any ι , constant on all flux tubes, an effective interchange can be carried out with respect to which the system is unstable.

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The situation is different if ι changes with changing ψ . In this case if we try to construct an interchange, the matching becomes worse and worse as we carry the mapping around the system. The first diagram in Fig. 1 indicates an attempt at assigning on a cross section, a mapping of the magnetic lines of force into each other indicated by the flow pattern. After carrying this mapping along the lines once around the system, the flow diagram becomes that of the second diagram. Here it is assumed that ι is larger on the outer surfaces than on the inner ones, and since the lines are the same in the flow pattern, the outer lines move farther, shearing the flow pattern. After a second time around the system, the flow pattern (for the mapping) becomes that illustrated in the third diagram. It is clear that the possibility of matching becomes more and more hopeless as we carry the mapping around more and more times. Further, even by carrying out displacements differing from the interchange which manage to produce matching, the magnetic energy cannot be made negligible. Thus an ι which depends on ψ inhibits the unstable interchanges.

Achievement of stabilization by use of "shear" magnetic fields, whose direction rotates out of the $\mathbf{r}, \mathbf{B}$ plane with increasing separation $\mathbf{r}$ from the magnetic axis, was suggested by Spitzer⁵ in 1955. A preliminary calculation,⁶ in which the twisting of the magnetic field lines is produced by an axial current on the surface of the plasma, was encouraging.

Stabilization by means of plasma currents is not applicable for a steady state machine (such as a stellarator) since an emf is necessary to drive the stabilization current. For this reason Spitzer⁵ proposed that the stabilizing field be produced by pole pieces placed at right angles to the plasma. These pole pieces would be rotated in space in order to make the magnetic field depend on z as well as r and θ , where r , θ , and z are cylindrical coordinates with axes parallel to the main magnetic field. (A dependence on z is necessary in order to obtain an equilibrium situation.) These fields are to lowest order (in themselves) helically invariant and proportional to $\sin(\theta - hz)$ or $\cos(\theta - hz)$. It can be seen that stabilizing fields that are the same as these to lowest order can equally well be produced by wrapping two wires helically around the system, with currents flowing in opposite directions in the wires. It had been pointed out earlier (in February 1955) by Koenig⁷ that a helical magnetic field of this type would produce a rotational transform, but it was not realized until later that this configuration would have a stabilizing effect because of the variation of transform

angle with radial distance. We examined the nature of these equilibria and obtained necessary and sufficient conditions⁸ for the system to be stable to all perturbations ξ which are periodic over the helical period $2\pi/h$. In the calculation, the equilibria and the perturbations were expanded in the parameters β and δ , where β characterizes the magnitude of the material pressure and δ characterizes the magnitude of the crossed fields. If β is of order δ the system is unstable. If $\beta \sim \delta^2$ a critical condition for stability is obtained, while if β is proportional to a higher power of δ the system is stable.

Spitzer⁹ pointed out in 1956 that the rate of change of the pitch of the magnetic lines of force, with increasing distance from the axis, could be increased by considering more general helically invariant fields which vary as $\sin(l\theta - hz)$ or $\cos(l\theta - hz)$, since if their pitch is small they vary to lowest order as r^{l-1} . Clearly larger values of l produce steeper dependences of ι on r for the same ι , but the advantage for stability of this steeper dependence is off-set by the steeper dependence of the fields on r , so that if the fields are applied by wrapping wires helically around the system some distance from the surface of the plasma, the fields in the plasma will be correspondingly smaller. In consideration of these factors it seems reasonable to suppose that either the $l = 3$ or $l = 4$ fields ($\iota \sim r^{l-2}$) are most favorable for stability, although the $l = 2$ fields produce a large rotational transform which may have advantages in the equilibrium situation.

Since the imposition of externally produced helically invariant fields makes the magnetic lines of force into small helices about their original position, an "interchange" perturbation will cause a larger decrease in the plasma energy with them present. There is, therefore, a critical value of β associated with a given helical field above which the system is inherently unstable. It is found in the minimization of δW that if one expresses the perturbation ξ of the equilibrium in terms of its Fourier components in $\exp i(m\theta + nkz)$, where k is 2π over the length once around the system, the worst ξ takes on its maximum near the radius for which $\iota = -2\pi n/m$. Such a radius exists for finite n and m only if ι is finite or, for infinitesimal ι , only if m is infinite. Since we wish to treat the stability of a system with infinitely small helically invariant fields superimposed on a finite axial field and avoid infinitely large values of m , it is necessary to consider the tube identified over an infinite length L in the lowest order ($L \sim 1/\delta^2$), so that ι can remain finite. We therefore obtained the stability criterion for systems in which there is a superposition of helically invariant fields imposed on a large axial field, with the ξ 's Fourier analysed over the length of the infinitely long machine.¹⁰ At the same time we determined the effect of these helical fields on the long-wavelength kink instability caused by the presence of a small axial current.

We therefore were able to show¹⁰ that by wrapping wires helically around the tube and passing current through them in opposite directions in alternate

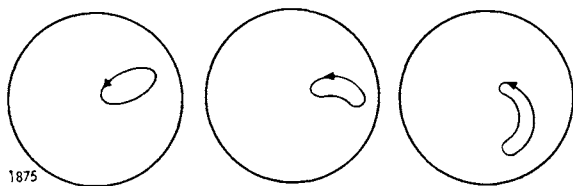


Figure 1. Distortion of the flow pattern if $d\iota/d\psi > 0$

wires, we can produce fields which stabilize the interchange instability. The same fields can also raise the limiting axial current which can be present without setting up a long-wavelength kink instability.

Equilibrium situations which would exist in a stabilized stellarator are discussed in the next section. In the following section, the integral δW , which arises in the treatment of stability by the energy principle,³ is minimized with respect to all components of the perturbation ξ except the lowest order radial component, ξ_r^0 . Finally, the "interchange" instability and the "kink" instability are treated separately and critical conditions for stability are obtained.⁴

EQUILIBRIUM

In the last section we suggested that certain classes of equilibria might be stable. In this section we calculate these equilibria and investigate those properties, such as the rotational transform angle ι ,⁴ which are related to stability.

We use the usual hydromagnetic equations,¹² assuming that the plasma is infinitely conducting, so that

$$\nabla p = \mathbf{j} \times \mathbf{B}, \quad (1)$$

as well as the usual Maxwell equations,

$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$

and

$$\nabla \times \mathbf{B} = \mathbf{j}, \quad (3)$$

must be satisfied.

In order to avoid the complications which are involved in the treatment of closed curved systems such as a torus, we imagine the system to be straightened out with the ends identified.

In the stellarator, a large axial magnetic field is employed to confine the plasma. The system is made unstable with respect to "interchanges" by bulges which, for instance, may be due to the finite spacing between the coils which produce the main field. In order to obtain stability we superimpose on this field a helically invariant field by wrapping wires in a helical fashion around the tube and passing currents of alternate sign through them. Clearly an integral number of periods of this field must fit between the identified ends of the tube. Since it is difficult to solve (1), (2), and (3) for this system if plasma is present ($\nabla p \neq 0$), we will assume that the amplitude of this helically invariant field is small compared to the main axial field. Further it will be shown that a small helical field can stabilize the system only if little plasma is present. We therefore expand all field quantities as power series in the small parameters δ and β which give a measure to the magnitude of the helical fields and the amount of plasma present, respectively. We define δ and β as follows:

The zero-order situation consists of a perfectly conducting right circular cylinder of radius S in which

there is a uniform axial magnetic field B^0 and in which no plasma is present.

For the purpose of this calculation, the fields produced by the helically invariant wires are considered to be produced by deforming this cylindrical boundary with the condition that the normal component of the magnetic field must remain zero. In particular, we consider the simple deformation

$$r(\theta, z) = S + \sigma^\delta \cos u, \quad (4)$$

where

$$u = l\theta - hz, \quad (5)$$

and define

$$\delta \equiv \sigma^\delta / S. \quad (6)$$

We will later generalize to the case where many helically invariant deformations are superimposed. For simplicity, no two different values of l are associated with the same helical wavelength, $2\pi/h$. Note that "bulges" in the magnetic field correspond to the choice $l = 0$.

We define the quantity β as

$$\beta = 2p(0)/B^{02}, \quad (7)$$

where $p(0)$ is the material pressure at the axis of the tube. (Note that we are using rationalized Gaussian units with $c = 1$ so that the usual factor 4π in the numerator of the right-hand side of (7) is omitted. Also, as we increase the value of the pressure at the axis, keeping δ fixed, we cannot expect the pressure to scale linearly since the magnetic surfaces are changing. Thus p is not directly proportional to β and we expect $p^{\beta\delta}$, $p^{\beta\beta\delta}$, ... to exist. The pressure must be zero at the surface of the plasma since it is contained by the magnetic field.) Since we do not take diffusion into account, the pressure distribution is arbitrary, subject only to the condition

$$\mathbf{B} \cdot \nabla p = 0, \quad (8)$$

which easily follows from (1).

One other parameter must be specified to determine the system, the net axial current. For steady state operation it is necessary to require that¹³

$$\int \int j_z r dr d\theta = 0, \quad (9)$$

the integral being taken over any cross section contained within a magnetic surface; that is, there shall be no net axial current in the volume contained by any magnetic surface. During the initial time of operation of the stellarator, before it has been heated to its steady-state condition, an axial current is set up in the plasma to heat it ohmically.¹⁴ Since during ohmic heating all inertial terms are small, we may calculate the quantities by the equilibrium equations (1), (2), and (3). Further, since these change slowly in time, we may in treating stability regard the situation as an equilibrium one. We introduce the parameter η to represent the magnitude of this ohmic heating current. Due to the nature of our zero-order situation, the j_z^η can be made independent of the pressure.

[†] Rationalized Gaussian units with $c = 1$ are used throughout this paper. A more detailed treatment of this problem may be found in the Project Matterhorn report PM-S-34.¹¹

We assume in this work that two distinct regions are present. All material pressure and currents in the equilibrium situation are confined inside an inner region bounded by the surface

$$r = R + \varrho^\delta + \varrho^{\delta\delta} + \varrho^{\beta\delta} + \varrho^{\eta\delta} + \dots \quad (10)$$

In the outer region bounded by this surface and the one given by (4), only vacuum fields exist. The boundary conditions which must be imposed are

$$\mathbf{B} \cdot \mathbf{n} = 0 \quad (11)$$

on both the outer surface and the interface (10), and

$$[\![\phi + (B^2/2)]\!] = 0 \quad (12)$$

across the interface. Note that (11) determines the interface (10). Here $\mathbf{n}$ is the normal to the surface and $[\![\]\!]$ indicates the jump in the direction of $\mathbf{n}$.

In the discussions of stability which follow we shall consider two physically different situations which have the same equilibrium quantities. In the first we assume that there are enough particles present in the exterior region so that it is infinitely conducting and therefore can support a current when it is perturbed. However, no currents are assumed in the external region in the equilibrium. Since not enough particles are present to represent a significant material pressure, we denote the exterior region a "pressureless plasma". In the second situation, the external region is assumed to be a vacuum and the magnetic field is determined by the usual boundary conditions.³

For equilibria to exist, ϕ and $\mathbf{j}$ must be constant on magnetic surfaces. However, in this expansion technique magnetic surfaces first appear in the equations in the δ^2 order where a rotational transform is present, and hence the first-order quantities ϕ^β and $\mathbf{j}^\eta$ must be influenced by second-order considerations. Mathematically, the higher-order equations are incompatible unless the first order quantities satisfy certain relationships. In the expansion of the equilibria we find that in calculating the first-order quantities, ϕ^β and $\mathbf{j}^\eta$ are arbitrary functions of r and θ . However, if ϕ^β and $\mathbf{j}^\eta$ are chosen as functions of r alone, the higher-order quantities can be calculated satisfactorily except in the case in which two helically invariant fields with different values of l have the same period, in which case ϕ^β and $\mathbf{j}^\eta$ must depend on θ (in a well-determined way)¹⁵ in order that the second-order quantities satisfy the periodicity conditions. This removal of a degeneracy is similar to that in the expansion of the case where a helically invariant field is applied to a toroidal system.¹⁶ For simplicity, we exclude in this paper the case where two helically invariant fields with different values of l have the same period. We use a prime to denote differentiation with respect to the argument and use the usual cylindrical coordinate system. Analytic continuation of the fields to the perturbed surface is assumed. The fields are thus to order zero:

$$\mathbf{B}^0 = \mathbf{e}_z B^0 \text{ (a constant), } \mathbf{j}^0 = 0, \quad \phi^0 = 0; \quad (0 \leq r \leq S), \quad (13)$$

and to order β :

$$\mathbf{B}^\beta = -\mathbf{e}_z \frac{\beta B^0}{2} \left[1 - f\left(\frac{r}{R}\right) \right], \quad \mathbf{j}^\beta = \mathbf{e}_\theta \frac{\beta B^0}{2R} f'\left(\frac{r}{R}\right),$$

$$\phi^\beta = \frac{\beta B^{02}}{2} \left[1 - f\left(\frac{r}{R}\right) \right]; \quad (0 \leq r \leq R)$$

$$\mathbf{B}^\beta = 0, \quad \mathbf{j}^\beta = 0, \quad \phi^\beta = 0; \quad (R \leq r \leq S) \quad (14)$$

where $f(r/R)$ is arbitrary, such that $f(0) = 0$, $f(1) = 1$, $|f(r/R)| \leq 1$. The quantity β is given by Eq. (7).

To order η ,

$$\mathbf{B}^\eta = \mathbf{e}_\theta \eta B^0 g\left(\frac{r}{R}\right),$$

$$\mathbf{j}^\eta = \mathbf{e}_z \frac{\eta B^0}{R} \left[g'\left(\frac{r}{R}\right) + \frac{R}{r} g\left(\frac{r}{R}\right) \right], \quad \phi^\eta = 0; \quad (0 \leq r \leq R)$$

$$\mathbf{B}^\eta = \mathbf{e}_\theta \frac{\eta B^0 R}{r}, \quad \mathbf{j}^\eta = 0, \quad \phi^\eta = 0; \quad (R \leq r \leq S) \quad (15)$$

where $g(r/R)$ is arbitrary, such that $g(0) = 0$, $g(1) = 1$.

The quantity $\eta \equiv B^\eta(R)/B^0$.

To order δ ,

$$\begin{aligned} \mathbf{B}^\delta = \sum_s \left[\mathbf{e}_r \delta_s B^0 X_s \frac{I'_l(x_s)}{I'_l(X_s)} \sin u_s \right. \\ \left. + \mathbf{e}_\theta \delta_s l_s B^0 \frac{X_s^2}{x_s} I^l(X_s) \frac{I_l(x_s)}{I_l(X_s)} \cos u_s \right. \\ \left. - \mathbf{e}_z \delta_s B^0 X_s^2 I^l(X_s) \frac{I_l(x_s)}{I_l(X_s)} \cos u_s \right]; \\ (0 \leq r \leq S) \\ \mathbf{j}^\delta = 0; \quad \phi^\delta = 0. \end{aligned} \quad (16)$$

The perturbed surface is $S + \sum_s \sigma_s^\delta \cos u_s$, the $\delta_s \equiv \sigma_s^\delta/S$ are all of the same order, $\delta u_s \equiv l_s \theta - h_s z$, $x_s \equiv h_s r$, $X_s \equiv h_s S$, $I_l(x)$ is the modified Bessel function of the first kind,¹⁷ $I^l(x) \equiv I_l(x)/x I'_l(x)$, and the subscript s has been dropped from the l 's when they are themselves used as subscripts or superscripts.

For the purpose of the following stability analysis only these lowest-order fields need be written down since all the properties of the higher-order fields which must be used can be obtained easily from these by (1), (2) and (3). Nevertheless, higher-order fields must be obtained to understand the equilibrium situation. These are given in Appendix I.

We now proceed to find the rotational transform angle ι in the blunt, but straightforward and pictorial, way by finding the average angle of rotation¹³ of a magnetic line of force about the magnetic axis as one proceeds along the z axis. Since the length over which we specify the average angle of rotation is arbitrary we shall choose it to be the period of the helically invariant field $2\pi/h$, and denote this average angle by ι_h . (The rotational transform angle ι is usually defined over the identification length of the system.)

We first compute the angle of rotation $\Delta\theta$ for a length $\tilde{z}$ such that a tube of force passing through a fixed constant- z cross section of a magnetic surface shall have sampled once every point in this cross section when it is screw-projected along the z axis at the helical rate, so that

$$(h/l)z - \Delta\theta = 2\pi. \quad (17)$$

The magnetic surfaces have cross sections perpendicular to the z axis which rotate about this axis at the constant rate,

$$d\theta/dz = h/l, \quad (18)$$

as we proceed along this axis, whereas the lines of force rotate around the z axis at a much slower average rate

$$\overline{d\theta/dz} \sim \delta^2. \quad (19)$$

Hence we must proceed in the z direction until a line of force first comes back to the same relative position in a cross section that it originally occupied. We are now assured that the average angle of rotation for all lines in a magnetic surface is the same as that of any one line. We thus have¹³

$$\iota_h = (\Delta\theta/\tilde{z})(2\pi/h) + 2\pi n, \quad (n = 0, \pm 1, \pm 2, \dots) \quad (20)$$

or, making use of (17) and adopting the convention $n = 0$,

$$\iota_h + 4\pi^2/h\tilde{z} = 2\pi/l. \quad (21)$$

We can use (17) and (21) to compute ι_h . Since the fields are known as a power series in the expansion parameters, the equations of lines of force and thus ι_h will be given as power series in the expansion parameters. We find to the lowest significant order

$$\iota_h^{\delta\delta} = \delta^2 \pi l \left(\frac{X I_V'(x)}{x I_V'(X)} \right)^2 [1 - 2I^l(x) + (l^2 + x^2)I^l(x)^2] \quad (22)$$

and

$$\begin{aligned} \bullet S d\iota_h^{\delta\delta}/dr &= -\delta^2 2\pi l \left(\frac{X}{x} \right)^3 \left(\frac{I_V'(x)}{I_V'(X)} \right)^2 \\ &\quad [3 - 2(2 + l^2 + x^2)I^l(x) + (3l^2 + 2x^2)I^l(x)^2]. \end{aligned} \quad (23)$$

For convenience, $\iota_h^{\delta\delta}$ and $R d\iota_h^{\delta\delta}/dr$ are tabulated for the limit $x \leq X \ll 1$ in Table 1 for several values of l . We now see from Table 1 that the rotational transform angle ι_h actually does depend on radius and therefore the helical wires do produce a varying rotational transform as surmised in the introduction.

Higher orders in the power series representation of ι_h as well as relations between ι_h and other important equilibrium quantities¹³ have been obtained.¹¹ In particular, if $V(\psi)$ is the volume enclosed by a surface of longitudinal flux ψ ,

$$V''(\psi) = \frac{h^2}{2\pi k B_{02} l r^3} \frac{d}{dr} (r^4 \iota^{\delta\delta}) + \frac{j^\beta}{k r B_{03}}. \quad (24)$$

Here $k = 2\pi/L$, where L is the length of the machine.

We note from (22) and (23) that $V''(\psi)$ is finite even for $l = 0$.

If only one helically invariant field is present, an important and elegant formulation of the problem of obtaining equilibria, which is especially valuable when solutions for finite β and δ are desired, can be given by utilizing the two-dimensional nature¹⁸ of the problem.

We introduce the scalar functions Ψ and f , such that

$$\mathbf{B} \cdot \nabla \Psi = 0 \quad (25)$$

$$f = l B_z + h r B_\theta. \quad (26)$$

We find from (25) that

$$\Psi \partial/\partial u = \partial r B_r, \quad \partial \Psi/\partial r = -\partial(l B_\theta - h r B_z), \quad (27)$$

where u is given by (5) and ∂ is an arbitrary function. It follows from (2) that ∂ is a function of Ψ only. For simplicity we choose $\partial(\Psi) = 1$, any other choice merely constituting a relabeling of the constant Ψ surfaces. It follows from (1), (3) and (27) that f and the pressure p are functions of Ψ only. The fields and currents are then given by

$$\begin{aligned} \mathbf{B} &= \mathbf{e}_r \frac{\partial \Psi/\partial u}{r} + \mathbf{e}_\theta \frac{h r f - l(\partial \Psi/\partial r)}{l^2 + (h r)^2} \\ &\quad + \mathbf{e}_z \frac{h r (\partial \Psi/\partial r) + l f}{l^2 + (h r)^2}, \end{aligned} \quad (28)$$

$$\mathbf{j} = \mathbf{e}_r \frac{f'}{r} \frac{\partial \Psi}{\partial u} + \mathbf{e}_\theta$$

$$\begin{aligned} &\times \left[-h r \mathcal{L} \Psi - \frac{l f' (\partial \Psi/\partial r)}{l^2 + (h r)^2} + \frac{2 h^2 r l f}{[l^2 + (h r)^2]^2} \right] \\ &+ \mathbf{e}_z \left[-l \mathcal{L} \Psi + \frac{h r f' (\partial \Psi/\partial r)}{l^2 + (h r)^2} + \frac{2 h l^2 f}{[l^2 + (h r)^2]^2} \right], \end{aligned} \quad (29)$$

where

$$\mathcal{L} \Psi + \frac{f f'}{l^2 + (h r)^2} - \frac{2 h l f}{[l^2 + (h r)^2]^2} = -p' \quad (30)$$

$$\mathcal{L} \equiv \frac{1}{r} \frac{\partial}{\partial r} \frac{r}{l^2 + (h r)^2} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial u^2} \quad (31)$$

and the prime denotes differentiation with respect to Ψ . Clearly Ψ and f , and therefore the fields, are determined by (30) if $p(\Psi)$ and a condition on the axial current such as (9) are given.

We can show¹¹ that Ψ is a flux related to ψ , the flux the long way, and χ , the flux the short way over one helical period around the system¹³ by

$$\frac{2\pi}{h} \Psi = \psi - l \chi. \quad (32)$$

MINIMIZATION OF δW

We determine whether or not our magnetostatic equilibria are stable by means of the energy principle,³ which reduces the question of stability to the problem: Can the quadratic functional of ξ ,

Table 1. Limiting Values of ι and $d\iota/dr$ for Small hR

	$\iota_h \delta\delta$ (per length $2\pi/h$)	$Rd\iota_h \delta\delta/dr$ (per length $2\pi/h$)
1	$\delta^2 \pi X^2$	$\delta^2 (s/s) \pi x X^3$
2	$\delta^2 2\pi$	$\delta^2 2\pi x X$
3	$\delta^2 4\pi x^2/X^2$	$\delta^2 8\pi x/X$
4	$\delta^2 6\pi x^4/X^4$	$\delta^2 24\pi x^3/X^3$

$$2\delta W = \int \{ \mathbf{Q}^2 + \mathbf{j} \cdot \boldsymbol{\xi} \times \mathbf{Q} + \boldsymbol{\xi} \cdot \nabla p \nabla \cdot \boldsymbol{\xi} + \gamma p (\nabla \cdot \boldsymbol{\xi})^2 \} d\tau, \quad (33)$$

where

$$\mathbf{Q} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}), \quad (34)$$

be made negative for any choice of $\boldsymbol{\xi}(\mathbf{r})$? Here $\mathbf{B}$, $\mathbf{j}$, and p represent the equilibrium values, $\boldsymbol{\xi}$ is an arbitrary virtual displacement from the equilibrium, subject only to the condition

$$\boldsymbol{\xi} \cdot \mathbf{n} = 0 \quad (35)$$

on a rigid boundary and δW is the resulting second order (in $\boldsymbol{\xi}$) change in the potential energy. If δW can be made negative for some $\boldsymbol{\xi}$, it can be made negative for a normal mode $\boldsymbol{\xi}$, some potential energy can be converted into kinetic energy and the system is unstable.

We shall first order our equilibrium parameters in such a way that we can carry through the stability analysis in terms of a single parameter λ . We shall then expand δW as a power series in λ and carry through the minimization order by order. We shall find that the zero order δW is non-negative and can be made to vanish, the first order is then automatically zero, the second order is again non-negative and can be made to vanish, and the third order is again automatically zero. The fourth order δW contains the desired critical condition. We minimize it here over all components of $\boldsymbol{\xi}$ except ξ_r^0 . In the next section we carry through this final minimization for specific cases.

We obtained equilibria in the last section as power series in several independent parameters. It is much easier, however, to solve the stability problem for a one-parameter equilibrium, and we express our equilibrium in terms of the single expansion parameter λ by assuming the ordering $\beta \sim \eta \sim \delta^2 \sim \lambda^2$. The reason for this particular choice can be seen later. It is clear that different values of λ characterize a family of equilibria. A $\boldsymbol{\xi}$ which makes δW negative for one value of λ need not make it negative for another. Since we wish to determine the stability conditions for this entire class of equilibria, we must allow $\boldsymbol{\xi}$ to depend arbitrarily on λ as well as $\mathbf{r}$. We therefore expand $\boldsymbol{\xi}$ as a power series in λ .

In this treatment of the stability of the stellarator, we neglect any effect of the curvature of the tube. However, to keep partially the effect of the closed machine we demand that our equilibrium and our perturbation $\boldsymbol{\xi}$ be periodic over a length $L = 2\pi/k$ equal to the length of the machine. We must choose $kR \sim \lambda^2$ (R is taken finite) to keep the rotational

transform angle ι , defined over the whole machine, finite as λ goes to zero.

It should be observed that any finite situation may be approached by a system expanded in λ in any number of ways. For instance, if $k \sim \lambda^2$, k will be finite when λ becomes finite. Alternatively, the same finite situation could have been approached by keeping k fixed (independent of λ), as λ is increased from zero. The best expansion is the one that leads to the most rapidly converging scheme. Since we do not examine the situation beyond the lowest significant order, we cannot apply this criterion to our scheme. However, our choice of orders leads to an expansion which remains "uniformly" valid as we arbitrarily shift our choice of orders, while another choice does not. In particular, we have shown¹¹ that the results remain valid when hR is made transcendentally small compared to λ . Further, our expansion readily yields itself to physical interpretation so in some sense it is the "best" expansion.

We cannot expand our $\boldsymbol{\xi}$'s at a fixed point since they will have wave lengths of order L and the expansion would not be uniform over L . To get around this difficulty we first Fourier analyze $\boldsymbol{\xi}$ in z . Thus

$$\boldsymbol{\xi} = \sum_{n,s} \boldsymbol{\xi}(s,n) e^{i(sh + nk)z}, \quad (36)$$

with s and n finite integers. We have shown¹¹ that the assumption that $\boldsymbol{\xi}$ can have no other Fourier components cannot affect the question of stability since the non-vanishing of such components always makes δW positive. Now $\boldsymbol{\xi}(s,n)$ can be expanded uniformly in λ . It should be noted that the operator $\partial/\partial z$ acting on $\boldsymbol{\xi}$ can change its order.

We now expand δW as a power series in λ and examine the lowest order term which gives a decisive answer as to stability.

The zero-order term, δW^0 , is positive definite, and can be made zero by restricting $\boldsymbol{\xi}^0(s,n)$, so that

$$\boldsymbol{\xi}_\perp^0(s,n) = 0, \quad (s \neq 0) \quad (37)$$

$$\nabla \cdot \boldsymbol{\xi}_\perp^0 = 0. \quad (38)$$

Here we have introduced the notation

$$\mathbf{A}_\perp \equiv \mathbf{e}_r A_r + \mathbf{e}_\theta A_\theta; \quad \mathbf{A}_\parallel \equiv \mathbf{e}_z A_z. \quad (39)$$

We temporarily relax the restriction given by (35). It will be shown that if δW can be made negative, the $\boldsymbol{\xi}$ which does this can be chosen to satisfy this condition to all orders.

The next order term, δW^λ , vanishes trivially. By performing an integration by parts we show that $\delta W^{\lambda\lambda}$ is positive definite. It can be made zero by choosing

$$\boldsymbol{\xi}_\parallel^0(s,n) = 0, \quad (s \neq 0) \quad (40)$$

in the region where the pressure is not zero, and

$$ishB^0 \boldsymbol{\xi}_\perp^\lambda(s,n) = -[\nabla \times (\boldsymbol{\xi}^0 \times \mathbf{B}^\lambda)]_\perp(s,n), \quad (41)$$

$$\mathbf{B}^0 \nabla \cdot \boldsymbol{\xi}_\perp^\lambda = [\nabla \times (\boldsymbol{\xi}^0 \times \mathbf{B}^\lambda)]_\parallel. \quad (42)$$

That (41) and (42) are compatible can be seen by taking the divergence of the first and the derivative

with respect to z of the second, and subtracting. We have shown¹¹ that at this point to the order we are interested, we can set $\xi_z^0(0, n)$ equal to zero everywhere with no loss in generality. We show that $\delta W^{\lambda\lambda\lambda}$ is zero by two straightforward integrations by parts.

In the next order we can now write

$$2\delta W^{\lambda\lambda\lambda} = \int_U \{ (\mathbf{Q}^{\lambda\lambda})^2 - j^\beta \xi_r^0 Q_z^{\lambda\lambda} + j^\eta (\xi_r^0 Q_\theta^{\lambda\lambda} - \xi_\theta^0 Q_r^{\lambda\lambda}) + \gamma p^\beta (\nabla \cdot \xi^\lambda)^2 + [\xi \cdot \nabla p (\nabla \cdot \xi)]^{\lambda\lambda\lambda} \} d\tau + \int_{\Delta U} \{ \xi_\perp^0 \cdot \nabla p^\beta (\nabla \cdot \xi^\lambda) \} d\tau, \quad (43)$$

where U is the volume in the unperturbed cylinder and ΔU is the perturbed volume. This can be minimized with respect to ξ_z^λ and $\xi_\perp^{\lambda\lambda}$ by setting

$$ish \xi_z^\lambda(s, n) = -\nabla \cdot \xi_\perp^\lambda(s, n) \quad (44)$$

and

$$ish B^0 \xi_\perp^{\lambda\lambda}(s, n) = -[\nabla \times (\xi^0 \times \mathbf{B}^{\lambda\lambda} + \xi^\lambda \times \mathbf{B}^\lambda)]_\perp(s, n) \quad (45)$$

$$B^0 \nabla \cdot \xi_\perp^{\lambda\lambda} = [\nabla \times (\xi^0 \times \mathbf{B}^{\lambda\lambda} + \xi^\lambda \times \mathbf{B}^\lambda)]_\parallel - j^\beta \xi_r^0. \quad (46)$$

As before, (45) and (46) can be shown to be compatible. We can now use (41) to express δW (henceforth we delete the superscripts) in terms of $\xi_\perp^0$ only. At this point it is convenient to Fourier analyze in terms of θ . We write

$$\xi_\perp^{(0, n)} = \sum_m \xi_\perp(m; n) e^{im\theta} \quad (47)$$

and note that δW breaks up into a sum over m and n with the $m; n^{th}$ term involving only $\xi_\perp(m; n)$. We can use (38), so that except for those terms for which $m = 0$, $\delta W_{m; n}$ is a function of $\xi_r^0(m; n)$ alone. Further $\delta W_{0; n}$ is positive definite, and can be made zero by setting

$$\xi_\theta^0(0; n) = 0. \quad (48)$$

We therefore obtain

$$2\delta W = \sum_{m>0} \sum_n 2\delta W_{m; n} \quad (49)$$

$$2\delta W_{m; n} = \frac{8\pi^2}{k} \int_0^S r dr \left\{ \left(\frac{d\mu_{m; n}}{dr} \right)^2 + \frac{\alpha_{m; n}}{r^2} (\mu_{m; n})^2 \right\}, \quad (50)$$

where

$$\mu_{m; n} = v_{m; n} |\xi_r^0(m; n)|, \quad (51)$$

$$v_{m; n} = \frac{nkrB^0}{m} + B^\eta(r) + \sum_s \frac{lsh(\sigma_s^\delta)^2 B^0}{2r} \left[\frac{I_l'(shr)}{I_l'(shS)} \right]^2 \times [1 - 2I^l(shr) + (l^2 + s^2 h^2 r^2) I^l(shr)^2] \quad (52)$$

and

$$\alpha_{m; n} = m^2 + \frac{d\eta}{dr} \frac{r^2}{v_{m; n}} - \sum_s \frac{j^\beta B^0}{v_{m; n}^2} (sh\sigma_s^\delta)^2 r \left[\frac{I_l'(shr)}{I_l'(shS)} \right]^2 \times \{ 1 - 2[l^2 + (shr)^2] I^l(shr) + l^2 I^l(shr)^2 \}. \quad (53)$$

The desired expression for δW in terms of an arbitrary $\xi_r^0(m; n)$ is given by Eq. (50). Here, as usual, the prime denotes differentiation with respect to the argument and $I^l(x)$ is given in Eq. (16). The letter l should be written as l_s to denote that for a given helical field period it is arbitrary. We have shown¹¹ that ξ 's which satisfy our minimization condition also satisfy (35) if $\xi_r^{\lambda\lambda}(s \neq 0, n)$ is accordingly restricted.

We are frequently interested in equilibria in which there is a sharp discontinuity at some surface in the plasma. We can treat the stability of such equilibria by including surface terms in Eq. (33),³ or by regarding the surface of discontinuity as a volume region in which physical quantities vary very rapidly and then letting the thickness shrink to zero. It is necessary to make the jump in $\xi \cdot \mathbf{n}$ across the discontinuity zero to prevent cavitation or interpenetration. We have considered¹¹ an equilibrium which has a discontinuity in the pressure, a surface longitudinal current and a discontinuity in the volume longitudinal current. We find that, denoting the radius at which the discontinuity occurs by R , Eq. (50) is replaced by the more general result

$$2\delta W_{m; n} = \frac{8\pi^2}{k} \left\{ \int_0^S r dr \left[\left(\frac{d\mu_{m; n}}{dr} \right)^2 + \frac{\alpha_{m; n}}{r^2} (\mu_{m; n})^2 \right] - \frac{R j^\eta (\mu_{m; n})^2}{v_{m; n}} \right\} \Big|_{r=R_-} + \left([B^\beta] \sum_s (sh\sigma_s^\delta)^2 B^0 \left(\frac{I_l'(shR)}{I_l'(shS)} \right)^2 \times [1 - 2[l^2 + (shR)^2] I^l(shR) + l^2 I^l(shR)^2] - [\beta^\eta] \sum_s \frac{lsh(\sigma_s^\delta)^2}{R} B^0 \left(\frac{I_l'(shR)}{I_l'(shS)} \right)^2 \times \{ 3 - 2[2 + l^2 + (shR)^2] I^l(shR) + [3l^2 + 2(shR)^2] I^l(shR)^2 \} - [B^{\eta 2}] \right) \left(\frac{\mu_{m; n}}{v_{m; n}} \right)^2 \Big\}, \quad (54)$$

where $\mu_{m;n}$, $v_{m;n}$ and $\alpha_{m;n}$ are again given by (51), (52) and (53).

In (50) and (54) we can treat $\mu_{m;n}$ as the arbitrary function in $\delta W_{m;n}$ instead of $\xi(m;n)$ with the provisions: $\mu(S)$ is zero, μ vanishes at all points where v does and μ' is continuous where v vanishes. With this in mind, we can minimize δW in the region $R \leq r \leq S$ over μ such that $\mu(R)$ is prescribed. If v does not vanish in this region, we get for this part of $2\delta W$

$$\Lambda_{m;n} \equiv \left(\frac{8\pi^2 m}{k} \right) \frac{1 + (R/S)^{2m}}{1 - (R/S)^{2m}} (\mu_{m;n}(R))^2. \quad (55)$$

If v does vanish in this region let aR be the smallest value of r at which it does. The contribution to $2\delta W$ from the integral between R and aR has the minimum value

$$\Lambda_{m;n} \equiv \left(\frac{8\pi^2 m}{k} \right) \frac{a^{2m} + 1}{a^{2m} - 1} (\mu_{m;n}(R))^2. \quad (56)$$

Since μ' is continuous at aR , the contribution to δW from the region between aR and S is not zero. Since it is positive it can be made as small as desired, for instance by taking

$$\mu = \frac{\mu'(aR)}{\kappa} \sin \kappa(r - aR) \quad \left(R \leq r \leq aR + \frac{2\pi}{\kappa} \right) \quad (57)$$

$$\mu = 0 \quad \left(aR + \frac{2\pi}{\kappa} \leq r \right) \quad (58)$$

and letting κ approach infinity.

In the preceding work we have assumed that the external region $R < r < S$ is filled with a pressureless plasma, capable of supporting a current. The energy principle³ can be generalized to consider equilibria in which this region is a true vacuum. Instead of minimizing $\mathbf{Q}^2$ in this region, we minimize $(\nabla \times \mathbf{A})^2$, subject to the conditions

$$\mathbf{n} \times \mathbf{A} = (\mathbf{n} \cdot \boldsymbol{\xi}) \mathbf{B} \quad (59)$$

at the plasma interface and

$$\mathbf{n} \times \mathbf{A} = 0 \quad (60)$$

at the external boundary. When we carry through the minimization we find that the contribution to $\delta W_{m;n}$ from this region is again given by Eq. (55). The difference between the two cases is that the pressureless plasma can develop a sheet current at aR , while the vacuum cannot. The vacuum is always unstable if the corresponding pressureless plasma case is, but the converse need not be valid.

We can rewrite Eq. (54) in terms of the rotational transform angle, ι , $d\iota/dr$, and V'' , as where

$$\mu_{m;n} = \frac{k r B^0}{2\pi} v_{m;n}^* |\xi_r^0(m;n)|, \quad (62)$$

$$v_{m;n}^* = 2\pi n/m + \iota^\eta + \sum_s \iota_s^{\delta\delta} \quad (63)$$

$$\alpha_{m;n} = m^2 + \frac{1}{r v_{m;n}^*} \frac{d}{dr} \left(r^3 \frac{d\iota^\eta}{dr} \right) + \frac{4\pi^2 r B^0 j^\beta}{k v_{m;n}^{*2}} V''_{\text{vac}}, \quad (64)$$

and $\Lambda_{m;n}$ is the contribution to $2\delta W$ from the external region, given by Eq. (55) if the external region is a vacuum, or by (55) or (56) if it is a pressureless plasma depending on whether $v_{m;n}^*$ is zero in that region or not, respectively. Here the rotational transform angles, $\iota_s^{\delta\delta}$ from the helically invariant field which depends on u_s and ι^η from the axial current, and the term V''_{vac} are computed over the length of the machine ($2\pi/k$). The terms in j^β and $[B^\beta]$ are similar to those which in the axially symmetric case³ represent the energy released by the expansion of the gas. The terms $(d\mu/dr)^2$ and $(\mu m/r)^2$ represent the energy increase due to the change in the magnetic field, $(\delta B)^2/2$. The terms in ι^η represent the work done by the force term $\mathbf{j} \times \delta \mathbf{B}$ (computed at a fixed point).

The field energy terms can be made somewhat more transparent as follows: The quantity v^* represents the average rate at which the magnetic lines of force turn compared to the displacement $\boldsymbol{\xi}$, and thus it represents an effective field across the displacement. For simplicity we consider as a model a straight tube of length L , with a large uniform B_z , with a B_θ independent of θ and z , just sufficient to produce a rotational transform angle equal to v^* , but with no axial current (so that one of Maxwell's equations is violated). We find that with a displacement ξ_r independent of z but depending on θ as $\exp im\theta$ and ξ_θ such that $\nabla \cdot \boldsymbol{\xi}_\perp = 0$, the fields are changed by

$$\delta B_r = im v^* \xi_r L B_0, \quad (65)$$

$$\delta B_\theta = - \partial (v^* r \xi_r L B_0) / \partial r. \quad (66)$$

Squaring (65) and (66) and adding, we have the stabilizing terms of Eq. (61). The origin of the terms is made pictorial by considering δB_r as being due to compression of the lines of force by the displacement and δB_θ as being due to shearing of the lines. The

$$2\delta W_{m;n} = \frac{8\pi^2}{k} \left\{ \int_0^R r dr \left[\left(\frac{d\mu_{m;n}}{dr} \right)^2 + \frac{\partial_{m;n}}{r^2} (\mu_{m;n})^2 \right] - \frac{1}{R} \frac{d(r^2 \iota^\eta)}{dr} \frac{\mu_{m;n}^2}{v_{m;n}^{*2}} \right|_{r=R} + \left[-[\iota^\eta] + [\iota^\eta] \sum_s R \frac{d\iota_s^{\delta\delta}}{dr} - \frac{4\pi^2 B^0}{k} [B^\beta] V''_{\text{vac}} \right] \frac{\mu_{m;n}^2}{v_{m;n}^{*2}} + \Lambda_{m;n} \right\}, \quad (61)$$

actual situation and this model are analogous in the sense that the model is obtained by smoothing out the ripples in the lines of force (which the displacements in the actual situation automatically take care of) and untwisting the system so that the displacement is "untwisted".

One can see, by examining (61), (62), (63) and (64), the reason for the particular ordering of the parameters which we made. If $\iota^{\delta\delta}$ or ι^η were not finite over the length of the machine, one or more of the terms in $\nu^*_{m,n}$ would have entered δW in a different order. Similarly if we had not chosen $\beta \sim \delta^2$, the terms in β would have entered in a different order. In any case, any other choice of the ordering would have yielded a critical condition in which one of the parameters does not enter, or would have made the system stable or unstable with no critical condition.

STABILITY CRITERIA

In the last section we carried out the minimization of δW with respect to all components of ξ except $\xi_{r^0}(m; n) \exp(i(m\theta + nkz))$. Since the final minimization is too difficult to do in general, we shall in this section discuss several special cases. We treat "interchange" instabilities and their stabilization in the first part of this section and kink instabilities and their stabilization in the second part.

Interchange Instability

We wish to determine the critical β for stability against interchanges when helically invariant fields are present. We assume $j^\eta = 0$, a condition clearly satisfied for steady state operation of the stellarator and assume only helically invariant fields with $l = 0$ or 3 and small hR . We consider in detail only the case where the plasma fills the entire tube ($R = S$), so that $\mu(R)$ is zero.

In Appendix II we carry out the final minimization for this situation with an arbitrary pressure distribution which we then select to maximize the critical β . One would expect any other distribution to quickly attain this optimum distribution if sufficient material is injected. We find that the critical β is

$$\beta_c = [(\sum_i p_i \delta_i^2)^2 / \sum_i p_i^2 \delta_i^2] \times [1 - (3\phi/4) \ln(1 + 4/3\phi)] \quad (67)$$

for the optimum pressure distribution

$$f(t) = \frac{t^2 - (3\phi/4) \ln[1 + (4t^2/3\phi)]}{1 - (3\phi/4) \ln[1 + (4/3\phi)]}, \quad (68)$$

where

$$\phi = \sum_i q_i^2 \epsilon_i^2 / \sum_i p_i^2 \delta_i^2, \quad (69)$$

$$t = r/R, \quad \delta_i = \varrho_i^\delta / R, \quad \epsilon_i = \varrho_i^\epsilon / R; \quad (70)$$

ϱ_i^δ and ϱ_i^ϵ are respectively the amplitudes of the Fourier components of the distortions of the plasma surface with wavelengths such that p_i helical and q_i bulge fields can fit into the machine, and $f(t)$ is related to the pressure by Eq. (14). If no bulges

($l = 0$) are present the optimum pressure distribution is parabolic. If, in addition, only one helically invariant field is present the inherent stability is given (for $l = 3$) by

$$\beta_c = (\varrho^\delta/R)^2. \quad (71)$$

Since the maximum value of ϱ^δ/R in a system with $l = 3$ is about $\frac{1}{3}$, on the basis of the present theory, β_c would not exceed about 0.1. However, the theory is presumably not valid for so large a value of either ϱ^δ/R or of β .

We have shown previously¹¹ that if a pressureless plasma exists between the plasma boundary and the perfectly conducting walls of the system, the same results are obtained. If, however, this region is a vacuum, the system is unstable for interchanges in a very thin surface layer if $f'(1)$ is not zero. We therefore expect that the pressure distribution will adjust itself so that $f'(1)$ is zero and the critical β will be somewhat lower than the β_c given by Eq. (68), which would obtain if the external region were a pressureless plasma.

The problem can be carried through in the same way[†] for the case where hR is finite, any combination of helically invariant fields (with any l 's) are present (subject to the condition that no two fields with different values of l have the same wavelength), and the next axial current j^η is zero. We find that the optimum pressure distribution is

$$f(t) = \frac{k}{8\pi^2 B^2 \beta_c} \int_0^1 \frac{\sum_s d\iota_s^{\delta\delta}(t)/dt)^2 t dt}{V''(\psi(t))_{\text{vac}}} \quad (72)$$

and

$$\beta_c = \frac{k}{8\pi^2 B^2} \int_0^1 \frac{(\sum_s d\iota_s^{\delta\delta}(t)/dt)^2 t dt}{V''(\psi(t))_{\text{vac}}}, \quad (73)$$

where $\iota_s^{\delta\delta}$ and V'' are given by Eq. (22), Eq. (24), respectively, but here evaluated over the length of the machine.

We note that as the machine is made longer by adding long straight sections which contribute nothing to V'' for the entire machine, one might suppose that if du/dr per unit length is kept fixed, the stability would be unchanged. On the contrary, it is necessary to increase du/dr in proportion to the length of the machine, V'' always being the same, to preserve the same critical β . This is also clear from our intuitive picture of the interchange instability, since the interchanges can unwrap themselves in the long straight section after being curled up by the du/dr and can thus match themselves with less increase in the change in magnetic energy. With this caution in mind, we can produce the du/dr shear by wrapping helical wires only over part of the tube and still obtain stability.

In minimizing δW we found for $\beta > \beta_c$ that, as $\beta - \beta_c$ is made smaller, the minimizing ξ 's become more nearly singular in the neighborhood of the radius at which $\iota = -2\pi n/m$ and eventually change ap-

[†] We have not yet completed the argument in connection with (93) that μ does not vanish between a and 1 in this general case.¹¹

precipitously over a region small compared to an ion Larmor radius. The minimizing ξ thus represents a motion to which the theory no longer applies since it is based on equations which assume that the ion Larmor radius is the smallest length in the system under description. Thus we cannot assert that these systems ($\beta > \beta_c$), which are only unstable to such sharp ξ 's, are definitely unstable. It is therefore of interest to ask how large β can become before instability sets in, when we apply the added constraint that ξ varies slowly over an ion Larmor radius. This question can be answered by minimizing δW over all ξ_r^0 subject to the restriction that $d\xi_r^0/dr < (\xi_r^0)_{\max}/\lambda$, where λ is a length of the order of the Larmor radius. We find¹¹ that the critical β under this restriction is, under certain circumstances, as much as twice the critical β derived allowing unrestricted ξ_r^0 . The question of the validity of this constraint on ξ must be settled by a more refined theory.

Kink Instability

The instabilities associated with a longitudinal current have been extensively studied.¹⁹ It is found that if the axial field is much larger than the azimuthal field, the system is, for a uniform axial current distribution, unstable with respect to minimizing ξ 's which vary as $\exp(i(m\theta + nkz))$, if

$$-n/m < \iota/2\pi < -n/(m-1). \quad (74)$$

Here if ι (the rotational transform over the identification length, evaluated at the plasma radius R and due to the axial current) is positive, n must be negative for instability. If the external region is a pressureless plasma the system is stable for $m > 1$.

These "kink" instabilities are not so well understood physically as interchange instabilities. We shall first make plausible the existence of the kink instability for $m = 1$ by a simple force picture based on the fact that the lines of force are tied to the matter. We shall then consider the stabilizing effect that stabilizing fields can have on this instability.

We consider a long cylinder of pressureless plasma of radius R and length L embedded in a large axial uniform magnetic field B_0 and in which a small axial current of uniform density j is flowing. As usual, the ends are to be identified. The current density j produces a field B_θ and an ι , given in the plasma by

$$B_\theta = 2jr, \quad \iota = B_\theta/krB_0, \quad k \equiv 2\pi/L. \quad (75)$$

Note that ι is constant in the plasma.

Let us subject this plasma to a displacement ξ given by

$$\begin{aligned} \xi_r &= \hat{\xi} \cos(\theta - kz), \\ \xi_\theta &= \hat{\xi} \sin(\theta - kz), \quad \xi_z = 0. \end{aligned} \quad (76)$$

which moves each constant- z cross section rigidly a constant distance $\hat{\xi}$ perpendicular to the axis, so that the tube of plasma is distorted into a helix whose pitch is L . Consider two cross sections α and β a distance $L/4$ apart which are therefore displaced in

perpendicular directions. If $\iota = 2\pi$, lines of force rotate through $\pi/2$ between these cross sections. That is, any line of force through a point S in α passes through a point T in β such that QT makes an angle $\pi/2$ with QS (see Fig. 2). In these circumstances, it is clear that any line of force passing through A in α and B in β is displaced to a line of force passing through A' in α and B' in β where OA' and QB' make a right angle with each other. But this means that the line of force through A, B is displaced into the position of a line of force which passed through A', B' in the undisplaced equilibrium. Further, since the density of lines is unchanged because the displacement of each cross section is rigid, the field is unchanged by the displacement. Thus the situation characterized by $\iota = 2\pi$ is neutral with respect to perturbation (76).

Let us now consider the case in which $(\iota - 2\pi)$ is positive but small, subject the plasma to the same perturbation (76) and examine the same cross sections α and β , $L/4$ apart. Then any line of force of the equilibrium passing through a point S in α will pass through a point T in β such that QT makes an angle $\iota/4 > \pi/2$ with QS (see Fig. 3). A line of force through A in α and B in β , where OA and QB make an angle of $\iota/4$ with each other, will then be displaced to a position passing through A' and B' , where OA' and QB' no longer make an angle of $\iota/4$ but make a slightly smaller angle. Thus the displaced line of force is rotating about the axis OQ at a slower rate (in z) than the line of force which passed through B' in the undisplaced equilibrium. Hence we see that B_θ is weakened by an amount proportional to $(\iota/4 - \pi/2)\hat{\xi}$. If we consider other points in the cross section, we find that $\delta\mathbf{B}$ is constant in each cross section. Since the cylinder is long, $\delta\mathbf{j} = \nabla \times \delta\mathbf{B}$ is negligible and $\delta\mathbf{F} = \mathbf{j} \times \delta\mathbf{B} + \delta\mathbf{j} \times \mathbf{B} \cong \mathbf{j} \times \delta\mathbf{B}$. We see, therefore, that $\delta\mathbf{F}$ is in the same direction as ξ and tends to enhance the perturbation. Thus for $\iota > 2\pi$ the system is unstable with respect to perturbation (76).

We now consider the effect of a helical field on these instabilities. In the stellarator the axial current is applied only when β is small. We therefore set β equal to zero in this discussion. We have shown¹¹ that the results are continuous as β goes to zero. For simplicity, we assume that the conducting walls are infinitely far away, the axial current is uniform in the plasma region and only one helically invariant field, with $l = 3$ and a small hR , is present.

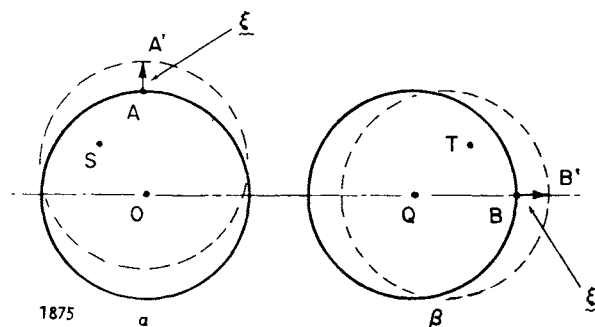
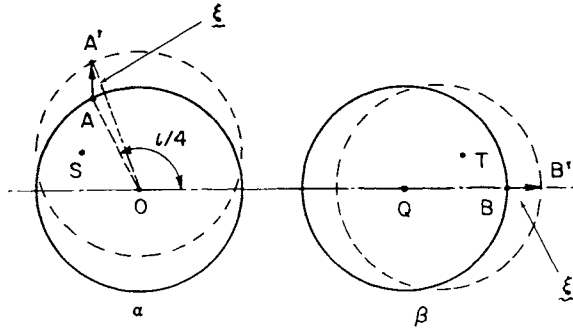


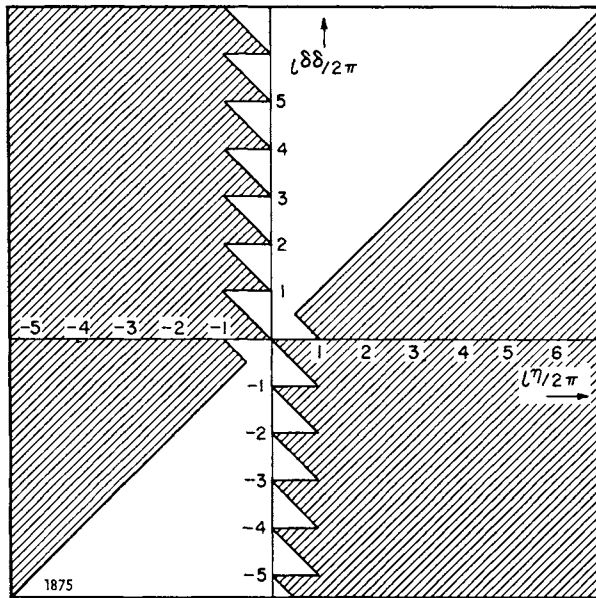
Figure 2. Kink displacement if $\iota = 2\pi$

Figure 3. Kink displacement if $\iota > 2\pi$

In Appendix III we carry through the calculation for the case where the external region is a vacuum. The results are exhibited in Fig. 4. The unshaded region is stable for $m = 1$. The regions in the first and third quadrants in which $|\iota^{\delta\delta}| > |\iota^{\eta\eta}|$ are stable for all m . The other regions have not been completely investigated for higher m 's.

We have also carried through the calculation¹¹ for the case where the external region is a pressureless plasma. The results are exhibited in Fig. 5. The unshaded region is stable for $m = 1$. Instabilities for higher values of m can occur only in the parts of the first and third quadrants for which $|\iota^{\delta\delta}| < |\iota^{\eta\eta}|$.

It can be easily seen¹¹ that if the axial current is a sheet current at the surface of the plasma, the system is stable for all m if $|\iota^{\eta\eta}| < 4|\iota^{\delta\delta}|$ and if both have the same sign.

Figure 4. Stabilization of the kink instability if external region is a vacuum. The unshaded region is stable for $m = 1$

We note that the helical fields have three effects on the kink instability which might lead to an understanding of these $\iota^{\delta\delta}$ vs. $\iota^{\eta\eta}$ diagrams. The first effect (1) is to increase the effective ι by $\iota^{\delta\delta}$ (which may be negative) so that for $m = 1$ the kink is made more unstable if $\iota^{\delta\delta}$ is positive (or less if $\iota^{\delta\delta}$ is negative);

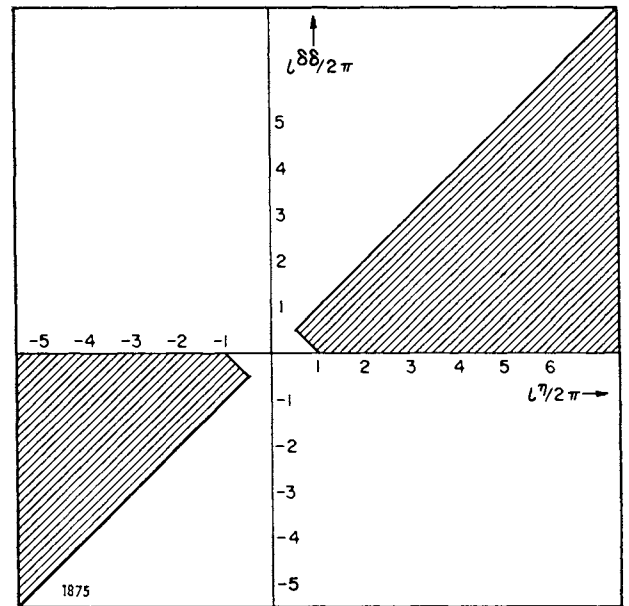
(2) the helically invariant fields affect the stability by introducing a shear (di/dr) in ι inside the plasma and by increasing or decreasing the shear outside the plasma according to the sign of di/dr . This effect is most prominent for the higher values of m ; finally, (3) the helically invariant fields may cause a surface to exist on which the effective $\iota = \iota^{\delta\delta} + \iota^{\eta\eta}$ is $2\pi n/m$. On this surface the electric field $\mathbf{E}$, due to the perturbation, is parallel to $\mathbf{B}$ and along a line of force is always in the same direction. Therefore, this electric field leads to large currents on the surface so that the surface acts like a rigid perfectly conducting wall to the displacement. This effect can lead to increased stability. For $m = 1$ the first effect outweighs the latter two and the critical current is decreased. For higher m 's the latter effects dominate and the stability is improved.

SUMMARY

We have thus been able to find and investigate the properties of equilibria which are hydromagnetically stable. These equilibria can be obtained, for example, by wrapping conductors helically around the stellarator tube. Systems with $l = 3$ or 4 are indicated to be optimum for stability purposes. In some cases an admixture of $l = 2$ fields can be advantageous for achieving equilibrium.

ACKNOWLEDGEMENTS

We wish to express our deepest gratitude to Dr. Lyman Spitzer Jr. for suggesting many of the problems treated in this paper and to Dr. Martin Kruskal for his constant advice and encouragement. We wish to thank all our colleagues in the Theoretical Division of Project Matterhorn for many helpful suggestions.

Figure 5. Stabilization of the kink instability if external region is a pressureless plasma. The unshaded region is stable for $m = 1$

APPENDIX I

HIGHER ORDER EQUILIBRIUM QUANTITIES

For simplicity, we give the higher-order quantities for the case where $f(r/R) = (r/R)^2$ (a parabolic pressure distribution), $g(r/R) = r/R$ (uniform axial current) and only one helically invariant field is present.

To order $\beta\beta$:

$$\begin{aligned} \mathbf{B}^{\beta\beta} &= -\mathbf{e}_z \frac{\beta^2 B^0}{8} \left[1 - \left(\frac{r}{R} \right)^2 \right]^2; \quad \mathbf{j}^{\beta\beta} = -\mathbf{e}_\theta \frac{\beta^2 B^0 r}{2 R^2} \left[1 - \left(\frac{r}{R} \right)^2 \right]; \quad p^{\beta\beta} = 0. \quad (0 \leq r \leq R) \\ \mathbf{B}^{\beta\beta} &= 0; \quad \mathbf{j}^{\beta\beta} = 0; \quad p^{\beta\beta} = 0. \quad (R \leq r \leq S). \end{aligned} \quad (77)$$

To order $\eta\eta$:

$$\begin{aligned} \mathbf{B}^{\eta\eta} &= \mathbf{e}_z \eta^2 B^0 \left[1 - \left(\frac{r}{R} \right)^2 \right]; \quad \mathbf{j}^{\eta\eta} = \mathbf{e}_\theta \frac{2\eta^2 B^0 r}{R^2}; \quad p^{\eta\eta} = 0. \quad (0 \leq r \leq R) \\ \mathbf{B}^{\eta\eta} &= 0; \quad \mathbf{j}^{\eta\eta} = 0; \quad p^{\eta\eta} = 0. \quad (R \leq r \leq S). \end{aligned} \quad (78)$$

To order $\delta\delta$:

$$\begin{aligned} \mathbf{B}^{\delta\delta} &= \mathbf{e}_r \frac{\delta^2 X B^0}{2} [1 - 2(l^2 + X^2) I^l(X)] \frac{I_{2l}'(2x)}{I_{2l}'(2X)} \sin 2u \\ &\quad + \mathbf{e}_\theta \delta^2 l \frac{X^2}{x} B^0 I^{2l}(2X) [1 - 2(l^2 + X^2) I^l(X)] \frac{I_{2l}(2x)}{I_{2l}(2X)} \cos 2u \\ &\quad - \mathbf{e}_z \frac{\delta^2 B^0}{2} [1 - 2(l^2 + X^2) I^l(X)] [1 + 2X^2 I^{2l}(2X)] \frac{I_{2l}(2x)}{I_{2l}(2X)} \cos 2u; \\ \mathbf{j}^{\delta\delta} &= 0; \quad p^{\delta\delta} = 0. \quad (0 \leq r \leq S) \end{aligned} \quad (79)$$

To order $\beta\eta$:

$$\mathbf{B}^{\beta\eta} = 0; \quad \mathbf{j}^{\beta\eta} = 0; \quad p^{\beta\eta} = 0. \quad (0 \leq r \leq S) \quad (80)$$

To order $\beta\delta$:

$$\begin{aligned} \mathbf{B}^{\beta\delta} &= \mathbf{e}_r \left[-\beta\delta l^2 B^0 \frac{S^2}{R^2} I^l(X) \frac{I_l(x)}{x I_l(X)} + \hat{C}^{\beta\delta} I_l'(x) \right] \sin u \\ &\quad + \mathbf{e}_\theta \left[-\beta\delta l B^0 \frac{S^2}{R^2} \frac{I_l'(x)}{X I_l'(X)} + \hat{C}^{\beta\delta} \frac{l}{x} I_l(x) \right] \cos u - \mathbf{e}_z \hat{C}^{\beta\delta} I_l(x) \cos u; \\ \mathbf{j}^{\beta\delta} &= \mathbf{e}_r \beta\delta l B^0 \frac{S}{R^2} \frac{I_l'(x)}{I_l'(X)} \sin u + \mathbf{e}_\theta \beta\delta l^2 B^0 \frac{S^2}{R^2} \frac{I^l(X)}{r} \frac{I_l(x)}{I_l(X)} \cos u \\ &\quad - \mathbf{e}_z \beta\delta B^0 h \frac{S^2}{R^2} I^l(X) \frac{I_l(x)}{I_l(X)} \cos u; \\ p^{\beta\delta} &= \beta\delta B^0 \frac{S r}{R^2} \frac{I_l'(x)}{I_l'(X)} \cos u. \quad (0 \leq r \leq R + \varrho^\delta + \varrho^{\beta\delta}) \\ \mathbf{B}^{\beta\delta} &= \mathbf{e}_r [\bar{C}^{\beta\delta} I_l'(x) + \bar{D}^{\beta\delta} K_l'(x)] \sin u + \mathbf{e}_\theta \frac{l}{x} [\bar{C}^{\beta\delta} I_l(x) + \bar{D}^{\beta\delta} K_l(x)] \cos u \\ &\quad - \mathbf{e}_z [\bar{C}^{\beta\delta} I_l(x) + \bar{D}^{\beta\delta} K_l(x)] \cos u; \quad \mathbf{j}^{\beta\delta} = 0; \quad p^{\beta\delta} = 0. \quad (R \leq r \leq S), \end{aligned} \quad (81)$$

where

$$\begin{aligned} \hat{C}^{\beta\delta} &= \frac{A}{I_l'(hR)} + \frac{B}{I_l'(X)} + hR \left[\frac{K_l'(X)}{I_l'(X)} - \frac{K_l(hR)}{I_l(hR)} \right] [C I_l(hR) + A I_l'(hR)], \\ \bar{C}^{\beta\delta} &= \frac{B}{I_l'(X)} + hR \frac{K_l'(X)}{I_l'(X)} [C I_l(hR) + A I_l'(hR)], \quad \bar{D}^{\beta\delta} = -hR [C I_l(hR) + A I_l'(hR)]; \\ A &= \beta\delta B^0 \frac{S}{R} \frac{I_l'(hR)}{I_l'(X)}, \quad B = \beta\delta B^0 \frac{X}{2}, \quad C = -\beta\delta B^0 \frac{l^2 S^2}{hR^3} I^l(X) \frac{I_l(hR)}{I_l(X)}; \end{aligned}$$

and $K_l(x)$ is the modified Bessel function of the second kind as defined by Watson.¹⁷

To order $\eta\delta$:

$$\begin{aligned}
 \mathbf{B}^{\eta\delta} &= \mathbf{e}_r \left[-\eta\delta 2lB^0 \frac{S^2}{Rr} I'(X) \frac{I_l(x)}{I_l(X)} + \hat{C}^{\eta\delta} I_l'(x) \right] \sin u \\
 &\quad + \mathbf{e}_\theta \left[-\eta\delta 2B^0 \frac{S}{r} \frac{I_l'(x)}{I_l'(X)} + \hat{C}^{\eta\delta} \frac{l}{x} I_l(x) \right] \cos u - \mathbf{e}_z \hat{C}^{\eta\delta} I_l(x) \cos u; \\
 \mathbf{j}^{\eta\delta} &= \mathbf{e}_r \eta\delta 2B^0 \frac{X}{R} \frac{I_l'(x)}{I_l'(X)} \sin u + \mathbf{e}_\theta \eta\delta 2B^0 l X I'(X) \frac{S}{Rr} \frac{I_l(x)}{I_l(X)} \cos u \\
 &\quad - \mathbf{e}_z \eta\delta 2B^0 \frac{X^2}{R} I'(X) \frac{I_l(x)}{I_l(X)} \cos u; \quad p^{\eta\delta} = 0. \quad (0 \leq r \leq R) \\
 \mathbf{B}^{\eta\delta} &= \mathbf{e}_r [\bar{C}^{\eta\delta} I_l'(x) + \bar{D}^{\eta\delta} K_l'(x)] \sin u + \mathbf{e}_\theta \frac{l}{x} [\bar{C}^{\eta\delta} I_l(x) + \bar{D}^{\eta\delta} K_l(x)] \cos u \\
 &\quad - \mathbf{e}_z [\bar{C}^{\eta\delta} I_l(x) + \bar{D}^{\eta\delta} K_l(x)] \cos u; \quad \mathbf{j}^{\eta\delta} = 0; \quad p^{\eta\delta} = 0. \quad (R \leq r \leq S) \\
 \hat{C}^{\eta\delta} &= -\eta\delta \frac{lB^0 R}{S I_l'(X)} + \eta\delta 2lB^0 \frac{S^2}{R^2} I'(X) \frac{I_l(hR)}{I_l(X)} \left[\frac{1}{I_l'(X)} + \frac{hR K_l'(hR) I_l(hR)}{I_l'(hR)} - \frac{hR K_l'(X) I_l(hR)}{I_l'(X)} \right], \\
 \bar{C}^{\eta\delta} &= -\eta\delta \frac{lB^0 R}{S I_l'(X)} - \eta\delta 2lB^0 \frac{SX}{R} I'(X) \frac{(I_l(hR))^2 K_l'(X)}{I_l'(X) I_l(X)}, \quad \bar{D}^{\eta\delta} = \eta\delta 2lB^0 \frac{SX}{R} I'(X) \frac{(I_l(hR))^2}{I_l(X)}. \quad (82)
 \end{aligned}$$

APPENDIX II

INTERCHANGE INSTABILITY

If $j^\eta = 0$, only helically invariant fields with $l = 0$ or 3 and small hR are present, and the plasma fills the entire tube ($R = S$), (50), (51), (52), and (53) can be written as

$$2\delta W_{m;n} = \frac{8\pi^2}{k} \int_0^1 \left\{ \mu'^2 + \left(\frac{m^2}{t^2} - \frac{\beta f'(t)\Gamma(t)}{t^2(t^2 - a^2)^2} \right) \mu^2 \right\} t \, dt \quad (83)$$

and

$$\mu = 2kRB^0 \left(\sum_i p_i \delta_i^2 \right) (t^2 - a^2) t |\xi_{r^0}(m;n)|, \quad (84)$$

where

$$a^2 \equiv -n/[2m \sum_i p_i \delta_i^2] \quad (85)$$

$$\Gamma(t) \equiv t[\sum_i 2(p_i \delta_i t)^2 + \frac{3}{2} q_i^2 e_i^2] / [2 \sum_i p_i \delta_i^2]^2, \quad (86)$$

and the other quantities are defined by (70). Clearly $\mu(0)$ and $\mu(1)$ are zero, $\mu(t)$ must go to zero at a at least as fast as $t - a$, and $\mu'(a)$ must be continuous.

We shall determine the critical β for each value of m and n for which a is in the range $0 \leq a \leq 1$. We shall show later that other values of m and n do not make the situation worse so that the true critical β is the least of these. The Euler equation which is obtained by varying (83) with respect to μ is

$$\mu'' + \frac{1}{t} \mu' + \left\{ \frac{\beta f'(t)\Gamma(t)}{t^2(t^2 - a^2)^2} - \left(\frac{m}{t} \right)^2 \right\} \mu = 0. \quad (87)$$

(We need not carry a normalization condition explicitly in this discussion.) In the vicinity of the singularity at a , (87) behaves like

$$\mu'' + \frac{1}{a} \mu' + \left\{ \frac{\beta f'(a)\Gamma(a)}{4a^4(t-a)^2} - \left(\frac{m}{a} \right)^2 \right\} \mu = 0. \quad (88)$$

The contributions to δW from the two regions $t < a$ and $t > a$ can be considered separately.

For $t > a$, the solutions of (88) are

$$\mu = (t - a)^{\frac{1}{2}} \exp[\pm \frac{1}{2}(1 - A)^{\frac{1}{2}} \ln(t - a)] \quad (89)$$

where

$$A = \beta f'(a)\Gamma(a)/a^4. \quad (90)$$

We now show that for values of m and n under consideration, the critical β is determined by setting A equal to 1.

If $A > 1$, the solution μ of (87) which vanishes at $t = 1$, is

$$\mu = (t - a)^{\frac{1}{2}} \cos \left\{ \frac{1}{2}(A - 1)^{\frac{1}{2}} \ln(t - a) + \gamma \right\}, \quad (91)$$

for t sufficiently near a . It therefore must possess at least one zero for $t > a$. For a particular value of A , $A_1 > 1$, we let t_1 be the largest zero below 1, and consider for this A_1 a function μ_1 , which we define to be identically zero for $t < t_1$ and to be a solution of (87) for $t > t_1$. It follows from (83) and (87) that for this μ_1 , $\delta W = 0$ if $A = A_1$ and $\delta W < 0$ if $A > A_1$. Therefore, we have shown that the critical β corresponds to a value of $A \leq 1$.

In cases where $A < 1$, we first consider any μ which is identically zero for all t less than some $t_1 > a$ and vanishes at $t = 1$. We now show that for μ 's associated with this t_1 , an $A_1 > 1$ exists such that $\delta W > 0$ for all $A < A_1$, and therefore for $A < 1$. We consider the pressure distribution given by (101), which we later shall find leads to the largest true critical β . For this distribution with m and $A = \beta/\beta$ set equal to 1, the general solution of (87)

$$\mu = [1 - (\alpha/t)^2]^{\frac{1}{2}} [C + D \ln(t^2 - \alpha^2)], \quad (92)$$

cannot vanish between α and 1 since C/D must be chosen to make it vanish at $t = 1$. Since δW is a continuous function of β , the position t_1 , at which the minimizing μ must first vanish for a given $A_1 > 1$, must approach α as A_1 approaches 1. Since the solution of the Euler equation with no zeros between t_1 and 1 yields the minimum value of δW over all μ 's which vanish identically for $\alpha < t < t_1$ and for $A = A_1$ this value is zero, all such μ 's make $\delta W \geq 0$ for $A = A_1$. We therefore have shown that, for this pressure distribution, for any $t_1 > \alpha$, an $A_1 > 1$ exists below which $\delta W > 0$ for any μ which is identically zero for $t < t_1$ and vanishes at $t = 1$.

We must still show that for $A < 1$ no other μ which vanishes at $t = \alpha$ and 1 (i.e., $t_1 = \alpha$) can lead to instability. To do this, we assume that such a μ exists which makes $\delta W < 0$, say $\delta W = -\epsilon$. We first show that for any sufficiently small δ , the contribution to δW from the region between α and $\alpha + \delta$ is positive, so that the integral from $t = \alpha + \delta$ to $t = 1$ must be more negative than $-\epsilon$. We shall then show that this integral from $t = \alpha + \delta$ to $t = 1$ differs from a positive definite integral by an amount which can be made as small as we desire by taking δ sufficiently small. This contradiction will complete the proof of stability for $A < 1$.

The contribution to δW from the region between α and $\alpha + \delta$ is, from (83),

$$I = \frac{8\pi^2\alpha}{k} \int_0^\delta \left\{ (y^{*'})^2 + \frac{1 + (2mx/\alpha)^2 - A}{4x^2} y^{*2} + \frac{1}{2x} (y^{*2})' \right\} x dx, \quad (93)$$

where

$$x = t - \alpha, \quad (94)$$

$$y^* = \mu^*/\sqrt{x}, \quad (95)$$

and the prime indicates differentiation with respect to x . The first two terms are positive since $A < 1$. Since $y^*(x)$ must go to zero as x goes to zero, we see that the last term is also positive for any y^* .

We now consider the value of δW which corresponds to a particular μ , that is $\hat{\mu}$, which is zero for $t < \alpha + \frac{1}{2}\delta$, increases linearly over the region between $\alpha + \frac{1}{2}\delta$ and $\alpha + \delta$, and from there to $t = 1$ is the same as the previous μ^* , which was assumed to make $\delta W < 0$. Since A is less than the A_1 which determines the solution of (87) which first vanishes at $t_1 = \alpha + \frac{1}{2}\delta$, this value of δW must be positive. It differs from the one for μ^* , which had to be less than $-\epsilon$, by the amount

$$I = (8\pi^2\alpha/k) \int_{\frac{1}{2}\delta}^{\delta} \left\{ (\hat{\mu}')^2 + [(m/\alpha)^2 - A/4x^2] \mu^2 \right\} dx. \quad (96)$$

This integral is less than a fixed positive quantity times δ , so that we can make it smaller than ϵ by a suitable choice of δ . This completes the proof that no μ exists which can make δW negative if $A < 1$ and,

therefore, if only the region $t > \alpha$ is considered, the critical β is determined by setting $A = 1$ in (90).

We can replace (89) by

$$\mu = (\alpha - t)^{\frac{1}{2}} \exp \left[\pm \frac{1}{2} (1 - A)^{\frac{1}{2}} \ln(\alpha - t) \right] \quad (97)$$

in the region $t < \alpha$ and repeat the entire argument to show that $A = 1$ again determines the critical β . If α is zero or one, the argument can still be carried through. Before considering values of m and n for which α^2 is not in the range $0 \leq \alpha^2 \leq 1$, we shall determine the pressure distribution which maximizes the critical β with α^2 in this range. We shall then show that for this optimum pressure distribution, instabilities for which α^2 is not in this range lead to higher critical values of β .

If $0 \leq \alpha^2 \leq 1$, the critical β is

$$\beta_c = \min_{\alpha} [\alpha^4/f'(\alpha)\Gamma(\alpha)]. \quad (98)$$

We now determine the pressure distribution, i.e., $f(t)$, so that for the worst volume of α , the critical β is as large as possible. For any particular $f(t)$, (98) requires that

$$\beta_c f'(\alpha) \leq \alpha^4/\Gamma(\alpha). \quad (99)$$

We consider large enough values of m and n that we can treat α as a continuous variable. Integrating both sides of (99) with respect to α and using the boundary conditions (14) on $f(t)$, we find that for any f ,

$$\beta_c \leq \int_0^1 d\alpha \alpha^4/\Gamma(\alpha). \quad (100)$$

However, for

$$f'(t) = t^4/\Gamma(t) / \int_0^1 [t^4/\Gamma(t)] dt, \quad (101)$$

$$\beta_c = \min_{\alpha} \int_0^1 dt t^4/\Gamma(t) = \int_0^1 dt t^4/\Gamma(t), \quad (102)$$

and this f is the optimum distribution. Since $\Gamma(t)$ is given by (86) the optimum pressure and critical β are given by (67), (68) and (69).

We must still show that instabilities for which α^2 is not in the range $0 \leq \alpha^2 \leq 1$ do not lead to lower critical values of β . It can be seen at once, by inserting the pressure distribution (101) into (83), that $\delta W > 0$ if $\alpha^2 < 0$ and $\beta < \beta_c$. Similarly, if we set

$$y = \mu/(\alpha - t)^{\frac{1}{2}}, \quad (103)$$

it is immediately apparent that $\delta W > 0$ if $\alpha^2 > 0$ and $\beta < \beta_c$ for this pressure distribution.

APPENDIX III

STABILIZATION OF THE KINK INSTABILITY

If $\beta = 0$, if $S = \infty$, if j^y is constant in the plasma region, if the external region is a vacuum, and if only one helically invariant field with $l = 3$ and small hR is present, we find by minimizing (61) over the volume regions that

$$2\delta W_{m,n} = 4kR^2B^2 \left| \xi_r^0(m;n) \right|^2 \times \left[\frac{m}{1 - a^{2m}} (-q + \iota^\eta + \iota^{\delta\delta})^2 - \iota^\eta (-q + \iota^\eta + \iota^{\delta\delta}) \right]_{r=R}, \quad (104)$$

where

$$q \equiv -2\pi n/m, \quad (105)$$

$$a^2 = (q - \iota^\eta)/\iota^{\delta\delta}, \quad (0 \leq a^2 < 1) \quad (106)$$

and a is the value of r/R where μ must vanish because $q + \iota^\eta(r) + \iota^{\delta\delta}(r)$ does, and all quantities are evaluated at the plasma boundary, R . If a does not satisfy the inequality in (106), we set it equal to zero.

We first consider cases where $0 < \iota^\eta < \iota^{\delta\delta}$. If $q < \iota^{\delta\delta}$, the last term in (104) is less than the first so that $\delta W > 0$. If $\iota^{\delta\delta} < q < \iota^{\delta\delta} + \iota^\eta$, a is not zero and we can show that $\delta W > 0$ for all m . If $q > \iota^{\delta\delta} + \iota^\eta$, both terms of (104) are positive. We have thus shown that the system is stable for all values of m if $0 < \iota^\eta < \iota^{\delta\delta}$.

We next consider cases where $0 < \iota^{\delta\delta} < \iota^\eta$. As before $\delta W > 0$ unless q is in the range $\iota^{\delta\delta} < q < \iota^{\delta\delta} + \iota^\eta$. If $\iota^{\delta\delta} < q < \iota^\eta$, then $a = 0$, and

$$\delta W \sim [m(-q + \iota^{\delta\delta}) + (m-1)\iota^\eta] \times [-q + \iota^\eta + \iota^{\delta\delta}] \quad (107)$$

is negative for $m = 1$ for all q 's in the range. For

simplicity we shall not consider higher values of m . If $\iota^\eta < q < \iota^{\delta\delta} + \iota^\eta$, then a is not zero and

$$\delta W \sim \left[\frac{m(\iota^{\delta\delta})^m}{(\iota^{\delta\delta})^m - (q - \iota^\eta)^m} - \iota^\eta \right] \times [-q + \iota^{\delta\delta} + \iota^\eta] \quad (108)$$

is again negative for $m = 1$ for these q 's. We have therefore shown that if $0 < \iota^{\delta\delta} < \iota^\eta$, the system is unstable for $m = 1$ if and only if $\iota^{\delta\delta} < q < \iota^{\delta\delta} + \iota^\eta$. Conditions for instability have not been obtained for higher m .

We finally consider cases where $\iota^\eta < 0 < \iota^{\delta\delta}$. Again $\delta W > 0$ unless q is in the range $\iota^{\delta\delta} + \iota^\eta < q < \iota^{\delta\delta}$ (ι^η is negative). For these q 's, $a^2 > 1$ and must therefore be set equal to zero in (104), so that (107) holds. For $m = 1$, δW is negative for all these q 's. Again we do not here obtain stability conditions for higher m .

If we replace $\iota^{\delta\delta}$ by $-\iota^{\delta\delta}$, ι^η by $-\iota^\eta$, and q by q , (104) is unchanged. We have therefore shown that the system is stable for all m if $|\iota^{\delta\delta}| > |\iota^\eta|$ and $\iota^{\delta\delta}$ and ι^η have the same sign. Otherwise it is unstable for $m = 1$ for values of q which lie between $\iota^{\delta\delta}$ and $\iota^{\delta\delta} + \iota^\eta$. Higher m 's have not been considered in these regions. It should be remembered that these results apply to the case where the external region is a vacuum, incapable of supporting a current. We see from (105) that q is limited to the values $-2\pi n/m$, where n can be any integer and can thus construct Fig. 4.

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Equilibrium of a Magnetically Confined Plasma in a Toroid

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A. INTRODUCTION

A static equilibrium of plasma (or of conducting fluid) with scalar pressure p and magnetic field $\mathbf{B}$ is often described by the magnetostatic equations

$$\nabla p = \mathbf{j} \times \mathbf{B}, \quad (\text{A1})$$

$$\nabla \times \mathbf{B} = \mathbf{j}, \quad (\text{A2})$$

$$\nabla \cdot \mathbf{B} = 0, \quad (\text{A3})$$

where $\mathbf{j}$ is the electric current density. In particular, these equations apply to many proposed controlled thermonuclear reactors and their prototypes, especially the stellarator¹ and the recently much discussed stabilized pinch. In sections B, C, and D there are derived² a variety of properties possessed by solutions of (A1–3).

One of these properties is that if p is constant on the boundary of its region of definition then, under some mild additional assumptions, that boundary must be topologically toroidal. However, prescribing such a boundary surface and the value of p on it by no means determines a unique solution, even though there are as many equations as unknowns (two vector and one scalar). One of our objects is to establish additional conditions which together with the magnetostatic equations (and the boundary prescription) do determine a unique solution. This is achieved in several different ways, the additional conditions always amounting to the specification of two numbers for each surface of constant p .

An experiment is imagined (section E) in which an ideal viscous hydromagnetic fluid exhibits a damped motion until coming to rest in an equilibrium configuration. A number of invariants with respect to any such motion are described in section F. These lead to constraints on the admissible trial states in a variational principle (sections G, H, and I) suggested by the experiment. The quantity varied is the potential energy which is the sum of the magnetic and the internal fluid energies. The variational principle provides a potentially powerful tool for proving the existence of solutions of the magnetostatic equations as well as for obtaining them numerically. It also

provides a characterization of solutions by their values of the invariants.

In Section J equations governing a steady state of magnetic field and slowly diffusing plasma are introduced.³ These amount to the magnetostatic equations together with two auxiliary conditions (K3, 7) for each surface of constant pressure (section K). The system of equations obtained is expected to have a solution which is unique, and this is verified in section L for the limiting case of low pressure.

B. MAGNETIC SURFACES

The magnetostatic equations (A1–3) have the simple consequences

$$\nabla \cdot \mathbf{j} = 0, \quad (\text{B1})$$

$$\mathbf{B} \cdot \nabla p = 0, \quad (\text{B2})$$

$$\mathbf{j} \cdot \nabla p = 0, \quad (\text{B3})$$

$$\mathbf{B} \cdot \nabla \mathbf{B} = \nabla(p + \frac{1}{2}\mathbf{B}^2), \quad (\text{B4})$$

$$\nabla \cdot (\mathbf{B} \times \nabla p) = 0, \quad (\text{B5})$$

$$\mathbf{B} \cdot \nabla \mathbf{j} = \mathbf{j} \cdot \nabla \mathbf{B}, \quad (\text{B6})$$

$$\mathbf{B} \cdot \nabla (\mathbf{B} \cdot \mathbf{j}) = \mathbf{j} \cdot \nabla \mathbf{B}^2 \quad (\text{B7})$$

Here (B1) follows from (A2), (B2, 3) from (A1), (B4) from (A1, 2), (B5) from (A2) and (B3), (B6) from the curl of (A1) in view of (A3) and (B1), and finally (B7) from (A1) and (B5) in view of (A3) and (B1).

If p is reasonably smooth and not constant in any (small) region, the equation $p = P$ determines a family of surfaces characterized by their values of the parameter P . By (B2) they are “magnetic surfaces”, in the sense that they are made up of lines of magnetic force, and similarly by (B3) they are “current surfaces”. If such a surface lies in a bounded volume of space and has no edges (because of not intersecting the edge of the region of definition of p) and if either $\mathbf{B}$ or $\mathbf{j}$ nowhere vanishes on it, then by a well-known theorem⁴ it must be either a toroid (by which we mean a topological torus) or a Klein’s bottle. The latter, however, is not realizable in physical space.

Under normal circumstances each surface $p = P$ (excepting a set of values of P of measure zero) is

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traversed ergodically and consequently determined by any line of force contained in it. Even when this is not so, however, we shall call it a magnetic surface.

As suggested by the foregoing discussion, we now explicitly assume that the magnetic surfaces form a family of nested toroids. The innermost toroid is degenerate, consisting of a single closed curve called the (magnetic) axis. We shall usually also assume that ϕ increases monotonically going inward (as is proved for steady diffusing plasmas in section K) and indeed that $\nabla\phi \neq 0$ except on the axis.

C. SURFACE QUANTITIES

We now introduce two coordinate functions η and θ . Each is to be multi-valued, its values at any point differing by integers. The function η is to be continuous everywhere and to increase by unity during one traversal of the magnetic axis. The function θ is to be continuous everywhere except at the axis and is to increase by unity during one small loop around the axis. Finally, a pair of values of η and θ is to determine a unique point on each magnetic surface. For definiteness we assume that η , θ , ϕ form a left-handed coordinate system.

For each particular magnetic surface we now define

$$V \equiv \int d\tau \quad (C1)$$

$$U \equiv \int d\tau \mathbf{B}^2 \quad (C2)$$

$$K \equiv \int d\tau \mathbf{B} \cdot \mathbf{j} \quad (C3)$$

$$\psi \equiv \int d\tau \mathbf{B} \cdot \nabla \eta \quad (C4)$$

$$\chi \equiv \int d\tau \mathbf{B} \cdot \nabla \theta \quad (C5)$$

$$I \equiv \int d\tau \mathbf{j} \cdot \nabla \eta \quad (C6)$$

$$J \equiv \int d\tau \mathbf{j} \cdot \nabla \theta, \quad (C7)$$

where $d\tau$ is the volume element and the region of integration is always the interior of the particular surface. (The integrals are well defined, since $\nabla\eta$ and $\nabla\theta$ are single-valued.) We note that V is the enclosed volume and U is twice the enclosed magnetic energy. There seems to be no simple physical interpretation of K , but its vanishing will be seen to be significant (section K). The integrands of (C4-7) can be written as divergences by (A3) and (B1), so we may apply Gauss' theorem; however, since η and θ are not single-valued, it is necessary first to cut the region of integration, say at $\eta = 0$ or at $\theta = 0$ as appropriate. Since by (B2) or (B3) the boundary contribution vanishes except for the double boundary at the cut, we then obtain, for example,

$$\begin{aligned} \psi &= \int_{\eta=1} dS (\nabla\eta / |\nabla\eta|) \cdot \mathbf{B} \eta - \int_{\eta=0} dS (\nabla\eta / |\nabla\eta|) \cdot \mathbf{B} \eta \\ &= \int_{\eta=1} dS (\nabla\eta / |\nabla\eta|) \cdot \mathbf{B}, \end{aligned} \quad (C8)$$

where dS is the area element and the integrations are over those parts of the indicated surfaces of constant η which are interior to the particular magnetic surface. Thus ψ is the longitudinal magnetic flux inside the magnetic surface, i.e., the magnetic flux through any cross section of the interior. Similarly, I is the longitudinal current inside the particular surface. On the other hand, χ is what may be called the azimuthal magnetic flux inside the magnetic surface, since it is the flux through any ribbon-like surface of constant θ of which one edge is the magnetic axis and the other lies on the particular surface. Similarly, J is the azimuthal current inside the particular surface.

Functions of position (like ϕ) which are constant on magnetic surfaces will be called surface quantities. The quantities defined by (C1-7) may be interpreted as functions of position in an obvious way and are then surface quantities. Any surface quantity may be considered as a function of any other, and derivatives of one with respect to another are meaningful and are themselves surface quantities.

It may be noted that definitions (C4-7) are invariant under continuous deformation of the coordinate functions η and θ . All functions η with the same direction of increase along the axis are deformable into each other. The analogous statement does not hold for θ , however; two functions θ are continuously deformable into one another if and only if their ribbons of constancy wind around the axis the same number of times. Two functions θ differ by an integral multiple of an acceptable function η , the integral multiplier being the difference of the winding numbers. If θ is increased by an integral multiple of η , χ and J are increased by the corresponding integral multiples of ψ and I , respectively. The results of the next section are manifestly invariant under these changes.

D. RELATIONS AMONG SURFACE QUANTITIES

Let $\mathbf{w}$ be any single-valued vector field satisfying

$$\nabla\phi \cdot (\nabla \times \mathbf{w}) = 0. \quad (D1)$$

Let $\mathbf{z}(P)$ be a particular point on each surface $\phi = P$. For each point $\mathbf{x}$ in space define

$$v(\mathbf{x}) = \int_{\mathbf{z}(P)}^{\mathbf{x}} d\mathbf{x} \cdot \mathbf{w}, \quad (D2)$$

where the path of integration lies on the surface; in view of (D1) it follows from Stokes' theorem that the value of $v(\mathbf{x})$ is independent of the path joining $\mathbf{z}$ to $\mathbf{x}$ for all paths continuously deformable into each other. However, not all paths are deformable into each other, so v is multi-valued. It can clearly be written

$$v = \lambda + \eta \oint_{\phi=P}^{\eta=0} d\mathbf{x} \cdot \mathbf{w} + \theta \oint_{\phi=P}^{\theta=0} d\mathbf{x} \cdot \mathbf{w}, \quad (D3)$$

where λ is some single-valued function and the loop integrals are taken in the direction of increasing η and θ , respectively.

If it were not for the variability of the lower limit in (D2), we would have $\nabla v = \mathbf{w}$; as it is we have

$$\nabla p \times (\nabla v - \mathbf{w}) = 0 \quad (\text{D4})$$

or equivalently

$$\mathbf{B} \cdot (\nabla v - \mathbf{w}) = 0, \quad \mathbf{j} \cdot (\nabla v - \mathbf{w}) = 0. \quad (\text{D5})$$

Now introduce two general surface quantities

$$F \equiv \int d\tau \mathbf{B} \cdot \mathbf{w}, \quad G \equiv \int d\tau \mathbf{j} \cdot \mathbf{w}. \quad (\text{D6})$$

By (D5, 3) and (C4-7) we then obtain

$$\begin{aligned} dF &= d \int d\tau \mathbf{B} \cdot \nabla v \\ &= d\psi \oint_{\theta=0} d\mathbf{x} \cdot \mathbf{w} + d\chi \oint_{\eta=0} d\mathbf{x} \cdot \mathbf{w}, \end{aligned} \quad (\text{D7})$$

$$dG = dI \oint_{\theta=0} d\mathbf{x} \cdot \mathbf{w} + dJ \oint_{\eta=0} d\mathbf{x} \cdot \mathbf{w}. \quad (\text{D8})$$

We are now in a position to obtain various relations among the surface quantities by special choices of $\mathbf{w}$ satisfying (D1). In view of (A2) and (B3) we are justified in choosing $\mathbf{w} = \mathbf{B}$. In this case $F = U$ and $G = K$, while by Stokes' theorem applied to that part of the respective surface $\eta = 0$ or $\theta = 0$ which lies inside the magnetic surface we have

$$\oint_{\eta=0} d\mathbf{x} \cdot \mathbf{B} = -I, \quad (\text{D9})$$

$$\oint_{\theta=0} d\mathbf{x} \cdot \mathbf{B} = J + \oint_{\psi=0} d\mathbf{x} \cdot \mathbf{B}, \quad (\text{D10})$$

where the last integral is taken around the magnetic axis ($\psi = 0$) in the direction of increasing η . From (D7, 8) we, therefore, obtain

$$dU = \left(J + \oint_{\psi=0} d\mathbf{x} \cdot \mathbf{B} \right) d\psi - Id\chi, \quad (\text{D11})$$

$$dK = \left(J + \oint_{\psi=0} d\mathbf{x} \cdot \mathbf{B} \right) dI - IdJ. \quad (\text{D12})$$

Another choice for $\mathbf{w}$ is the vector potential $\mathbf{A}$ ($\nabla \times \mathbf{A} = \mathbf{B}$), justified by (B2). This leads analogously to

$$d \int d\tau \mathbf{B} \cdot \mathbf{A} = \left(\chi + \oint_{\psi=0} d\mathbf{x} \cdot \mathbf{A} \right) d\psi - \psi d\chi, \quad (\text{D13})$$

$$d \int d\tau \mathbf{j} \cdot \mathbf{A} = \left(\chi + \oint_{\psi=0} d\mathbf{v} \cdot \mathbf{A} \right) dI - \psi dJ. \quad (\text{D14})$$

Our next choice is

$$\mathbf{w} = (\mathbf{B} \times \nabla p) / (\nabla p)^2, \quad (\text{D15})$$

which may be justified by observing that

$$\nabla p \times \mathbf{w} = \mathbf{B} \quad (\text{D16})$$

in view of (B2) and then using (A3). In this case we have $F = 0$ and $G = V$ in view of (A1). Also,

$$\begin{aligned} \oint_{\eta=0} d\mathbf{x} \cdot \mathbf{w} &= \oint_{\eta=1} d\mathbf{x} \cdot \mathbf{w} \eta - \oint_{\eta=0} d\mathbf{x} \cdot \mathbf{w} \eta \\ &= - \int_{p=P} dS (\nabla p / |\nabla p|) \cdot [\nabla \times (\mathbf{w} \eta)] \\ &= \int_{p=P} dS (\nabla p / |\nabla p|) \cdot (\mathbf{w} \times \nabla \eta) \\ &= \int_{p=P} dS (\mathbf{B} \cdot \nabla \eta) / |\nabla p| = -d\psi/dp. \end{aligned} \quad (\text{D17})$$

where the first step is trivial, the second follows from Stokes' theorem applied to the cut magnetic surface, the third follows from (D1), the fourth from (D16), and the last from (C4) and the fact that

$$d\tau = -dS dp / |\nabla p|$$

(since $|dp|/|\nabla p|$ is the distance between two neighboring magnetic surfaces). Similarly

$$\oint_{\theta=0} d\mathbf{x} \cdot \mathbf{w} = d\chi/dp. \quad (\text{D18})$$

Thus (D7) is tautological, but (D8) gives

$$dp dV = d\chi dI - d\psi dJ. \quad (\text{D19})$$

Another possibility is to choose $\mathbf{w}$ to be a gradient, thus satisfying (D1) trivially. Indeed, let $\mathbf{w} = \nabla \mathbf{q}$ be a gradient of a vector field and, therefore, a dyadic, and note that nothing in the derivation of (D7, 8) is invalidated. If $\mathbf{q}$ is single-valued, the loop integrals in (D7, 8) vanish and we may conclude that F and G themselves (now vectors) vanish, which is also obvious from Gauss' theorem. Since $\nabla \mathbf{x}$ is the unit dyadic, taking $\mathbf{q} = \mathbf{x}$ gives

$$\int d\tau \mathbf{B} = 0, \quad \int d\tau \mathbf{j} = 0. \quad (\text{D20})$$

Taking $\mathbf{q} = \mathbf{B}$ instead and using (B4) gives

$$\begin{aligned} 0 &= \int d\tau \nabla(p + \frac{1}{2}\mathbf{B}^2) = \\ &\quad - \int_{p=P} dS (\nabla p / |\nabla p|) (p + \frac{1}{2}\mathbf{B}^2), \end{aligned} \quad (\text{D21})$$

and since the first term of the integrand contributes nothing (take p outside the integral and convert back to a volume integral),

$$\int_{p=P} dS (\nabla p / |\nabla p|) \mathbf{B}^2 = 0. \quad (\text{D22})$$

E. AN IMAGINED EXPERIMENT

Suppose that everywhere in a given rigid toroidal tube T with perfectly conducting walls there is a viscous perfectly conducting fluid with an adiabatic equation of state, and also a magnetic field tangent to the tube walls at the walls. Suppose that any heat generated by the viscosity is somehow magically removed so that each element of fluid is isentropic. The system can then lose but not gain energy since there can be no energy flux through the walls.

Let the fluid be initially at rest. In general, it will not be in equilibrium and will start to move. As long as it moves it loses energy, so it must eventually come to rest in a state with less energy than its initial state. Clearly, an initially resting state of minimum energy cannot start moving at all and so must be in equilibrium; i.e., satisfy the magnetostatic equations.

Since we are comparing resting states we are interested in the non-kinetic (i.e., potential) energy W

$$W = \int_T d\tau \left(\frac{1}{2} \mathbf{B}^2 + p/[\gamma - 1] \right) \quad (\text{E1})$$

where the first and second terms of the integrand are the energy densities of the magnetic field and of the fluid, respectively; γ being the ratio of specific heats of the fluid. (If $\gamma = 1$ the second term should be $p \ln p$.)

F. INVARIANTS

It now appears that minimizing W should provide equilibrium solutions. However, we must be careful. If we minimize W outright we obtain $\mathbf{B} = 0$, $p = 0$, which, although certainly an equilibrium solution, is of no interest and is clearly not a state which will be reached eventually by every initial state. We have neglected to observe that any motion of our fluid is subject to certain constraints. By a constraint here is meant a condition that some quantity be an invariant during any motion, an invariant being a constant of motion which depends only on the instantaneous state and not on the velocity. (All constraints here are holonomic.) Only states with the same invariants can possibly be transformed into each other by a motion. We should therefore not minimize W among all states, but only among states with the same values of the invariants as the initial state.

We must therefore find invariants. Since the fluid is a perfect conductor it carries lines of force with it.⁵ Therefore any topological property of the lines of force is an invariant. For instance, if in the initial state there were a line of force ergodic in T then this would have to be carried into a similarly ergodic line in the final state. But the final state has to satisfy (B2), so that p will be constant on the line and hence constant everywhere (if it is to be continuous). Such a state is not of interest. This example shows that we must choose the initial magnetic field to have precisely those topological properties possessed by equilibria of interest.

Accordingly, we choose an initial magnetic field which has a family of nested toroidal magnetic surfaces which are, however, not necessarily surfaces of constant p . The quantities ψ and χ are then defined, and since the lines of force are carried with the fluid, ψ and χ for the magnetic surface formed by some definite set of fluid particles are invariant during a motion. The quantities V , U , K , I and J are also defined but need not be invariant.

The way in which a line of force intertwines with itself as it is continued around its magnetic surface many times is a topological property and therefore

another invariant, but it turns out to be describable in terms of ψ and χ and therefore does not provide an independent constraint. Indeed, this intertwining is characterized by the limit of the ratio of the number of loops around the magnetic axis to the number of traversals around the length of the toroid made by a line of force indefinitely prolonged; i.e., the limit of θ/η following the line. This limit is usually denoted by $\iota/2\pi$ and is equal to $d\chi/d\psi$.

Let ρ be the mass density of the fluid. Then $\rho d\tau$ is the mass of a little element of fluid and is therefore invariant during a motion. Furthermore, the adiabatic law assumed amounts to requiring that p/ρ^γ be invariant for a fluid element. We thus have two purely hydrodynamic constraints. But ρ is of no interest since it enters neither into the magnetostatic equations nor into the potential energy W . Eliminating ρ we have only one invariant $p^{1/\gamma} d\tau$ for each element of fluid.

The invariants we have found (ψ and χ for magnetic surfaces, $p^{1/\gamma} d\tau$ for fluid elements) apply if we know which fluid element in the final state corresponds to each element in the initial state. However, there is no reference to this correspondence in (E1). We wish to minimize W for all states p , $\mathbf{B}$ which could possibly be reached by a motion from the initial state; i.e., for which there exists some correspondence preserving the value of the invariants. The correspondence not being known ahead of time, it now makes no sense to require that ψ and χ are individually preserved. Nevertheless, the correspondence must be chosen to preserve ψ and that same correspondence must preserve χ . Then χ considered as a function of ψ (for example) must be the same in the final state as in the initial state. In short, the functional relationship between χ and ψ is an invariant which can be specified without knowing the correspondence ahead of time. What has been done, in effect, is to label the surfaces with their values of ψ , after which the only magnetic invariant left is χ , now as a function of the label ψ .

These considerations so far only eliminate consideration of the correspondence of surfaces as a whole. To eliminate consideration of the correspondence of fluid elements within a given magnetic surface, we must use the invariant $p^{1/\gamma} d\tau$ to form a label. Assuming that lines of force are ergodic on almost all surfaces and choosing the correspondence for one particular fluid element as a reference point arbitrarily, we may use the integral of $p^{1/\gamma} d\tau$ along a little flux tube going from some point on the surface to the reference point as a label for that point. This label is clearly invariant and hence extends the correspondence from the arbitrary reference point to the whole surface (since the line of force through the reference point is assumed ergodic on the surface and hence covers a dense set of points).

However, in establishing this labeling we have not exhausted the information available from the invariance of $p^{1/\gamma} d\tau$. There remains the condition that the integral of $p^{1/\gamma} d\tau$ over the shell-like volume

bounded by two neighboring surfaces must obviously be invariant. Introducing the surface quantity

$$M = \int d\tau p^{1/\gamma} \quad (\text{F1})$$

(the integral being taken over the interior of the surface), we may require equivalently that M be invariant. (It may be observed that M is just proportional to the mass contained within the surface if ρ happens to be such that the fluid is isentropic; i.e., if p/ρ^γ is the same for all fluid elements. The invariance of M then represents conservation of mass.) As with χ , by considering M now to be a function of ψ , we eliminate any reference to the correspondence.

G. STATEMENT OF VARIATIONAL PRINCIPLE

The preceding considerations suggest the following variational principle: a function p and a solenoidal magnetic vector field $\mathbf{B}$ in T , forming nested toroidal magnetic surfaces and having a fixed total longitudinal flux and no normal component at the walls, make W stationary among all such pairs with the same invariant functions $\chi(\psi)$ and $M(\psi)$ if and only if $\nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B}$.

Before proving this it is desirable to reformulate it so as to include ψ explicitly in the characterization of a state, since otherwise it is difficult to tell whether a neighboring field $\mathbf{B}$ has nested toroidal surfaces. Correspondingly, we have an additional constraint and an additional variational condition. Thus we propose the following variational principle:

Consider all triples p , $\mathbf{B}$, and ψ in T satisfying the constraints

$$(a) \quad \psi \text{ has toroidal level surfaces, } \psi = C \text{ at the walls, } \min \psi = 0, \max \psi = C,$$

$$(b) \quad \nabla \cdot \mathbf{B} = 0, \quad (\text{G1})$$

$$(c) \quad \mathbf{B} \cdot \nabla \psi = 0, \quad (\text{G2})$$

$$(d) \quad \int_{\psi \leq c} d\tau \mathbf{B} \cdot \nabla \eta = c, \quad (\text{G3})$$

$$(e) \quad \int_{\psi \leq c} d\tau \mathbf{B} \cdot \nabla \theta = \chi(c), \quad (\text{G4})$$

$$(f) \quad \int_{\psi \leq c} d\tau p^{1/\gamma} = M(c); \quad (\text{G5})$$

here C is a constant and $\chi(c)$ and $M(c)$ are arbitrary fixed functions defined for $0 \leq c \leq C$, which is also the range of the last three conditions. Then a particular triple makes W stationary among all such triples if and only if it satisfies the variational conditions

$$(y) \quad p \text{ is a function of } \psi \text{ alone,}$$

$$(z) \quad \nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B}. \quad (\text{G6})$$

It should be noted that in non-degenerate cases in which (almost) any magnetic line of force covers a complete magnetic surface ergodically, (z) implies (y) in view of (B2) and (G2). In degenerate cases (y) is (partly) independent. But in these cases ψ is not itself physically significant, only p and $\mathbf{B}$, which

are consistent with a variety of functions ψ . Just in these degenerate cases we could have found additional constraints in our imagined experiment (the integral of $p^{1/\gamma} d\tau$ in each thin closed flux tube) and by omitting these from our formulation of the variational principle we force upon ψ a physical significance (the only constraint on the flow of fluid elements is that they stay on surfaces of constant ψ) which is reflected in the variational condition (y).

H. PROOF

Given a function ψ satisfying (a), it is easily seen by the methods used at the beginning of section D that the most general field $\mathbf{B}$ satisfying (b) and (c) is given by

$$\mathbf{B} = \nabla \psi \times \nabla \nu, \quad (\text{H1})$$

where ν is a multivalued function such that (H1) determines $\mathbf{B}$ uniquely; i.e., on each surface of constant ψ the various branches of ν differ only by constants. Furthermore, (d) and (e) are then satisfied if and only if ν can be written

$$\nu = \lambda + \eta \chi'(\psi) - \theta \quad (\text{H2})$$

with λ single-valued, as can be seen upon comparing (H1) with (D16, 4) and referring to (D3, 17, 18).

Let us first assume that W is stationary and derive (y) and (z). For the moment hold ψ and λ fixed and vary only p . Then

$$\begin{aligned} 0 = \delta W &= (\gamma - 1)^{-1} \int_T d\tau \delta p \\ &= (\gamma - 1)^{-1} \int_0^C dc \int_{\psi=c} (dS/|\nabla \psi|) \delta p \end{aligned} \quad (\text{H3})$$

for any perturbation δp satisfying

$$\gamma^{-1} \int_{\psi=c} (dS/|\nabla \psi|) p^{1/\gamma-1} \delta p = 0, \quad (\text{H4})$$

which is obtained from (f) by varying p and also differentiating with respect to c . Picking δp so that the integrand of (H4) approximates to the difference of two Dirac delta functions with peaks at two points of the same surface, we satisfy (H4) and see from (H3) that p must have the same value at the two points. Thus (y) is established. We now have $p = P(\psi)$, whereby (f)

$$P(c) = [M'(c)/\int_{\psi=c} (dS/|\nabla \psi|)]^\gamma. \quad (\text{H5})$$

Next we vary only λ , obtaining

$$\begin{aligned} 0 = \delta W &= \int_T d\tau \mathbf{B} \cdot \delta \mathbf{B} = \int_T d\tau \mathbf{B} \cdot (\nabla \psi \times \nabla \delta \lambda) \\ &= - \int_T d\tau \delta \lambda \nabla \cdot (\mathbf{B} \times \nabla \psi), \end{aligned} \quad (\text{H6})$$

$$(\nabla \times \mathbf{B}) \cdot \nabla \psi = 0. \quad (\text{H7})$$

Finally we wish to vary only ψ . When we do so, $p = P(\psi)$ varies at a fixed point not only on account of the argument ψ but also because $P(c)$ does. To compute the contribution of $\delta P(c)$ to δW we note that

$$\delta \int_{\psi=c} \frac{dS}{|\nabla\psi|} = \delta \frac{d}{dc} \int_{\psi \leq c} d\tau = - \frac{d}{dc} \int_{\psi=c} dS \frac{\delta\psi}{|\nabla\psi|}, \quad (\text{H8})$$

$$\begin{aligned} \int_T d\tau \delta P(\psi) &= \int_T d\tau P \delta \log P \\ &= \int_0^c dc \int_{\psi=c} \frac{dS}{|\nabla\psi|} P \gamma \left(\frac{d}{dc} \int_{\psi=c} dS \frac{\delta\psi}{|\nabla\psi|} \right) \\ &\quad / \int_{\psi=c} \frac{dS}{|\nabla\psi|} \\ &= \gamma \int_0^c dc P(c) \frac{d}{dc} \int_{\psi=c} dS \frac{\delta\psi}{|\nabla\psi|} \\ &= - \gamma \int_0^c dc P' \int_{\psi=c} dS \frac{\delta\psi}{|\nabla\psi|} \\ &= - \gamma \int_T d\tau P'(\psi) \delta\psi, \end{aligned} \quad (\text{H9})$$

in which we have used the fact that $\delta\psi = 0$ at the walls in view of (a). Accordingly we have

$$\begin{aligned} 0 = \delta W &= \int_T d\tau [\mathbf{B} \cdot \delta \mathbf{B} + (P' \delta\psi + \delta P)/(\gamma - 1)] \\ &= \int_T d\tau [\mathbf{B} \cdot (\nabla \delta\psi \times \nabla \nu + \nabla \psi \times \nabla \delta\nu) - P' \delta\psi] \\ &= \int_T d\tau \delta\psi [\nabla \cdot (\mathbf{B} \times \nabla \nu) - P'], \end{aligned} \quad (\text{H10})$$

in which we have used (H7); there is no trouble with the discontinuity of $\nabla\theta$ at the magnetic axis because $\delta\psi$ vanishes there (since ψ and $\nabla\psi$ both do). Thus we obtain

$$(\nabla \times \mathbf{B}) \cdot \nabla \psi = P'. \quad (\text{H11})$$

Taking the cross-product of (H1) with $\nabla \times \mathbf{B}$ and using the variational conditions (H7, 11) gives

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = P' \nabla = \nabla p, \quad (\text{H12})$$

which establishes (z).

It is clear that all the steps can be reversed to show that (y) and (z) imply that $\delta W = 0$ for all perturbations.

I. REMARKS

There is a noteworthy modification of the variational principle. Suppose we omit condition (e). We are then free to vary $\chi(c)$ so we obtain an additional variational condition from

$$\begin{aligned} 0 = \delta W &= \int_T d\tau \mathbf{B} \cdot \delta \mathbf{B} \\ &= \int_0^c dc \delta \chi'(c) \int_{\psi=c} (dS/|\nabla\psi|) \mathbf{B} \cdot (\nabla\psi \times \nabla\eta), \end{aligned} \quad (\text{I1})$$

$$\begin{aligned} 0 &= \int_{\psi=c} dS (\nabla\psi/|\nabla\psi|) \cdot (\mathbf{B} \times \nabla\eta) \\ &= \int_{\leq c} d\tau \nabla \cdot (\mathbf{B} \times \nabla\eta) = \int d\tau \mathbf{j} \cdot \nabla\eta = I. \end{aligned} \quad (\text{I2})$$

That is, we obtain just the additional condition appropriate for the steady-state of a diffusing plasma (see section K).

Our variational principle characterizes equilibria as stationary states of the potential energy W ; i.e., states for which the first variation of W vanishes. The stability of such equilibria has been investigated elsewhere⁶ by examining the positive-definiteness of the second variation of W .

Two limiting choices of γ are particularly simple. The first is $\gamma \rightarrow \infty$ (incompressibility) for which $M \rightarrow V$, so that we prescribe the volume to be enclosed by each surface and vary the magnetic energy alone. The second choice is $\gamma \rightarrow 0$ (pressure completely independent of density) for which $M'\gamma$ approaches the maximum value of p on the surface, so that we prescribe $p(\psi)$ and vary the integral⁷ of $\frac{1}{2} \mathbf{B}^2 - p$.

J. THE STEADY SLOWLY DIFFUSING PLASMA

We now wish to obtain a complete set of equations governing a steady-state plasma slowly diffusing through a magnetic field to the containing walls of the toroidal tube T . We assume that the walls are perfect electric conductors with purely tangential magnetic field and also perfect plasma absorbers ($p = 0$ there) and that new plasma is somehow introduced or injected into T (necessary to maintain a steady state) at the source density rate S , which may depend on position. We assume that there is no temperature gradient and ignore a variety of complicating factors, such as nuclear reactions and radiation, which might occur in applications of interest.

Our steady state is almost static so to lowest order in the diffusion velocity $\mathbf{v}$ we have the magnetostatic equations (A1-3). We also have the (first order) equation of continuity

$$\nabla \cdot (p\mathbf{v}) = S, \quad (\text{J1})$$

where we have taken the plasma density to be p in view of the assumed isothermality. In addition we have Maxwell's equation

$$\nabla \times \mathbf{E} = 0 \quad (\text{J2})$$

and Ohm's law⁸ which we take in the form

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{j}/\sigma + (\nabla p/p)\alpha, \quad (\text{J3})$$

where $\mathbf{E}$ is the electric field, σ the conductivity (assumed constant and scalar) and α a physical constant.

Now (J2) holds everywhere in space (not just in T) so $\mathbf{E}$ is the gradient of a single-valued scalar. Since $\nabla p/p$ is also such a gradient we can introduce a single-valued scalar ϕ in T satisfying

$$\mathbf{E} = \nabla\phi + (\nabla p/p)\alpha, \quad (\text{J4})$$

so that (J3) becomes

$$\nabla\phi + \mathbf{v} \times \mathbf{B} = \mathbf{j}/\sigma. \quad (\text{J5})$$

Let us consider (J5) as an equation for $\mathbf{v}$. The condition that it have a solution is

$$\mathbf{B} \cdot \nabla\phi = (\mathbf{B} \cdot \mathbf{j})/\sigma \quad (\text{J6})$$

and if this is satisfied the general solution of (J5) is

$$\mathbf{v} = \mathbf{B}^{-2}(\nabla\phi - \mathbf{j}/\sigma) \times \mathbf{B} + a\mathbf{B}, \quad (\text{J7})$$

where a is an arbitrary scalar function. Eliminating $\mathbf{v}$ by (J7) and using (A1), we have for (J1)

$$\nabla \cdot (p a \mathbf{B}) = S + \nabla \cdot [(p/\mathbf{B}^2)(\mathbf{B} \times \nabla\phi + \nabla p/\sigma)], \quad (\text{J8})$$

which may be viewed as a diffusion equation of sorts for p , the diffusion coefficient being $p/\mathbf{B}^2\sigma$.

K. THE TWO AUXILIARY CONDITIONS

By (A3) and (B2) the left-hand side of (J8) may be written $p\mathbf{B} \cdot \nabla a$. Thus (J8) as well as (J6) are what we may call "magnetic differential equations"; namely equations of the type

$$\mathbf{B} \cdot \nabla r = s, \quad (\text{K1})$$

with scalar r and s . Viewed as an equation for r with s given, (K1) determines how r varies along a magnetic line of force. In the nondegenerate case that the line covers a magnetic surface ergodically, (K1) and the assignment of a value to r at one point determine r at a set of points dense in the surface. A necessary condition that the values of r so obtained be expandable to a continuous single-valued function over the whole surface is easily derived by integrating (K1) over the shell volume between two neighboring magnetic surfaces $p = P$ and $p = P + dP$. By (A3), Gauss' theorem, and (B2) the left-hand side then vanishes, while after dividing by dP the right-hand side becomes

$$\int_{p=P} s dS/|\nabla p| = 0. \quad (\text{K2})$$

It is plausible to assume (and we shall) that, in the nondegenerate case, (K2) is also a sufficient condition for (K1) to have a continuous single-valued solution r . It is then clear that (K1) determines r up to a surface quantity.

In accordance with the foregoing paragraph, the conditions that (J6) permit a solution ϕ and (J8) a solution a are

$$\int_{p=P} (dS/|\nabla p|) \mathbf{B} \cdot \mathbf{j} = 0, \quad (\text{K3})$$

$$\int_{p=P} (dS/|\nabla p|) \{ S + \nabla \cdot [(p/\mathbf{B}^2)(\mathbf{B} \times \nabla\phi + \nabla p/\sigma)] \} = 0. \quad (\text{K4})$$

The latter may be considerably simplified by multiplying by dP and integrating over all magnetic surfaces interior to a particular one; using Gauss' theorem then leads to

$$\int d\tau S \mp \int dS (\nabla p/|\nabla p|) \cdot (\mathbf{B} \times \nabla\phi + \nabla p/\sigma) p/\mathbf{B}^2 = 0, \quad (\text{K5})$$

where the minus or plus sign is to be adopted accordingly as p decreases or increases going outward (so that $\mp \nabla p/|\nabla p|$ is the unit outward normal to the magnetic surface). Eliminating ∇p by (A1) except

in the denominator, expanding out the inner products of the cross products, and using (J6) gives

$$\int d\tau S \mp \int (dS/|\nabla p|) (-\mathbf{j} \cdot \nabla\phi + \mathbf{j}^2/\sigma) p = 0. \quad (\text{K6})$$

By (B1, 3) the term involving ϕ can be written as a divergence; converting the surface integral to a shell volume integral, we see by Gauss' theorem and (B3) that the contribution of that term is zero. Since S , σ , and p are essentially positive, we must take the minus sign. Thus p decreases going outward (as assumed at the end of section B) and (K6) may be written

$$\int_{p \geq P} d\tau S - (P/\sigma) \int_{p=P} (dS/|\nabla p|) \mathbf{j}^2 = 0. \quad (\text{K7})$$

Our system of equations now consists of the magnetostatic equations (A1–3) together with the auxiliary conditions (K3, 7). For any solution p , $\mathbf{B}$, $\mathbf{j}$ of this system we can find ϕ and a and therefore $\mathbf{E}$ and $\mathbf{v}$, which together with the solution represent a slowly diffusing equilibrium. (The arbitrariness of a surface quantity each in a and ϕ corresponds to a physical arbitrariness in the total fluid flow along lines of force and in the total charge on a magnetic surface, respectively.)

Condition (K7) can be shown to be equivalent to the energy balance equation, which could have been written down *a priori*. Condition (K3) may be written $-dK/dp = 0$ or, integrating, $K = 0$. When (K3) holds (D12) can be integrated to show that I is proportional to $J + \oint d\mathbf{x} \cdot \mathbf{B}$. Since I and J but not the loop integral vanish at the axis, the constant of proportionality vanishes and $I = 0$. (Conversely the vanishing of I for all surfaces entails that of K .)

Thus (K3) is equivalent to (I2), the extra variational condition obtained by minimizing W without prescribing χ . In other words, in the diffusing plasma the azimuthal magnetic flux adjusts to give the lowest energy; the lines of force associated with the longitudinal magnetic flux are permanently trapped by the perfectly conducting walls, but "untwist" themselves locally as much as possible.

It is not hard to see that the vanishing of dI at a particular magnetic surface implies that the lines of electric current there are closed curves and, indeed, closed curves topologically like (deformable into) curves of constant η . The converse is even easier to see (choose η to be constant on the current lines).

L. THE LOW PRESSURE LIMIT

It is physically quite plausible to suppose that our system of equations has a solution p , $\mathbf{B}$, $\mathbf{j}$, and a unique one for any reasonable general prescription of the tube T , the source function S , the conductivity σ and the total trapped longitudinal magnetic flux C . This supposition is further borne out by the variational principle which shows that two auxiliary conditions for each magnetic surface are just the appropriate number. We now proceed to prove the supposition in the limiting case of low pressure under an

assumption on the geometry of T which is apparently necessary if the solution is to behave regularly in the limit. This assumption is that the unique vacuum magnetic field which is purely tangential at the walls and has a prescribed total longitudinal flux C vanishes nowhere and determines a nondegenerate family of nested toroids. (For a wide class of toroidal geometries this is assured to a high approximation by rotational transform theory.⁹)

For ϕ small (A1) becomes, in the limit, $\mathbf{j} \times \mathbf{B} = 0$ or

$$\mathbf{j} = g\mathbf{B} \quad (\text{L1})$$

with g a scalar function. From (A3) and (B1) we obtain the magnetic differential equation

$$\mathbf{B} \cdot \nabla g = 0, \quad (\text{L2})$$

which implies that g is a surface quantity. But then using (L1) in (K3) and taking g out of the integral gives $g = 0$ and, therefore, $\mathbf{j} = 0$. Thus $\mathbf{j}$ must be small if ϕ is small.

Starting over now with ϕ and $\mathbf{j}$ both small, we see from (A2, 3) that to lowest order $\mathbf{B}$ must be the unique magnetic field of our assumption. Now (A1) is equivalent to the pair of equations obtained by taking the inner and the cross products with $\mathbf{B}$; namely, (B2) and

$$\mathbf{j} = [\mathbf{B} \times \nabla \phi] / \mathbf{B}^2 + h\mathbf{B}, \quad (\text{L3})$$

with h a scalar function. The only restriction (A2) places on our remaining unknowns ϕ and $\mathbf{j}$ is (B1). We now adopt the point of view that (L3), (B1), and (K3) are conditions on $\mathbf{j}$, while (B2) and (K7) are conditions on ϕ .

Now (L3) expresses $\mathbf{j}$, in terms of what we may consider a new scalar unknown h . But then (B1) is equivalent to the magnetic differential equation

$$\mathbf{B} \cdot \nabla h = -(\mathbf{B} \times \nabla \phi) \cdot \nabla (1/\mathbf{B}^2) \quad (\text{L4})$$

by (A3) and (B5), while (K3) becomes

$$\int_{\phi=P} (dS/|\nabla \phi|) h \mathbf{B}^2 = 0 \quad (\text{L5})$$

and may be viewed as determining the additive surface quantity left arbitrary in h by (L4).

Of course by (B2) ϕ is a surface quantity of the magnetic surfaces of the vacuum field and is therefore a function of ψ only. It is now obvious from (L4, 5) that h is determined completely independently of ϕ except for being proportional to $\phi'(\psi)$, and by (L3) the same is true for $\mathbf{j}$. Indeed, setting

$$\mathbf{j} = \phi' \mathbf{y}, \quad h = \phi' k, \quad (\text{L6})$$

we see that $\mathbf{y}$ and k are uniquely determined independently of ϕ by the three equations obtained from (L3-5) by replacing ϕ , $\mathbf{j}$, and h everywhere by ψ , $\mathbf{y}$ and k , respectively.

We can now write (K7) in the form

$$\int_{\psi \leq c} d\tau S + \phi' / \sigma \int_{\psi=c} \mathbf{y}^2 dS / |\nabla \psi| = 0. \quad (\text{L7})$$

(Note that S is of the second order of smallness compared with ϕ and $\mathbf{j}$.) Thus $(\phi^2)'$ is determined and, since ϕ vanishes at the wall, ϕ is determined, whereupon $\mathbf{j}$ is determined by (L6). This completes the proof.

It may be noted that $(\phi^2)'$ remains finite at the walls. Thus ϕ' becomes infinite there, ϕ varying as the square root of the distance to the walls.

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Hydromagnetic Disturbances of Large Amplitude in a Plasma

By J. H. Adlam and J. E. Allen *

Experiments on the magnetic compression of a plasma are being carried out in many laboratories, largely because of the possible ultimate application as a fusion reactor. The theory of the rapid compression of a pinched gas discharge has been discussed by several authors. The collapse time of a simple pinched discharge is determined essentially by the inertia of the ionised gas, as shown by Leontovich and Osovets.¹ The detailed behaviour, however, depends on the collision frequencies. When many collisions take place during the collapse, a hydrodynamic description will suffice, and shock waves are generated as described by Allen.² In very high temperature plasmas, however, the collision frequency is low and a rapid compression may be essentially "collision-free". Rosenbluth's treatment³ refers to such a "collision-free" compression.

The calculations presented in this paper refer to the collision-free collapse of a plasma containing a magnetic field, since the latter was not included in Rosenbluth's theory. The compressing field is assumed to be in the same direction as the internal field, for the sake of simplicity. Figure 1 is a schematic diagram illustrating this configuration in a practical case.

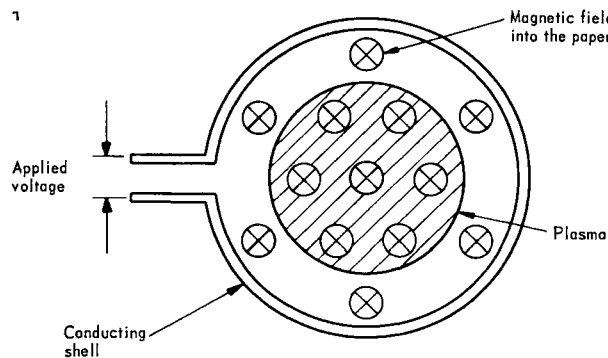


Figure 1. Compression of a plasma by a magnetic field parallel to the internal field

The following calculations refer to a plane plasma, an approximation that suffices for the early stages of the compression.

FUNDAMENTAL EQUATIONS

The equations are the same as those used in previous work on collision-free hydromagnetic waves.⁴ The coordinate system is chosen so that the magnetic field is in the z direction and the plasma initially occupies the space represented by positive values of x . There are no variations in the y or z directions. The equations of motion for the electrons and ions can be written as follows, where u and v are the velocity components in the x and y directions, and suffixes 1 and 2 refer to the electron and ion respectively,

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = \frac{e_1}{m_1} \left[E_x + \frac{v_1 B_z}{c} \right] \quad (1)$$

$$\frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial x} = \frac{e_1}{m_1} \left[E_y - \frac{u_1 B_z}{c} \right] \quad (2)$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} = \frac{e_2}{m_2} \left[E_x + \frac{v_2 B_z}{c} \right] \quad (3)$$

$$\frac{\partial v_2}{\partial t} + u_2 \frac{\partial v_2}{\partial x} = \frac{e_2}{m_2} \left[E_y - \frac{u_2 B_z}{c} \right] \quad (4)$$

The original thermal velocities are ignored, since they are small compared with the directed velocities which the particles acquire.

The equations of continuity are

$$\left. \begin{aligned} \frac{\partial n_1}{\partial t} &= - \frac{\partial}{\partial x} (n_1 u_1) \\ \frac{\partial n_2}{\partial t} &= - \frac{\partial}{\partial x} (n_2 u_2) \end{aligned} \right\} \quad (5)$$

where n is the particle density, and the field equations, neglecting the displacement current, are

$$\frac{\partial H_z}{\partial x} = - \frac{4\pi}{c} [n_1 e_1 v_1 + n_2 e_2 v_2] \quad (6)$$

$$\frac{\partial E_y}{\partial x} = - \frac{1}{c} \frac{\partial B_z}{\partial t} \quad (7)$$

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and

$$\frac{\partial E_x}{\partial x} = \frac{4\pi}{k} [n_1 e_1 + n_2 e_2]. \quad (8)$$

The numerical values of μ and k (the permeability and dielectric constant of a vacuum) are both unity, but they are retained in the equations used here. The latter will be simplified, as in the previous work,⁴ by using the fundamental property of a plasma, namely, $|n_1 - n_2| \ll n$. The difference between n_1 and n_2 is ignored, except in Poisson's equation, Eq. (8); this small difference produces the electrostatic field E_x which dominates the motion of the massive positive ions. It has been assumed that the latter are singly charged, so that $e_1 + e_2 = 0$.

The current must flow in the y direction, since the magnetic field is in the z -direction and varies only with x . Thus $u_1 = u_2 = u$ (say), since $n_1 = n_2$, and the relative velocity between the electrons and ions may be written $v = v_1 - v_2$. Dropping the subscript from B_z and eliminating E_x from Eqs. (1) and (3) gives

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \frac{e_2 v B}{(m_1 + m_2) c} \quad (9)$$

and equations (2) and (4) yield

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} = - \frac{e_2 (m_1 + m_2)}{m_1 m_2} \left[E_y - \frac{u B}{c} \right]. \quad (10)$$

Equations (5) degenerate to form one equation:

$$\frac{\partial n}{\partial t} = - \frac{d}{dx} (nu) \quad (11)$$

and the field equations become

$$\frac{\partial H}{\partial x} = \frac{4\pi n e_2 v}{c} \quad (12)$$

and

$$\frac{\partial E_y}{\partial x} = - \frac{1}{c} \frac{\partial B}{\partial t}. \quad (13)$$

Equations (9)–(13) form a set of five simultaneous differential equations relating the five unknowns, u , v , B , E_y and n .

Change of Co-ordinate System

A new variable, h , will now be introduced, which is constant for any particular particle, and which is equal to the x co-ordinate of that particle at $t = 0$. Then

$$\left(\frac{\partial}{\partial t} \right)_h = \left(\frac{\partial}{\partial t} \right)_x + \left(\frac{\partial x}{\partial t} \right)_h \left(\frac{\partial}{\partial x} \right)_t$$

and

$$\left(\frac{\partial}{\partial x} \right)_t = \left(\frac{\partial h}{\partial x} \right)_t \left(\frac{\partial}{\partial h} \right)_t$$

but $(\partial x / \partial t)_h$ is equal to u , the velocity of the particle in the x direction, and $(\partial h / \partial x)_t = (n/n_0)$ so that

$$\left(\frac{\partial}{\partial t} \right)_h = \left(\frac{\partial}{\partial t} \right)_x + u \left(\frac{\partial}{\partial x} \right)_t$$

and

$$\left(\frac{\partial}{\partial x} \right)_t = \left(\frac{n}{n_0} \right) \left(\frac{\partial}{\partial h} \right)_t.$$

The differential equations (9)–(13), when expressed in terms of the independent variables, h and t , become

$$\frac{\partial u}{\partial t} = - \frac{e_2 v B}{(m_1 + m_2) c} \quad (9a)$$

$$\frac{\partial v}{\partial t} = - \frac{e_2 (m_1 + m_2)}{m_1 m_2} \left[E_y - \frac{u B}{c} \right] \quad (10a)$$

$$\frac{\partial}{\partial t} \left(\frac{n_0}{n} \right) = \frac{\partial u}{\partial h} \quad (11a)$$

$$\frac{\partial H}{\partial h} = \frac{4\pi n_0 e_2 v}{c} \quad (12a)$$

and

$$\frac{\partial E_y}{\partial h} = \frac{1}{c} \left[u \frac{\partial B}{\partial h} - \frac{n_0}{n} \frac{\partial B}{\partial t} \right]. \quad (13a)$$

Introduction of Dimensionless Quantities

The following dimensionless quantities will now be introduced: $h_n = h/d$, $t_n = t/\tau$, $u_n = u/v_A$, $v_n = v/v_*$, $B_n = B/B_0$, $E_{yn} = E_y/E_0$ and $n' = n_0/n$, where

$$d = \left[\frac{m_1 m_2 c^2}{4\pi n_0 \mu e_2^2 (m_1 + m_2)} \right]^{1/2}, \quad \tau = \frac{(m_1 m_2)^{1/2} c}{e_2 B_0}$$

$$v_A = \left[\frac{B_0^2}{4\pi n_0 \mu (m_1 + m_2)} \right]^{1/2}, \quad v_* = \left[\frac{B_0^2 (m_1 + m_2)}{4\pi n_0 \mu m_1 m_2} \right]^{1/2}$$

and $E_0 = v_A B_0 / c$. Equations (9a)–(13a) can now be written as follows:

$$\frac{\partial u_n}{\partial t_n} = - v_n B_n \quad (9b)$$

$$\frac{\partial v_n}{\partial t_n} = - E_n + u_n B_n \quad (10b)$$

$$\frac{\partial n'}{\partial t_n} = \frac{\partial u_n}{\partial h_n} \quad (11b)$$

$$\frac{\partial B_n}{\partial h_n} = v_n \quad (12b)$$

and

$$\frac{\partial E_n}{\partial h_n} = u_n \frac{\partial B_n}{\partial h_n} - n' \frac{\partial B_n}{\partial t_n}. \quad (13b)$$

On eliminating u_n , v_n and E_n from these equations, the following pair of simultaneous equations is obtained for B_n and n' :

$$\frac{\partial^2 n'}{\partial t_n^2} = - \frac{\partial}{\partial h_n} \left(B_n \frac{\partial B_n}{\partial h_n} \right) \quad (14)$$

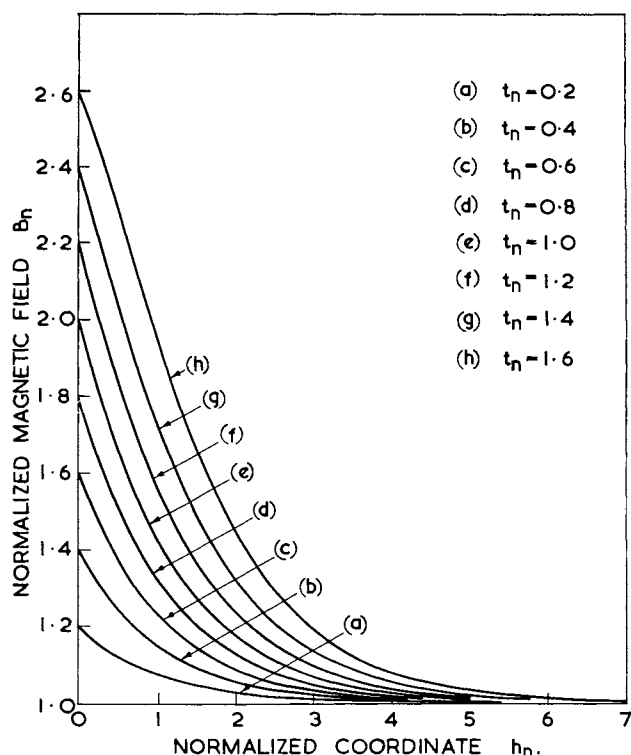


Figure 2. Magnetic field distributions computed for various times after initiation of the plasma collapse

and

$$\frac{\partial^3 B_n}{\partial h_n^2 \partial t_n} = \frac{\partial}{\partial t_n} (n' B_n). \quad (15)$$

Boundary Conditions

The plasma is initially uniform, so that $B_n = 1$ and $n' = 1$ when $t_n = 0$. Using this boundary condition, direct integration of equation (15) yields

$$\frac{\partial^2 B_n}{\partial h_n^2} = n' B_n - 1. \quad (16)$$

It is convenient to specify the current, rather than the voltage, since the former is directly proportional to the increase in the magnetic field strength (outside the plasma). A linear increase with time is assumed in this paper, viz., $(B_n - 1) = \alpha t_n$, and the results which are presented below refer to the particular case where $\alpha = 1$.

RESULTS

Figures 2 and 3 show the values of B_n and n' as functions of h_n for different values of t_n . These have been obtained by solving equations (14) and (16) numerically. Figure 4 shows B_n and n/n_0 plotted against x_n , for different values of t_n ; these curves have been obtained from the previous ones by using the relation $dx = n' dh$. Figure 5 shows the position of the plasma boundary as a function of time.

Shortly after $t_n = 1.6$, $n' \rightarrow 0$ at $h = 0$, i.e., $n \rightarrow \infty$ at the plasma boundary. Physically this means that

the outermost ions are overtaking their neighbours, thus forming two streams of ions at some points. The equations are no longer valid after this time, since it has been assumed that the electron (or ion) velocity is single-valued at all points.

Other cases are being computed at present, in which the external magnetic field obeys the law

$$B_n = 1 + C[1 - \exp(-\alpha t_n)]$$

In some circumstances (those of the less violent compressions) hydromagnetic waves are launched into the plasma, and double streams do not occur.

DISCUSSION

It was shown in the previous work⁴ that the equations are valid if three conditions are satisfied:

(a) $(v_*/c)^2 \ll 1$. This condition must hold if $|n_1 - n_2| \ll n$; it is also the condition for non-relativistic equations to be valid. An alternative form is $B_0^2/4\pi n_0 \mu \ll m_1 c^2$.

(b) $kT_1 \ll B_0^2/4\pi n_0 \mu$. The initial random energy is negligible if this condition holds, since the electrons are given a velocity of the order of v_* . Combination of (a) and (b) gives $kT_1 \ll B_0^2/4\pi n_0 \mu < m_1 c^2$.

(c) $\tau \ll (n_0 v Q)^{-1}$ where v is the random velocity and Q is the positive ion cross section for momentum transfer. This is the criterion for collisions to be unimportant.

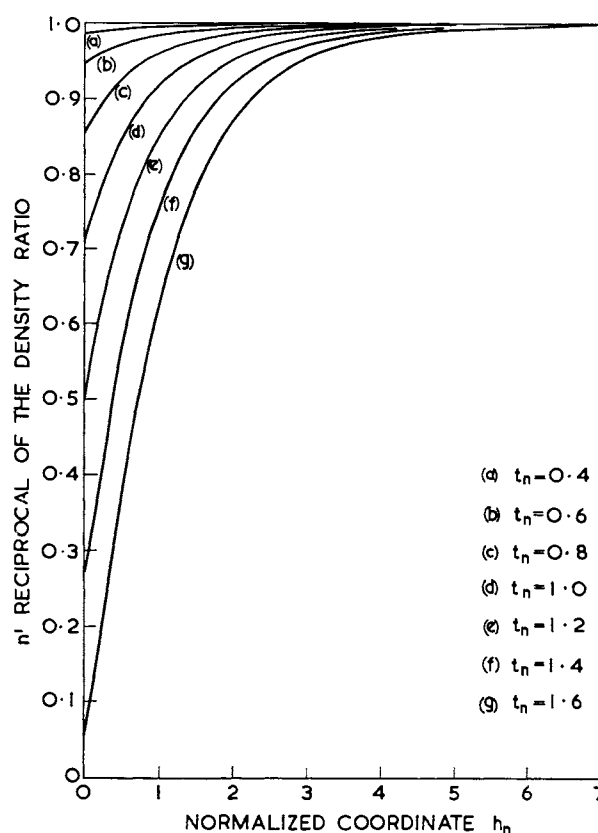


Figure 3. Density distributions computed for various times after initiation of the plasma collapse

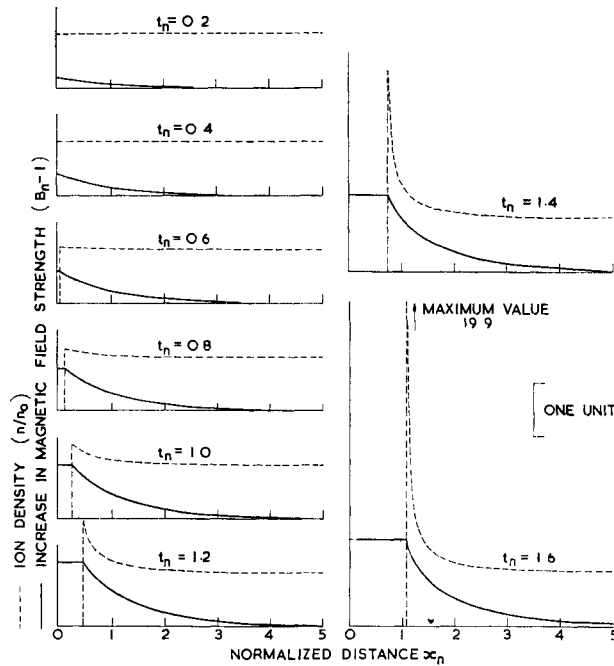


Figure 4. The rapid compression of a plasma containing a magnetic field; the current rises linearly with time at such a rate that $B_r = 1 + t_n$

It is easily shown, using Equations (1)–(4), that the electrostatic field is given by

$$E_x = - \left(\frac{m_2 - m_1}{m_2 + m_1} \right) \frac{vB}{c}.$$

This quantity is finite at the plasma boundary, which means that a surface charge must exist. It can be shown, however, that the thickness of the space-charge layer is of the order of $(v_*/c)^2 d$, which is small compared with d if condition (a) is satisfied.

The formation of double streams of ions is a result which might have been anticipated, since the plasma boundary is continuously accelerated in the present case (as shown in Fig. 5), whereas the previous work⁴ has shown that simple hydromagnetic waves cannot be propagated at speeds greater than twice the Alfvén velocity. Also, it is known that two streams must exist if the compressing field is sufficiently large compared with the internal field, since double streaming occurs when the latter is zero.³

It has been mentioned that hydromagnetic waves are propagated into the plasma if the compression is

not too violent. If the time of compression is long compared with the transit time of a hydromagnetic wave, the plasma is uniform and Eq. (16) shows that $n/n_0 = B/B_0$, i.e. the electrons are “tied” to the magnetic lines of force. In a rapid compression, however, the electrons are not tied to the lines of force, but “slip” relative to them.

A considerable amount of energy is imparted to the particles in a rapid compression; the massive positive ions are accelerated inwards by the electrostatic field, while the electrons attain high velocities in the direction of current flow. These high electron velocities result from the “slipping” of the electrons relative to the magnetic lines of force, and the mean energy is comparable to that of a positive ion.

ACKNOWLEDGEMENT

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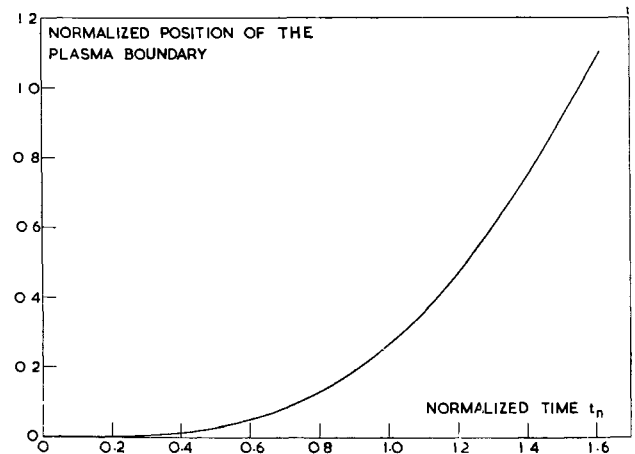


Figure 5. Position of the plasma boundary as a function of time

Simple Waves and Shock Waves in Magnetohydrodynamics

By A. I. Akhiezer, G. J. A. Lubarski and R. V. Polovin

As is well known, the equations of hydrodynamics are valid if the mean-free path l is small in comparison with the characteristic length L involved in the problem. For a plasma in a strong magnetic field, the hydrodynamic approximation can also be used even in those cases when this criterion is not fulfilled. This follows from the fact that in a strong magnetic field there is a second characteristic length R_L , the Larmor radius. If this length is much less than the characteristic size of the system, $R_L \gg L$, the equations of magnetohydrodynamics will be valid in the plane perpendicular to the magnetic field provided that the frequency of the field oscillation is much less than both the cyclotron frequency $\omega_c = eH/Mc$ and the plasma frequency $\omega_p = (4\pi e^2 n/M)^{1/2}$, which are characteristic frequencies of the plasma. The equations of magnetohydrodynamics can also be used to describe the motion along the magnetic field if the extent of the particle motion along the magnetic lines of force is finite. One should bear in mind that in this case, if collisions do not occur frequently, the pressure entering into the magnetohydrodynamic equations is anisotropic and depends not only upon the density but upon the magnetic field as well. Collisions, however, will tend to reduce the degree of the pressure anisotropy.

In deducing the magnetohydrodynamic equations from the equations of motion for electrons and ions, one usually supposes the validity of Ohm's law and infinite conductivity. In reality, however, these assumptions are not required. It is possible to prove that in order to deduce the magnetohydrodynamic equations it is only necessary to suppose that the fields change slowly in space and time.

This paper gives some results of an investigation of non-linear plasma effects in the hydromagnetic approximation when the pressure is isotropic.

SIMPLE MAGNETOHYDRODYNAMIC WAVES

Generally speaking, non-linear equations do not admit solutions in which all the variables depend upon the quantity $x - Vt$, because the wave form varies during its motion*; here, x is a space coordinate,

t is the time and V is the constant phase velocity. A natural generalization of this type of wave is the simple wave, introduced for the first time by Riemann, which has the property that all the hydrodynamic variables u_k , $k = 1, 2, \dots, n$, are functions of one of them, for example u_1 . The importance of simple waves is connected with the fact that in the absence of shock waves only simple waves can bound domains of constant flow. This statement, well known in the case $n = 2$, appears to be valid for any n .

In this section we shall enumerate all the simple hydromagnetic waves and establish their connection with linearized plane waves.

We write the system of hydromagnetic equations schematically as

$$\sum_{j=1}^n (X_{ij} \partial u_j / \partial x + T_{ij} \partial u_j / \partial t) = 0, \quad (1)$$

where T_{ij} and X_{ij} are some functions of u_j .

Let us consider first the linearized plane waves which correspond to small amplitude oscillations of the variables u_j about the constant value u_j^0 . There are n types of such linear waves,

$$\delta u_j = U_j^\mu (u_1^0, u_2^0, \dots, u_n^0) \exp i(kx - \omega t), \\ \mu = 1, 2, \dots, n$$

where U_j^μ is the wave amplitude, $k = \omega/V_\mu$ is the

the phase velocity V is arbitrary, while the oscillation frequency is a complex function of the amplitude:¹

$$\omega = \pi \omega_0 (1 - k^2)^{1/2} / 2 [2E - (1 - k^2)K]$$

where

$$\omega_0^2 = 4\pi n_e e^2 / m$$

$$k^2 = \frac{1 - (1 - U_{\max}^2)^{1/2}}{1 + (1 - U_{\max}^2)^{1/2}}$$

$$K = \int_0^1 \frac{dx}{[(1 - x^2)(1 - k^2 x^2)]^{1/2}}$$

$$E = \int_0^1 \left(\frac{1 - k^2 x^2}{1 + k^2 x^2} \right)^{1/2} dx,$$

and $U_{\max}$ denotes the maximum value of v/c , where v is the electron velocity. The quantity n_e is the electron density.

Original language: Russian.

* Non-linear oscillations of a plasma at low pressures are an important example of such types of wave motion. In this case

wave vector, and $V_\mu = V_\mu(u^0)$ is the phase velocity of the μ th wave determined by the equation

$$\det |X_{ij}(u^0) - VT_{ij}(u^0)| = 0. \quad (2a)$$

To each linearized wave corresponds the simple wave determined by the equations

$$\frac{du_1}{U_1^\mu(u_1, \dots, u_n)} = \dots = \frac{du_n}{U_n^\mu(u_1, \dots, u_n)},$$

$$\partial u_i / \partial t + V_\mu(u_1, u_2, \dots, u_n) (\partial u_i / \partial x) = 0. \quad (2b)$$

The general solution of (2) is given by

$$x - V_\mu(u_1, u_2, \dots, u_n)t = f(u_i), \quad (3)$$

where $f(u_i)$ is an arbitrary function determined by the initial conditions. It follows that the displacement velocity of a point with specified values of the variables $u_1, u_2, \dots, u_n$ coincides with the phase velocity of the corresponding linear wave. Therefore, the simple and linear waves are closely connected with each other. In addition, notice that if we put $f(u_i) = 0$ in Eq. (3) we obtain similarity [avtomodel'nie] waves.

The simple waves may now be enumerated:

1. Hydromagnetic waves:

$$H_y^2 + H_z^2 = \text{constant}, \quad v_y = \mp V_y, \quad v_z = \mp V_z,$$

$$\varrho = \text{constant}, \quad s = \text{constant},$$

$$x - (v_x \pm V_x)t = f(H_z)$$

where s is the entropy and $\mathbf{V} = \mathbf{H}/(4\pi\varrho)^{1/2}$ is the Alfvén velocity.

2. Magnetosonic waves:

$$dv_x/d\varrho = u/\varrho,$$

$$dv_y/d\varrho = -H_x H_y u / 4\pi\varrho^2 (u^2 - V_x^2),$$

$$dv_z/d\varrho = -H_x H_z u / 4\pi\varrho^2 (u^2 - V_x^2),$$

$$dH_y/d\varrho = u^2 H_y / \varrho (u^2 - V_x^2),$$

$$dH_z/d\varrho = u^2 H_z / \varrho (u^2 - V_x^2), \quad (5)$$

$$x - (v_x \pm u)t = f(\varrho),$$

where $u^2 = d(p + {}^2\varrho V^2)/d\varrho^{1/2}$ is a root of the equation

$$u^4 + (V^2 + c_s^2)u^2 + c^2 V_x^2 = 0, \quad (6)$$

and c_s is the velocity of sound propagation.

3. Entropy waves:

$$x - v_x t = f(s),$$

with $\mathbf{v}, \mathbf{H}, \varrho = \text{constant}$.

An important property of simple magnetosonic waves⁵ is that their phase velocity depends upon quantities $(\mathbf{v}, \mathbf{H}, \varrho)$ which vary from point to point. Therefore these waves change their form during propagation. In this respect they are different⁴ from simple hydromagnetic waves which have a constant phase velocity and therefore propagate without changing their form.

Equation (5) shows that the phase velocity of a simple magnetosonic wave increases with increasing density if

$$(\partial^2 \varrho^{-1} / \partial p^2)_s > 0. \quad (7)$$

This condition is valid for most substances.

Since similarity waves are a particular case of simple magnetosonic waves they are always rarefaction waves. Another important consequence of Eq. (7) is that in the compression region the density gradient increases and this leads to the formation of shock waves.

POSSIBLE TYPES OF SHOCK WAVES

We now employ the general equations (1) to investigate the stability of plane hydromagnetic shock waves subjected to small disturbances which depend only upon the distance from the surface and the time.

Shock waves will be instable if the total number of hydromagnetic, magnetosonic and entropy waves propagating from the discontinuous front is not equal to six.

To prove this statement, let us denote by $u_k^{1,2}$ the magnitudes of the variable u_k on both sides of the surface of discontinuity. The index 1 corresponds to the region ahead of the shock front, $x < 0$, and the index 2 corresponds to the region behind the front, $x > 0$. Let us also describe the perturbation by $\delta u_j^{1,2}(x, t) = u_j^{1,2}(x, t) - u_j^{1,2}$. If the discontinuous solution $u_j^{1,2}$ is stable, these variables will stay small if they are small initially. In this case the linearized equations

$$\sum_{j=1}^n \{ X_{ij}^{1,2} \partial \delta u_j^{1,2}(x, t) / \partial x + T_{ij}^{1,2} \partial \delta u_j^{1,2}(x, t) / \partial t \} = 0 \quad (8)$$

are valid where

$$X_{ij}^{1,2} = X_{ij}^{1,2}(u^{1,2}), \quad T_{ij}^{1,2} = T_{ij}(u^{1,2}).$$

It may happen that this system of equations does not have a unique solution. On the other hand, Cauchy's problem in hydrodynamics must always have a unique solution. Therefore, the absence of a unique solution to the system (8) indicates that it is impossible to replace the exact equations with linearized equations, and this in turn implies that perturbations which are small initially will not stay small at later times.

A discontinuous change in $\delta u_j^{1,2}$ in time occurs only if the initial shock wave splits into several waves in the regions enclosed between the newly formed surfaces of discontinuity. Therefore, the criterion for the stability of a shock wave is the existence of a unique solution for the linearized Cauchy problem (8) corresponding to the system of Eq. (1).

The well-known general solution of Eq. (8) which satisfies the initial conditions is

$$\delta u_j^{1,2}(x, t) = \int_{-\infty}^{\infty} \sum_{\mu=1}^n C_{j\mu}^{1,2}(\omega, x) e^{(ik_\mu^{1,2}x - \omega t)} d\omega, \quad (9)$$

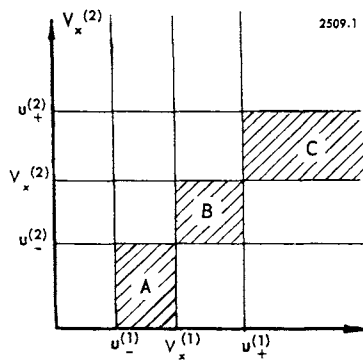


Figure 1. Stability diagram for magnetohydrodynamic shock waves. The shaded areas are regions of stability

where $k_\mu^{1,2} = \omega/V_\mu^{1,2}$ and μ enumerates the various plane waves. The integration is carried out in the upper half-plane along the straight line parallel to the real ω axis, $\text{Im}\omega > 0$. The amplitudes $C_{j\mu}^1(\omega, x)$ and $C_{j\mu}^2(\omega, x)$ are obtained by the method of variation of constants; they satisfy a system of n inhomogeneous differential equations. Therefore there are n arbitrary constants in the expression (9) for δu_j^1 and δu_j^2 . These constants must be selected so that the boundary conditions at the discontinuity surface are satisfied and also so that $\delta u_j^1(-\infty, t)$ and $\delta u_j^2(\infty, t)$ are finite.

Consider now the latter condition in detail. If the linearized wave with index μ moves to the right, ($V_\mu > 0$), then $\text{Im}k_\mu$ must be positive so that the perturbations approach zero as $x \rightarrow +\infty$. If the wave propagates to the left ($V_\mu < 0$) then the imaginary part of k_μ will be negative and in order for $\delta u_k^2(x, t)$ to be finite as $x \rightarrow +\infty$, it is necessary to set equal to zero the amplitudes $C_{j\mu}^2(\omega, x)$ that correspond to waves converging toward the discontinuity front. These conditions must also be applied on the opposite side of the discontinuity surface. If the total number of waves dispersing in both directions is equal to m , the number of convergent waves will be $(2n - m)$. Adding the n boundary conditions for the discontinuity surface to $(2n - m)$, we obtain in all $(3n - m)$ conditions which must be placed on the $2n$ arbitrary constants contained in $C_{j\mu}^{1,2}(\omega, x)$. In order to obtain a unique solution to the Cauchy problem, it is necessary to have the number of conditions, $3n - m$, equal to the number of unknown quantities, $2n + 1$. The unknown quantities consist of $2n$ integration constants and the perturbed velocity of the discontinuity front. The necessary condition for stability is therefore

$$m = n - 1; \quad (10)$$

i.e., the number of divergent waves must be one less than the number of boundary conditions at the discontinuity surface³. If this condition is met, one must still determine whether it is possible to satisfy the boundary conditions with the available arbitrary constants.

In magnetohydrodynamics there are seven types of plane waves, which can be classified according to their

phase velocities^{4, 5}: $v_x \pm V_x$ (hydromagnetic waves), $v_x \pm u_\pm$, where $u_\pm$ are positive roots of Eq. (6) (magnetosonic waves), and finally v_x (entropy waves).

A necessary condition for the stability of a shock wave, according to Eq. (10), is that the number of divergent waves be equal to six. Taking into account the inequality $u_-^{1,2} < V_x^{1,2} < u_+^{1,2}$, we find that a shock wave can be stable only under the following three conditions:

- A. $u_-^1 < v_x^1 < V_x^1, \quad v_x^2 < u_-^2;$
- B. $V_x^1 < v_x^1 < u_+^1, \quad u_-^2 < v_x^2 < V_x^2;$
- C. $u_+^1 < v_x^1, \quad V_x^2 < v_x^2 < u_+^2.$

These conditions are displayed in the stability diagram of Fig. 1.

If the magnetic field is parallel to the discontinuity front these conditions are reduced to the inequalities $(v_x^1)^2 > c_1^2 + (V_x^1)^2, \quad (v_x^2)^2 < c_2^2 + (V_x^2)^2$ (see Fig. 2).

When the magnetic field is perpendicular to the front there are additional considerations since then the boundary conditions split into three isolated groups. The stability regions for this case are shaded in Figs. 3a, 3b and 3c.

The necessary conditions for stability which have been obtained make it possible to carry out the classification of shock waves. In the general case, when the magnetic field is inclined relative to the plane of discontinuity, there are only three types of stable waves which correspond to the three regions A, B, and C in Fig. 1. If the jumps in all the variables at the discontinuity approach zero, the shock velocities approach the phase velocities of the linearized waves. Furthermore, in this limit, the velocity of a type A wave approaches the phase velocity $v_x \pm u_-$ of the magnetosonic wave and the velocities of types B and C waves approach the velocities $v_x \pm V_x$ and $v_x \pm u_+$ of the hydromagnetic and magnetosonic waves, respectively. In this connection, it is expedient to subdivide the shock waves into slow magnetosonic, hydromagnetic, and fast magnetosonic waves.

If the magnetic field is parallel to the plane of discontinuity, then only fast magnetosonic shock waves can be stable. If the magnetic field is perpendicular

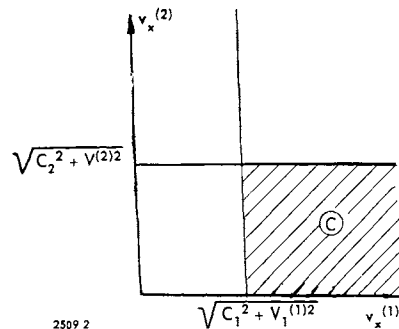


Figure 2. Stability diagram when the magnetic field is parallel to the discontinuous front

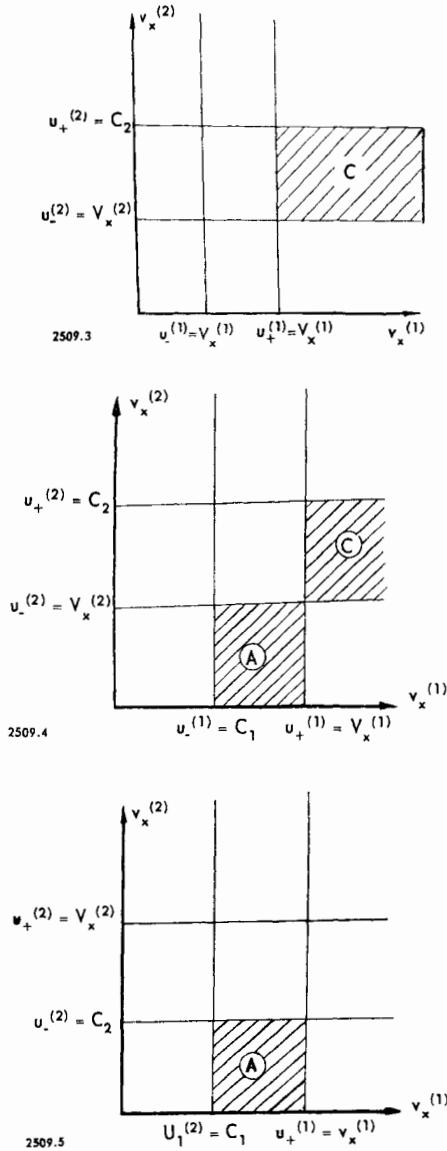


Figure 3 a, b, c. Stability diagrams when the magnetic field is perpendicular to the discontinuous front

to the discontinuity, only magnetosonic discontinuities can be stable (see Figs. 3a, 3b, 3c). Let us now note some consequences of the stability conditions for shock waves.

1. If two shock waves of the same type follow each other, the last wave will always catch up to the first wave.

2. If two waves of different types follow each other, a hydromagnetic shock wave will catch up to a slow magnetosonic wave while a fast magnetosonic wave will catch up to any shock wave.

3. A shock wave will catch up to a weak discontinuity of the same type as the shock wave or of a slower type.

4. A weak discontinuity will catch up to a shock wave of the same type or of slower types.

IMPOSSIBILITY OF RAREFACTION SHOCK WAVES IN MAGNETOHYDRODYNAMICS

In ordinary hydrodynamics, in conformity with Cemplen's theorem, rarefaction shock waves are impossible if

$$(\partial^2(\rho^{-1})/\partial p^2)_s > 0. \quad (11)$$

In magnetohydrodynamics, shock waves are also compression waves * if the condition

$$(\partial p/\partial T)_\rho > 0$$

is satisfied together with condition (11). Cemplen's theorem in magnetohydrodynamics follows from the fact that the shock adiabat in the (p, ρ) plane in the presence of a magnetic field is located below the shock adiabat in the absence of a magnetic field, for $\rho_2 < \rho_1$. This section of the adiabat ($\rho_2 < \rho_1$) cannot really exist since the entropy must decrease, $s_2 < s_1$ (see Fig. 4).

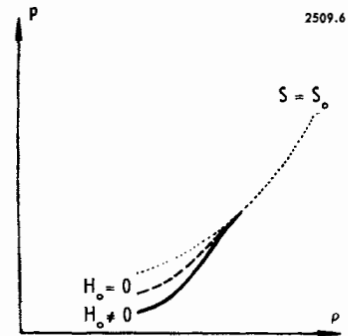


Figure 4. Shock adiabat in the (P, ρ) plane

From the boundary conditions at the discontinuity it follows that

$$H_\perp^2 = H_\perp^1 = \rho_2 \frac{(v_x^1)^2 - (V_x^1)^2}{\rho_1 (v_x^1)^2 - \rho_2 (V_x^1)^2}. \quad (12)$$

where $H_\perp$ is the projection of the magnetic field on the discontinuous surface. This equation leads us to the following conclusions:

1. If $(v_x^1)^2 < (V_x^1)^2$, the tangential component of the magnetic field does not change its direction and $|H_\perp^2| < |H_\perp^1|$.

2. If $(V_x^1)^2 < (v_x^1)^2 < 2\rho_2(V_x^1)^2/(\rho_1 + \rho_2)$, the tangential component of the magnetic field changes its direction and $|H_\perp^2| < |H_\perp^1|$.

3. If $2\rho_2(V_x^1)^2/(\rho_1 + \rho_2) < (v_x^1)^2 < \rho_2(V_x^1)^2/\rho_1$, the tangential component of magnetic field changes its direction and $|H_\perp^2| > |H_\perp^1|$.

* De Hoffman and Teller * have shown that a compressional shock wave in a perfect gas is thermodynamic stable if the magnetic field is parallel to the discontinuity surface. The question of the thermodynamic instability of a rarefaction shock wave was not decided.

4 Finally, if $\rho_2(V_x^1)^2/\rho_1 < (v_x^1)^2$, the tangential component of magnetic field does not change its direction and $|H_{\perp}^2| > |H_{\perp}^1|$

These inequalities demonstrate that a shock wave passing through a weak field $H_x < [2\pi(\rho_1 + \rho_2)\rho_1/\rho_2]^{1/2}v_x^1$

intensifies the magnetic field, while a shock wave passing through a strong magnetic field for which this inequality is reversed, reduces the strength of the magnetic field. This demonstrates that shock waves have some smoothing action when passing through a magnetic field.⁷

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Hydromagnetic Shock Waves in High-temperature Plasmas

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M. H. Rose and H. Rubin *

INTRODUCTION

Shock waves are one of the most important tools for heating ordinary gases and for creating a plasma. It is still an open question whether shock heating will have as vital a position for high temperature thermonuclear plasmas. The theoretical results are not quite complete, and experimental confirmation may take a while. However, the present results, summarized in this paper, do justify a considerable degree of optimism as to the suitability of a certain range of shocks for heating to thermonuclear temperatures.

In classical nondissipative gas dynamics, it is observed that a finite-amplitude compressive wavefront progressively steepens until it becomes multivalued and thereby physically meaningless. This mathematical calamity can be avoided by the introduction of dissipation, usually as heat flow and viscosity. The widening effect of the dissipative mechanism becomes more pronounced as the wavefront steepens, and a wave shape of permanent form is reached as a balance between the effects of nonlinearity and dissipation. The reference length for this transition zone is the mean free path, and its actual thickness ranges from one to several mean free paths except in the case of very weak shocks which are wider.¹ The shock layer is therefore experimentally indistinguishable from a discontinuous wavefront in all conventional gas dynamical applications. The state on one side of the shock can be computed from the state on the other side (and one additional parameter, the "strength" of the shock) by direct use of the laws of conservation of mass, momentum and energy. The particular nature of the dissipative mechanism determines the shape of the transition zone but not the end states. This fortunate circumstance makes it possible to solve gas dynamical flow problems with the relatively simple nondissipative fluid equations by piecing together smooth solutions at appropriately selected shock fronts.

In a conducting fluid (plasma), the results obtained by conventional theory are essentially the same.

In particular, with the inclusion of a third dissipative mechanism, finite electrical conductivity, the thickness of the transition layer remains of the order of the mean free path or larger.

The presence of a magnetic field complicates matters considerably, but the essential features are unaltered.² In high-temperature plasmas, the mean free path can be many times larger than the size of the apparatus. The apparent conclusion is that one should not be able to observe a shock wave in a thermonuclear plasma of laboratory scale. This would be unfortunate on two counts. Mathematically, it would necessitate the use of complicated differential equations to solve problems that could otherwise be solved explicitly algebraically. Physically, it would seem to eliminate one of the most promising means of irreversible heating of a plasma.

Fortunately, the conventional theory, i.e., the conventional modification of the stresses and heat flow in a magnetic field,³ is completely inapplicable when the gyro radius is smaller than the mean-free path.

With the mean free path eliminated, we naturally turn to the Debye length or the gyro radius as a suitable reference length.[†] The significant length parameter turns out to be neither of these; it is the speed of light divided by the plasma frequency. This is roughly intermediate between the electron and ion gyro radii. The problem is to exhibit an irreversible mechanism which operates on this scale of length.

To isolate the essentials of the problem we take the mean-free path to be infinite. An electron or ion feels only the electromagnetic field of the averaged distant charges which are represented by the source terms in Maxwell's equations. This is a so-called *self-consistent field* formulation. The crucial point to

[†] To be sure, the Debye length already exists as an alternative size even without the magnetic field. However, it can be shown that a compressive motion, e.g., one produced by a piston, will widen indefinitely in the absence of a magnetic field until particle collisions take over. Paradoxically, collisions which are the source of the conventional dissipative widening effect are required to effect a state in which gas dynamics becomes applicable, i.e., in which compression waves steepen.

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realize is that irreversibility has not been eliminated by removing the collision term in the Boltzmann equation; it has merely been concealed. We can still rely on the fundamental irreversible mechanism described heuristically by Gibbs. A special case in the application of this mechanism to plasmas has been referred to as Landau damping. The problem we consider is more subtle and the solution has been frequently held to be undamped. The basic property which distinguishes *every irreversible mechanism* is that two microscopic initial states which are so close as to be indistinguishable may eventually emerge very far apart. Short-range intermolecular collisions (which we have chosen to ignore) exhibit this property very clearly. This mechanism, sometimes called phase mixing, is intuitively simple but frequently offers great mathematical difficulties.

The practical question which we must answer is whether a violent irreversible change of state can be produced within a thin layer of some permanence. The theoretical question is, does the shock exist? In addition, the heating of ions and electrons separately must be found. It is necessary to solve the shock transition problem in a high-temperature plasma not only to establish the observability, i.e., the thickness of the shock layer, but even to find the correct jump conditions. In the absence of collisions, there is no reason for the state after the shock to be in thermodynamic equilibrium; in fact, the ion and electron temperatures can be different. Conservation of mass, momentum and energy can only predict the mean temperature. We find that it is possible to choose conditions such that the shock will heat the ion gas much more than the electron gas.

In the following sections, a number of approximate two-fluid theories are described, also several approximations to a complete self-consistent field solution, as well as combinations of fluid and particle analyses. The cumulative evidence seems to be quite strong for the existence of shock waves with the properties described.

FORMULATION OF THE PROBLEM

We consider a steady flow of gas, in the positive x direction, in which a shock transition occurs from a constant state at $x = -\infty$ (front of the shock) to another constant state at $x = +\infty$ (back of the shock). All macroscopic quantities are assumed to be functions of x alone. A magnetic field B points in the z direction. It is accompanied by a current J in the y direction†

$$\mu J = -dB/dx, \quad (1)$$

as shown in Fig. 1. The electric field E has a component E_y in the y direction and possibly a component E_x (charge separation) in the x direction. From $\partial B/\partial t = -\text{curl } E = 0$, we conclude that E_y is a constant, independent of x . Velocity components are (u, v) in the (x, y) directions. In a two-fluid theory,

the current J arises from the velocity component v . We distinguish between ion and electron quantities by subscripts, u_+ , u_- , etc. At $x = +\infty$ and $x = -\infty$ we have $v_+ = v_- = J = 0$ and $E_x = 0$. We assume charge neutrality for all x :

$$n_+ = n_- = n, \quad (2)$$

where n is the number density. This is valid if the Debye length is small compared to the scale of the shock transition. For most of the arguments in this paper this is a convenience rather than a necessity. From the two laws of conservation of mass (or number), we have

$$n_+ u_+ = \text{constant}, \quad n_- u_- = \text{constant}, \quad (3)$$

whence, for all x , we conclude that

$$u_+ = u_- = u. \quad (4)$$

At both $x = +\infty$ and $x = -\infty$ we must have

$$E_y = uB. \quad (5)$$

The state in front, i.e., at $x = -\infty$, is assumed to be not only constant but in thermodynamic equilibrium, $T_+ = T_-$. However, at $x = +\infty$ we may have

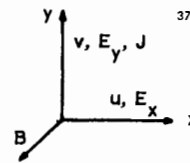


Figure 1

unequal temperatures, $T_+ \neq T_-$. Of course, collisions would eventually make the two temperatures equal, but this is not part of the problem considered.

We introduce the following self-explanatory notation:

$$p_i = nkT_i = \rho_i R_i T_i, \quad \rho_i = m_i n, \quad R_i = k/m_i; \quad (6)$$

$$\begin{aligned} p &= p_+ + p_-, \quad \rho = \rho_+ + \rho_-, \quad T = \frac{1}{2}(T_+ + T_-), \\ p &= \rho RT, \quad R = k/m, \quad m = \frac{1}{2}(m_+ + m_-); \end{aligned} \quad (7)$$

$$\begin{aligned} \rho u &= \rho_+ u_+ + \rho_- u_-, \quad \rho v = \rho_+ v_+ + \rho_- v_-, \\ J &= ne(v_+ - v_-). \end{aligned} \quad (8)$$

JUMP CONDITIONS

From a particle viewpoint, the z component of velocity of each particle is a constant of the motion; only the x and y components matter. Hence, any fluid theory which describes this situation must be strictly two-dimensional; i.e., the internal energy, $\mathcal{E}$, is given by RT rather than $\frac{3}{2}RT$, and the gas constant (ratio of specific heats) is $\gamma = 2$. The laws of conservation of momentum and energy—mass has already been taken care of by Eq. (3)—are expressible as

$$\begin{aligned} [\rho u^2 + p + B^2/2\mu] &= 0, \\ [\rho u(\mathcal{E} + \frac{1}{2}u^2) + pu + E_y B/\mu] &= 0; \end{aligned} \quad (9)$$

† We use rationalized mks units.

where the symbol $[Q] = Q_1 - Q_0$ is for the jump in Q across the shock. In terms of the "total pressure" p^* and "total energy" $\mathcal{E}^*$ defined as

$$p^* = p + B^2/2\mu, \quad \mathcal{E}^* = \mathcal{E} + B^2/2\mu\varrho, \quad (10)$$

the jump conditions (9) are identical to those with no magnetic field; i.e.,

$$[\varrho u^2 + p^*] = 0, \quad [\varrho u(\mathcal{E}^* + \frac{1}{2}u^2) + p^*u] = 0. \quad (11)$$

We also have the corresponding equations of state

$$p = \varrho\mathcal{E}, \quad p^* = \varrho\mathcal{E}^*. \quad (12)$$

In terms of the *reduced* variables ϱ , u , p^* , $\mathcal{E}^*$ (and $T^* = p^*/\varrho$, if desired), the jump conditions are the same as those for an ordinary gas with $\gamma = 2$.⁴ To find the jump in B and in the original variables p , $\mathcal{E}$, and T , it is only necessary to supplement the reduced jump conditions with the relation $[Bu] = 0$, see Eq. (5).

We introduce three speeds: the sound speed a , the Alfvén speed A , and the reduced sound speed a^* , with three corresponding Mach numbers M_a , M_A and M^* :

$$\begin{aligned} a^2 &= 2RT, & M_a &= u/a \\ A^2 &= B^2/\mu\varrho, & M_A &= u/A \\ (a^*)^2 &= a^2 + A^2, & M^* &= u/a^*. \end{aligned} \quad (13)$$

We recall that the value of any one of the shock strength parameters M_0^* , ϱ_1/ϱ_0 , u_0/u_1 , p_1^*/p_0^* , T_1^*/T_0^* fixes all the others. (Henceforth we shall write M^* instead of M_0^* .) In other words, there is only one dimensionless parameter characteristic of a gas shock. On the other hand, there are two dimensionless parameters for the magnetic shock. These can be chosen as M^* and

$$\beta_0 = p_0/(B_0^2/2\mu), \quad (14)$$

the ratio of gas to magnetic pressure in front. Alternatively, we can take two of the Mach numbers (13) to describe the shock.

Heating

Although the pressure and temperature ratios across a shock, p_1^*/p_0^* and T_1^*/T_0^* , can become arbitrarily large for strong shocks, the density and velocity ratios, ϱ_1/ϱ_0 and u_0/u_1 , are at most *three* for a gas with $\gamma = 2$. The same bound therefore holds for the magnetic field ratio $B_1/B_0 = u_0/u_1$. From the easily verified fact that $p_1^*/p_0^* > u_0^2/u_1^2 = B_1^2/B_0^2$ and from the relation

$$p_1^*/p_0^* = (B_1/B_0)^2(1 + \beta_1)/(1 + \beta_0), \quad (15)$$

we see that $p_1/p_0 > p_1^*/p_0^*$. Moreover, if we keep the strength M^* (or B_1/B_0) fixed and let β_0 approach zero, we obtain a limiting value for β_1 which is different from zero. In other words, if the magnetic field dominates in front of the shock, then even a shock of moderate strength can heat the gas enormously: $T_1/T_0 \rightarrow \infty$. The gas is heated to reach an energy per unit volume which is a certain fraction of the magnetic energy, independent of the original gas

energy. For example, a shock of medium strength, $M^* = 2$, impinging on a plasma at "zero" temperature, $\beta_0 = 0$, will heat it to a value $\beta_1 = \frac{1}{4}$.

In order to separate the ion and electron energies at the back of the shock, it is necessary to solve the flow problem for the transition or else find an additional jump condition. We shall find that under certain circumstances this extra condition is that the electron gas is approximately adiabatic. The word adiabatic is used in two different but equivalent senses. If the individual electron motion is adiabatic in the sense that its magnetic moment is approximately constant, we have $\frac{1}{2}m_e\bar{v}^2/B = \text{constant}$. For each electron we have energy $\sim B \sim \varrho$; therefore, for the whole electron gas we have $RT \sim \varrho$ or $p \sim \varrho^2$, which is the adiabatic gas law for $\gamma = 2$. We shall discuss the justification of this adiabatic assumption later. The result is that the electron temperature T_- can increase by at most a factor three; the ions are always heated more, and if β_0 is small, the ion temperature T_+ can increase very much more. Specifically, writing τ for the ratio of the electron temperature behind the shock to that ahead (so that $1 < \tau < 3$), we find

$$(T_+)_1/(T_+)_0 = \tau + 2(1 + 1/\beta_0)(\tau - 1)^3(3 - \tau)^{-\tau}. \quad (16)$$

ADIABATIC TWO-FLUID THEORY

This is perhaps the simplest theory that can be proposed which distinguishes between the ion gas and the electron gas. They are each individually assumed to obey conventional nondissipative fluid equations and supplemented by a Lorentz force. We know beforehand that we shall not find a solution which joins two end states compatible with the shock conditions since the latter imply an entropy increase. Nevertheless, the results are very illuminating.

The meaning of the term "nondissipative" gas is somewhat flexible. One can assume that there is a scalar pressure which satisfies the adiabatic law

$$p = \alpha\varrho^2. \quad (17)$$

Or one can take an energy conservation equation of conventional form involving a scalar pressure and no heat flow. In each case, there is some ambiguity in the term *scalar pressure* depending on whether the frame of reference is the speed of the individual gas or of the gas mixture; for simplicity we adapt the former definition. In the case of a simple gas, the energy equation has the adiabatic relation (17) as an integral. For the mixture of two gases, the two postulates are distinct. The equation of energy conservation would seem to be more fundamental. The results obtained from the two theories are qualitatively similar. Since only the qualitative features of the solution have any bearing on the shock problem as posed, we shall describe here the analytically simple adiabatic theory.

The total (i.e., sum of ion and electron) mass and x -momentum equations can be integrated:

$$\varrho u = m, \quad \varrho u^2 + p + B^2/2\mu = P. \quad (18)$$

Together with the adiabatic law (17), we find the relation between B and u :

$$B^2 = 2\mu[P - mu - a(m/u)^2]. \quad (19)$$

The difference between the two y -momentum equations,

$$\frac{d}{dx} (Q_i u v_i) = n e_i (E_y - uB), \quad (20)$$

can be combined with the Maxwell Eq. (1) to give

$$\begin{aligned} \frac{d^2 B}{dx^2} &= \mu n e^2 \left(\frac{1}{m_+} + \frac{1}{m_-} \right) (B - E_y/u) \\ &= \frac{\kappa^2}{u} (B - E_y/u) \equiv - \frac{d\Phi}{dB}. \end{aligned} \quad (21)$$

We recognize the dimensional factor in the first line to be the squared ratio of plasma frequency to the speed of light; κ^2 is a constant. Since u is known implicitly through Eq. (19), as a function of B , Eq. (21) can be integrated. The behavior of the solutions are immediately evident when we interpret $\Phi(B)$ as a potential for the "motion" $B(x)$. From Fig. 2, we see that there is a family of periodic motions of increasing amplitude around the potential minimum (back of the "shock") culminating in a pulse or solitary wave which starts at the front and eventually returns. These solutions are shown in the phase plane, Fig. 3, and are sketched as functions of x in Fig. 4.

The entropy can be chosen as is the parameter which distinguishes these curves. It should certainly be expected that a dissipative mechanism would produce the damped oscillation indicated in Fig. 4 by the dashed lines, producing a legitimate shock transition from front state to back state. This has actually been verified by inserting a friction term in the momentum equations (ohmic dissipation) proportional to the relative velocity, but the damping length is, of course, the mean free path.

The reference length for the periodic oscillations is essentially

$$L^2 = m_- / \mu n e^2. \quad (22)$$

The phase length or distance travelled by a particle in one gyro period is

$$L_i = m_i u_0 / e B. \quad (23)$$

We find that $L_-/L = \epsilon M_A$ and $L_+/L = M_A/\epsilon$ where $\epsilon^2 = m_-/m_+$. For M_A not too large (also not too

small—for weak shocks, the actual wavelength becomes large compared to the reference length L), we conclude that the individual electron orbits are quite accurately adiabatic ($L_- \ll L$), whereas the ion orbits definitely are not. Provided that there does exist a shock transition approximated by this theory, the electron gas will be adiabatic; the ion gas will be heated more and in some cases much more. For the thermonuclear application, the most interesting shocks would seem to be those of *medium* strength.

The value of this theory is that it strongly suggests that we look for an oscillatory shock transition rather than the classical monotone transition.

More elaborate fluid theories, including non-scalar stresses and heat flow, have been attempted, but the results are not yet definitive.

SELF-CONSISTENT FORMULATION

If the fields $E(x)$ and $B(x)$ are known, the motion of an individual particle is governed by the equations

$$\begin{aligned} \frac{du}{dt} &= \frac{e}{m} \{E_x(x) + vB(x)\}, & u &= \frac{dx}{dt}, \\ \frac{dv}{dt} &= \frac{e}{m} \{E_y - uB(x)\}, & v &= \frac{dy}{dt}. \end{aligned} \quad (24)$$

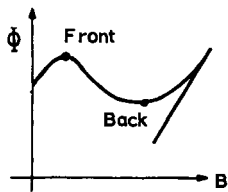
(There should be no confusion in using u and v for particle velocities as well as for gas velocities.) The second equation can be integrated once in terms of the "vector" potential,

$$\begin{aligned} \mathbf{A}(x) &= \int^x B(\xi) d\xi, \\ v &= (e/m) [E_y t - \mathbf{A}(x)] + \text{constant}. \end{aligned} \quad (25)$$

If the energy of the particle is bounded, we conclude that its position is never more than a bounded distance away from the solution of the implicit relation $E_y t - \mathbf{A}(x) = 0$, which is equivalent to $dx/dt = E_y/B$. In particular, all particles eventually move on to $x = +\infty$.

In principle, Eq. (24) can be solved for any given coefficients $E(x)$ and $B(x)$ and for any given initial values. Now suppose that there is a Maxwellian velocity distribution of particles for x very far to the left. A solution of Liouville's equation, which is equivalent to a solution of the particle trajectories for all initial states, yields the particle distribution for all x . By solving this equation for both the electron gas and the ion gas—with the given $E(x)$ and $B(x)$ —we can compute the densities, velocities and current as functions of x . If the current computed in this way is consistent with the original $B(x)$, and if the ion and electron densities turn out to be equal for all x , then we have obtained a self-consistent solution to the problem.

Any verification for the existence of a solution to the shock problem just posed must be extremely subtle. There is a very simple and plausible argument, based on Liouville's theorem, which apparently proves the nonexistence of a shock under the circumstances considered except that the argument



374

Figure 2

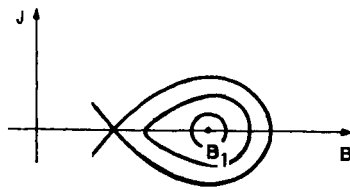


Figure 3

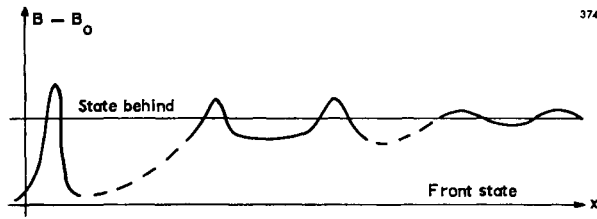


Figure 4

depends on certain unwarranted assumptions. Consider the molecular distribution function, $f(x, y, u, v)$. It satisfies Liouville's equation which states that f is a constant on every particle trajectory. At $x = -\infty$, f is Maxwellian; in particular, the lines of constant f are circles in the u, v plane (Fig. 5). The existence of a constant state at $x = +\infty$ implies that f is isotropic in velocity; the lines of constant f are again circles, but centered about u_1 rather than u_0 . Now, for some value of x far to the left, consider an element of the four-dimensional phase space: specifically, consider the product of an area element dA in physical space and a region $2\pi r dr$ between two circles in velocity space; f is essentially constant on this domain. Although the area element dA may be deformed in a very complicated way in moving to $x = +\infty$, the projection on to velocity space must again be circular at $x = +\infty$ since it must lie on a curve of constant f . The only question is which circle maps on to which. According to Liouville's theorem, the four-dimensional volume remains constant during the motion. Just as the over-all conservation relation $n_0 u_0 = n_1 u_1$ is obtained, a similar one can be obtained for any

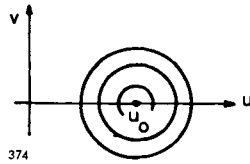


Figure 5

specially identified group of particles. We conclude that the area element dA is compressed by a factor n_1/n_0 in moving from $x = -\infty$ to $x = +\infty$. The area element in velocity space is expanded by the same factor. This implies that the radius of the original circle is expanded by a factor $(n_1/n_0)^{1/2}$, from which we conclude that the distribution at $x = +\infty$ is also Maxwellian at a temperature which is higher by the factor n_1/n_0 . But this is an *adiabatic* compression and is inconsistent with the shock jump conditions unless the two end states are identical.

This argument is fallacious in exactly the same sense as the conventional arguments which pretend to demonstrate that no irreversibility can appear in a Hamiltonian dynamical system. It has been tacitly assumed that there is pointwise convergence of the distribution function f to its assumed limit at $x = +\infty$. A little consideration will show that not only is the area element dA deformed, but the image

of the circle in velocity space becomes extremely deformed (cf. Fig. 6) and only converges *weakly* to its limit; two points in phase space which are very close at $x = -\infty$ can become widely divergent as $x \rightarrow +\infty$. A circle at $x = -\infty$ becomes smeared on to a two-dimensional region at $x = +\infty$.

It is possible to generalize the concept of constant state at the back of the shock in the following way. If the ion and electron distribution, $f(u, v)$ are symmetric about the mean velocity (even though not isotropic), there will be no net current. Consequently, there exist exact self-consistent solutions in which a non-constant microscopic state is coupled with a constant macroscopic state. One might think that such non-isotropic states are equally possible as limiting states behind the shock. However, it is easy to see by the same qualitative arguments as above, that such states are unstable unless they are isotropic.

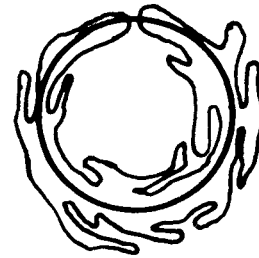


Figure 6

We have already observed that the case $\beta_0 = 0$ (zero temperature in front of the shock) is particularly interesting from the point of view of heating.⁸ At zero temperature, there is no dispersion of ion velocities, and only a single orbit has to be calculated. If, in the particle Eq. (24) we set $d/dt = u(d/dx)$, we observe that the equation of motion of a single particle is identical with the fluid equation obtained by setting $p = 0$. The exact self-consistent field problem has therefore been solved in this special case in Section 4. There exists *no* self-consistent shock solution (at least for shocks of moderate strength; for strong shocks the question is still open).[§] It is clear, *a priori*, that we could expect no shock in this special case. The damping mechanism which we have just discussed requires a dispersion of velocities about the mean, i.e., a finite, temperature. We do derive the useful piece of information that the shock width, i.e., the damping distance, will be much larger than the reference length, L , Eq. (22), if β_0 is very small.

In the remainder of this paper we shall concentrate mainly on medium strength shocks for which the electron motion is assumed to be adiabatic. This converts the problem to a combination of one essentially fluid equation (for the electrons) and one

[§] This zero-temperature solution has also been obtained by Davis, Lust and Schluter, also by Mittelman and probably by many others.

particle equation (for the ions). A particularly simple form of the adiabatic electron relations is:⁵⁻⁷

$$\begin{aligned} u(x) = u_-(x) = E_y/B(x), \quad v_-(x) = -E_x(x)/B(x) \\ n(x) = \text{constant}/u(x), \quad J_- = -nev_-. \end{aligned} \quad (26)$$

The x component of velocity and the electron contribution to the current (which, under certain circumstances, can be taken to be the entire current) are explicitly given in terms of $E_x(x)$ and $B(x)$. To this approximation, the magnetic field self-consistency relation becomes

$$B(x) = \text{constant} \int E_x(x) dx.$$

The problem is to find an $E_x(x)$ such that the resulting ion density agrees with the above computed electron density.

EXPANSION IN THE MASS RATIO

The results of this section are based on essentially two concepts: expansion of the particle trajectories and of the self-consistent fields in the electron-ion mass ratio,[¶] and separation of the Maxwellian ion distribution at $x = -\infty$ into velocity ranges which exhibit entirely different trajectories.

We introduce the parameter $\varepsilon^2 = m_-/m_+$. More precisely, we keep all macroscopic parameters fixed, including the Alfvén speed which fixes m_+ , and take $m_- \sim \varepsilon^2$. We have $L_+ \sim 1$, $L_- \sim \varepsilon^2$, and $L \sim \varepsilon$. An important point is that the charge separation field is assumed to scale as $1/\varepsilon$. Because of this, the motion of the electrons is a modification of the conventional guiding-center motion.⁵⁻⁷ In particular, the spiraling frequency is not the gyro frequency, $\omega = eB/m_-$, but is $\{\omega^2 + \omega(d/dx)(E_x/B)\}^{1/2} \equiv \omega^*$; the adiabatic invariant is $m\bar{v}^2/2\omega^*$ rather than the magnetic moment $m\bar{v}^2/2\omega$. The guiding-center drift velocities in the x and y directions are given by

$$v_- = -E_x/B, \quad u_- = E_y \left(B + \frac{m_-}{e_-} \frac{dv_-}{dx} \right). \quad (27)$$

To lowest order, the ions see only the large electrostatic field E_x if they are in the body of the shock, away from $x = \pm \infty$. This, however, is misleading. Ions of low energy can be trapped between two potential peaks of E_x . The number of trapped ions in each valley must be known in order to compute a self-consistent solution. But we know, from the exact equations, cf. Eq. (25), that all ions must have come from the Maxwellian at $x = -\infty$ and must eventually proceed to $x = +\infty$. It is therefore necessary to go to higher-order approximations in the ε expansion, approximations which allow the ions to pass through, to determine the parameters which are inserted in the lowest-order solution.

[¶] This is the zero-order guiding center approximation to the current; Eq. (26) can easily be improved.

^{¶¶} Some analysis of the problem on this basis has been made by M. Rosenbluth and C. Longmire (private communication).

This problem takes on entirely different orders of complexity depending on whether all the ion trajectories are monotone in x or whether some of the orbits loop. In the second case, the density and current contribution of an ion at a point is the sum of the individual contributions from each occasion when the ion passes that point. If the shock is not too strong, then an ion started at $x = -\infty$ with the mean fluid x velocity and no y velocity, i.e., at the peak of the Maxwellian, will not loop. Neither will those ions loop whose initial velocities lie within a certain circle C_0 : $(u - u_0)^2 + v^2 < R_0^2$ in velocity space surrounding the mean ion velocity. Consider the following modification of the problem originally posed. There are assumed to be no ions at $x = -\infty$ outside the circle C_0 ; the distribution is a Maxwellian which is cut off at R_0 . To lowest order in the expansion parameter, ε , this modified problem can be solved self-consistently. The result turns out to be qualitatively the same as that described in Section 4. There are periodic solutions and pulses, but no shocks.

Rebound as a Damping Mechanism

An ion lying outside the circle C_0 may be reflected back by the electric field E_x . The ions can be classified according to the number of reflections they make against the first potential barrier which they meet until they pass over it, as eventually they must. It is found, to lowest order in ε , that after having overcome the first barrier, an ion will not be turned back again for at least the distance L/ε . Now consider a concentric circle C_1 with radius $R_1 > R_0$, within which the ions make only a fixed number of bounces (the number is bounded, independent of ε).^{**}

This problem can also be solved self-consistently to lowest order in ε . The solution starts as a pulse and then becomes a periodic wave (Fig. 7). The reason for the distinction between the first wave and all the

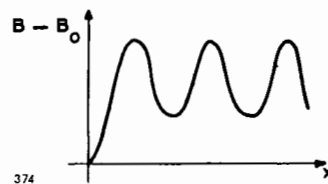


Figure 7

succeeding waves is that only within the first are any ions reflected back.

It is intuitively appealing to interpret these "collisions" of ions with the electric field as being similar in effect to conventional intermolecular collisions in an ordinary gas. Consider a circle $(u - u_0)^2 + v^2 = R^2$ on which some ions pass through the potential barrier without reflection while others pass through only on the second attempt. It is evident that this circle may be enormously deformed by the time it ultimately gets entirely past the potential barrier.

^{**} To lowest order in ε the circles C_0 and C_1 are independent of ε .

After passing over an infinite oscillatory wave train, it would not be surprising if this circle becomes dense in a two-dimensional domain. A smaller circle, on which there are no reflections at all, will be only slightly deformed after passing through a wave. This could be termed a "small-angle" collision of the ions with the electromagnetic field and is a much slower process.

It must be noted that the expansion of a particle trajectory in ϵ cannot be expected to be valid uniformly for an indefinite length of the trajectory; the error will eventually cumulate. In particular, the result shown in Fig. 7 is not valid for distances of the order of $1/\epsilon$ wavelengths from the first potential peak. This is about one ion phase length, Eq. (23); i.e., the distance the ions would go before being affected appreciably by the magnetic field. Consequently, it is not surprising that the second type of damping (termed "small-angle" above) does not emerge to this order in ϵ .

An interesting group of ions can be found which makes of the order of $1/\epsilon$ bounces before passing through. During this process, they gain enormous energy (of the order of $1/\epsilon^2$). Their effect is hard to assess quantitatively since each such particle contributes a large damping effect, but there are few such particles in a Maxwellian distribution. The importance of even a very few such highly energetic ions may be great in thermonuclear applications.

This mechanism is reminiscent of the Fermi mechanism for accelerating cosmic particles, except that in this case it is the electric field which is dominant.

It is clear that the determination of the width of a shock of the type envisioned here is a complex matter. The result depends very closely on the precise shape of the distribution at $x = -\infty$ (which in practice could be rather arbitrary, as for example the residue left by an earlier shock). There exists the strong possibility that the major part of the damping could occur across the first wave, in which case this could be taken as a practical measure of the shock width even though the final decay could be extremely slow.

It is possible to analyze the ultimate decay to the state behind the shock by utilizing certain stochastic approximations. It is important to realize that the damping mechanism is basically nonlinear and cannot be expected to produce an exponential decay. One can estimate the ultimate decay to be at a rate $1/x^{1/2}$. However, if one lets ϵ approach zero first and then lets x approach infinity, a decay at the faster rate $1/x^{1/2}$ is plausible. The damping length scale will also depend on the strength of the shock, and one can estimate it as becoming exponentially large as the strength approaches zero.

For shocks which are stronger than a certain critical value beyond which the fluid theory of Section 4 breaks down, there are indications that there are two stages in the transition. In the second stage there are oscillations whose effective wavelength becomes much smaller by a factor ϵ^2 . This would bring the wavelength close to or even under the Debye length,

and charge separation would become a dominant feature of the problem.⁸ The wavelength is smaller than the electron gyro radius and reverses the roles of electrons and ions with regard to heating in the second stage. It is, however, impossible to estimate qualitatively whether the ions, heated in the first stage, or the electrons, heated in the second stage, end up hotter.

A RELATED TRANSIENT PROBLEM

The question of the approach to equilibrium behind a shock is, in principle, better answered by studying the transient problem, e.g., of a piston moving into the plasma, than by studying the stationary problem. On the other hand, the transient problem is more difficult. In this section we discuss the solution which, roughly speaking, describes the compression wave produced by a slight piston motion. For simplicity we take the case of equal ion and electron masses, thereby eliminating the charge separation field E_x .^{††} The pressure is taken to be zero; consequently we are free to consider these equations as representing a fluid or a self-consistent particle stream. The x component of velocity is the same for ions as for electrons; the y components are negatives, $v_+ = -v_-$. The number of equations is thereby reduced to that in a single-fluid theory. For linear, small amplitude departures from a state of rest (as in Fig. 1, but with the unperturbed state $n_0 = 0$ and $E_{y0} = 0$ and $n = n_0$), the equations are

$$\begin{aligned} \partial n / \partial t + \partial u / \partial x &= 0 \\ m \partial u / \partial t + e B_0 v &= 0 \\ m \partial v / \partial t - e (E_y - u B_0) &= 0 \\ \partial B / \partial x + \mu e n_0 v &= 0 \\ \partial E_y / \partial x + \partial B / \partial t &= 0. \end{aligned} \quad (28)$$

It is possible to reduce this system to a single equation for E_y :

$$\frac{\partial^2 E_y}{\partial t^2} - A^2 \frac{\partial^2 E_y}{\partial x^2} = L^2 \frac{\partial^4 E_y}{\partial x^2 \partial t^2}. \quad (29)$$

A is the Alfvén speed (13) and L is essentially the same reference length as in (22), taking note that $m_+ = m_-$ and that the shock speed, in so far as it makes sense to speak of it in this problem, would be A .

It is convenient to impose, as a boundary condition at $x = 0$, the sudden application of a constant transverse electric field E_{y0} rather than a piston condition. If one imagines an originally quiescent cylindrical plasma, then a suddenly applied tangential electric field corresponds to a conventional pinch.

Equation (29), with the given initial and boundary conditions, can be solved by a Laplace transform with respect to t . The inverse of the transform cannot be found in closed form, but its asymptotic behavior for

^{††} Time-independent problems of this type have been treated by M. Mittelman and M. D. Kruskal (private communication).

large t can be determined by the method of steepest descents.

In this way we obtain

$$E_y(x, t) \sim E_{y0} \int_{c\alpha}^{\infty} A_i(s) ds \quad (30)$$

where $C = (\frac{2}{3})^{\frac{1}{3}} \alpha = (1/L^3)(x - At)/(At)^{\frac{1}{3}}$ is to be distinguished from α of Eqs. (19) and (20), and $A_i(s)$ is the Airy function, defined by

$$A_i(s) = \frac{1}{\pi} \int_0^{\infty} \cos(\frac{1}{3}y^3 - sy) dy$$

This result represents a wave train moving at the Alfvén speed. Figure 8 shows the qualitative behavior of E_y plotted against α , for $E_{y0} = 1$.

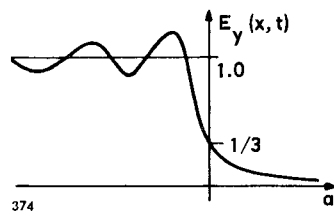


Figure 8

The presence of the cube root of time in the denominator of the scale parameter α shows that the width of the wave is increasing as the cube root of time. The height of the wave drops off exponentially ahead of the front $\alpha = 0$ or $x = At$. For negative α , the curve oscillates, and approaches unity. The amplitude of the oscillations decreases like $\alpha^{-\frac{1}{3}}$.

This result is not easily compared with the previous ones. By analogy with ordinary gas dynamics, one would guess that the spreading of the wavefront is due to the linearization. However, we know the

solution of the appropriate non-linear equations in the steady state, and there is no corresponding steady shock. Also, the damping mechanism is itself non-linear, so the damped shape of this linearized wave may not be representative of the answer to the preceding problem. In this context, we can refer again to the gas dynamical shock. The solution to a free-flow initial value problem (i.e., with no collisions) widens indefinitely even when of finite amplitude. Collisions are required to validate the fluid equations, which imply steepening, as well as to supply dissipation, the conventional widening mechanism.

CONCLUSIONS

The object of study has been a medium strength shock (measured by M^* or by the magnetic field rate). The conclusion that the major portion of the heating is applied to the ions follows trivially from the single assumption that the electrons are adiabatic. The goal of this entire complex self-consistent theory is only to verify this assumption as well as the implicit one that the shock transition zone is small enough to fit inside the laboratory. The most interesting case, producing a very large temperature rise, is when the plasma energy in front of the shock is a small fraction of the magnetic energy ($\beta_0 \ll 1$). However, this is exactly the case when the wave train is largest. If quantitative computations verify that a significant part of the entropy increase takes place across the first wave, then the macroscopically small length L , Eq. (22), can be taken as the effective width of the shock. If this is the case, it should be possible to obtain thermonuclear temperatures from a relatively cold plasma by the passage of a single shock wave. Whether experiment or theory will be the first to definitely verify or refute this possibility remains to be seen.

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Possibilities of Achieving Fusion Reaction with a 4π Focusing System

By G. Barsanti, B. Barsella, M. Camerini, U. Federighi, L. Musumeci and N. Talini *

INTRODUCTION

Giiti Iwata and his colleagues proposed¹ the idea of a fusion-type nuclear reactor based on the principle of 4π focusing by means of an electromagnetic field. For example, for a primary gas of deuterium ions, this method would make possible the focusing of the T^+ and He^{3+} ions produced during D-D reactions at the same point, in such a way as to also use the secondary reactions D-T and D- He^3 at this point.

The 4π focusing can be obtained at the origin (0, 0, 0) of an electrical field having a potential

$$V = \frac{1}{2}k(-x^2 - y^2 + 2z^2), \quad (1)$$

where $k > 0$, superimposed on a magnetic field (0, 0, $-B$).

The equations of motion of a mass m with charge $+e$ are:

$$\begin{aligned} \ddot{x} &= \frac{ek}{m}x - \frac{eB}{m}\dot{y} \\ \ddot{y} &= \frac{ek}{m}y + \frac{eB}{m}\dot{x} \\ \ddot{z} &= -\frac{2ek}{m}z. \end{aligned} \quad (2)$$

This system was integrated by G. Iwata *et al.* for the case of trajectories passing through the origin. By writing $x + iy = u \exp(i\omega s/2m)$, they obtain

$$\begin{aligned} \ddot{u} &= -((eB/2m)^2 - ek/m)u = -(\omega s)^2 u \\ \ddot{z} &= (2ek/m)z = -\omega^2 z. \end{aligned} \quad (3)$$

These equations show that the motion of each particle can be decomposed into simple harmonic motions.

To obtain focusing at the point ($u = 0, z = 0$), it is sufficient that s be a rational number. The collisions which take place at this point do not change the focusing conditions for the ions, while a suitable choice of s makes possible the approximate focusing of several different masses.

For the moment, we shall disregard the effect of the space charge and the losses due to collisions in the vicinity of the origin. We conclude that each particle is focused an indefinite number of times up to the time it is absorbed in a nuclear reaction. Accordingly, if a primary beam is focused at the origin, the ion density increases until equilibrium between the ions introduced and those consumed in the reaction is established.

From the practical standpoint, a number of questions arise which call for both an experimental and theoretical answer. To this end, we have considered the following points:

(a) The general solutions of system (2) valid for arbitrary initial conditions. This is necessary for the consideration of imperfectly focused primary trajectories, of the effect of collisions in the vicinity of the origin and of ions produced in the neutral gas which diffuses into the reaction chamber.

(b) Investigation of the primary ion injection system, of the density of the ions in the chamber and of the energy balance as a matter of principle.

(c) The experimental apparatus needed for 4π focusing of deuterons and a sketch of a fusion reactor.

GENERAL SOLUTION OF THE EQUATIONS OF MOTION

We adopt a system of cylindrical coordinates (0, u, z), the z axis coinciding with the same axis of the system (2). We can now write the electrical potential in terms of z and u

$$V = \frac{1}{2}k(-u^2 + 2z^2)$$

and Eq. (2) becomes

$$\begin{aligned} \ddot{\theta} &= (c/u)^2 + eB/2m \\ \ddot{u} &= (\ddot{\theta} - (eB/m)\dot{\theta} + ek/m)u \\ \ddot{z} &= -(2ek/m)z, \end{aligned} \quad (4)$$

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* CAMEN., Leghorn.

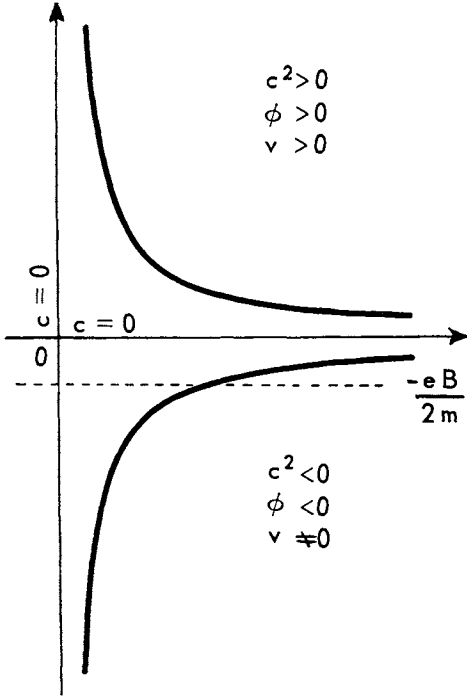


Figure 1

in which c^2 is a constant of integration. With the transformations

$$\begin{aligned}\phi &= \dot{\theta} - eB/2m \\ v &= u^2,\end{aligned}$$

the first equation of (4) becomes

$$\phi v = c^2. \quad (5)$$

In the case where $c^2 = 0$, (5) has the asymptotic solutions

$$\begin{aligned}\phi &= 0 \\ v &= 0\end{aligned} \quad \text{or} \quad \begin{cases} \dot{\theta} = eB/2m \\ u = 0. \end{cases}$$

The solution $u = 0$, together with the third equation of (4), represents a harmonic oscillatory motion along the z axis, the frequency of which, $\frac{1}{2}\pi(ek/m)^{1/2}$ is independent of B , whereas the solution $\dot{\theta} = eB/2m$ represents the particular case investigated by G. Iwata and his group. The above solution leads to the system (3). Generally speaking, $c^2 \neq 0$, u is real and $v > 0$. This leads us to consider the three cases illustrated in Fig. 1 and listed below:

$$(a) \quad c^2 > 0, \phi > 0 \text{ and } \dot{\theta} = eB/2m.$$

The argument θ always increases.

$$(b) \quad c^2 < 0, 0 \leq \dot{\theta} < eB/2m.$$

The argument θ always increases.

$$(c) \quad c^2 < 0, \dot{\theta}_0 \leq \dot{\theta} < eB/2m \text{ with } \dot{\theta}_0 < 0.$$

The argument θ alternately increases and decreases.

The direction of motion about the z axis is reversed when the particle goes through the cylindrical surface

whose axis is the z axis and whose radius is $(2mc^2/eB)^{1/2}$. On this surface $\dot{\theta} = 0$.

By substituting the first equation in (4) into the second, we obtain

$$\ddot{u} = c^4/u^3 - (\omega s)^2 u, \quad (6)$$

where $c^4 > 0$ and $u > 0$, always. Let us assume $(\omega s)^2 > 0$ and note that $\ddot{u} = 0$ when $u^4 = c^4/\omega^2 s^2$ and that $\ddot{u} \leq 0$ according as $u \leq |c(\omega s)^{-1/2}|$. Accordingly, the acceleration $\ddot{u}$ is always directed toward point $u = |c(\omega s)^{-1/2}|$ and the motion along the u axis is oscillatory.

If $(\omega s)^2 \leq 0$, that is, if $(eB/2m)^2 \leq ek/m$ then $\ddot{u} > 0$ always and, after a certain time, the particle recedes indefinitely from the origin.

This case is of no practical interest and, consequently, we must choose parameters B and k so that $(eB/2m)^2 > ek/m$. Equation (6) can be integrated once to obtain

$$\dot{u}^2 = c^4/u^2 - (\omega s u)^2 + 2c_1 \quad (7)$$

with $c_1 > 0$, and a second time to obtain

$$\begin{aligned}u^2 &= v = [c_1 \\ &+ (c_1^2 - \omega^2 s^2 c^4) \sin \{\omega s(2t + c_2)\}]/(\omega s)^2. \quad (8)\end{aligned}$$

Since v is real and positive, we have $c_1^2 \geq (\omega s c^2)^2$ and, therefore, we always have $c_1 \geq [c_1^2 - (\omega s c^2)^2]^{1/2}$.

Examination of Eq. (8) shows two limiting cases. The first is the case where $c^4 > 0$ and the equation becomes

$$v = (c_1/\omega^2 s^2)[1 + \sin \{\omega s(2t + c_2)\}],$$

which is the particular case we have just seen for $\dot{\theta} = eB/2m$. We shall have the extremes of the oscillation for

$$\begin{aligned}u &= 0, \\ u &= (2c_1)^{1/2}/\omega s.\end{aligned}$$

The second case is that where

$$c_1^2 = (\omega s c^2)^2.$$

The projection of the particle's path on the plane $z = 0$ is a circle with center at the origin and a constant radius

$$u = c(\omega s)^{1/2}.$$

Its angular velocity, which is always constant, is

$$\dot{\theta} = \omega s + eB/2m.$$

By integrating the first equation in (4), we can write

$$\begin{aligned}\theta &= \tan^{-1}[(p^2 + 1)^{1/2} + p \tan \{\omega s(t + \frac{1}{2} c_2)\}] \\ &+ eBt/2m + c_3 \quad (9)\end{aligned}$$

in which $p = c_1/\omega s c^2$. Equation (9) together with (8) gives the parametric equations of the trajectory.

REACTION CHAMBER AND ION INJECTION

The configuration of the electric field given by Eq. (1) is obtained by means of electrodes whose surfaces have the form of hyperboloids of revolution about

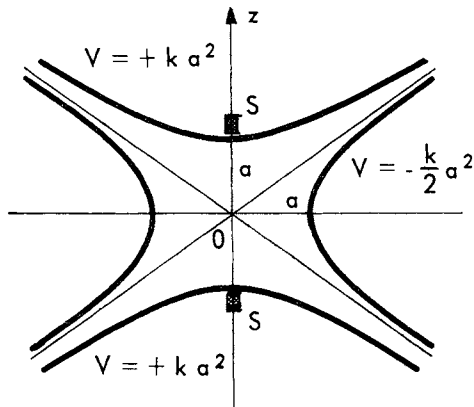


Figure 2. Reaction chamber

the axis. The walls of the chamber are the electrodes themselves (Fig. 2).

For a chamber in which the radius is a and the height $2a$, the potentials to be applied to the electrodes are $+ka^2$ and $-ka^2/2$. If the ion source is placed on the z axis in the position indicated by S and the emission energy of the particles is disregarded, then they reach the origin 0 with energy kea^2 . During the oscillatory motion along the z axis there is consequently a probability of collisions with a relative energy of $2kea^2$ at this point. Nevertheless, due to the finite dimensions of the source, to the initial divergence of the beam, and to the collisions themselves, the relative energy near the origin will, in fact, be between 0 and $2kea^2$.

A rough estimate of the distribution of the particle density in the chamber may be obtained if some simplifying assumptions are made. Let us suppose that the ion beam oscillating along the axis has a cross section $\Sigma(z)$.

The mean density at the corresponding height is

$$\rho_{av} = (n/\pi)\Sigma(z)(a^2 - z^2)^{1/2}, \quad (10)$$

where n is the total number of the ions present in the chamber at a given moment and $\omega = (2ek/m)^{1/2}$.

The parameters k and B of the magnetic field should be chosen in such a way as to have

$$s = [(eB^2/8km) - \frac{1}{2}]^{1/2} = 2q,$$

where q is a positive integer. The emission energy of the ions from the source is neglected. Given these conditions, the cross section of the beam in the vicinity of the origin is approximately equal to the area of the emitting surface of the source itself. This means $\Sigma(0) = \Sigma(a)$. Then we shall find, at the origin, $\rho_{av}(0) = n/\pi a \Sigma(a)$.

Under steady state conditions, the number n is a result of the balance between the ions produced by the source, the ions produced by ionization of the neutral gas in the chamber, the ions used in the nuclear reactions and those which are lost. The number of ions consumed per unit time and unit volume during the nuclear reaction is dependent on the density, on the effective cross section and on the energy. For an

accurate computation we should have to allow for several factors which influence the density distribution. These factors are essentially related to the presence of many ions (Coulomb diffusion, etc.) which may interact at the origin while diffusing in such a way as to destroy the density distribution of the primary beam. This may cause an increased isotropy of the ions at the origin. This distribution then would become a certain $\rho(x)$. The expression for N , the total number of reactions per unit time, would then become

$$N = \frac{1}{2} \int_{vol} d^3x \rho^2(x) \langle \sigma(v)v \rangle_{av}. \quad (11)$$

where $\sigma(v)$ is the reaction cross-section, v the velocity of the ions and $\langle \sigma(v)v \rangle_{av}$ the mean value in the sphere. Let us assume that radius r of the source is small. The mean density, as given in Eq. (10), is uniform over a sphere whose center is at the origin and the maximum cross section is equal to $\Sigma(a) = \pi r^2$. Inside this sphere, the number of reactions per unit time is

$$N = \frac{2}{3} \pi r^3 \rho^2(0) \langle \sigma(v)v \rangle_{av}. \quad (12)$$

We disregard reactions taking place outside the sphere.

The power developed would be EN , E being the energy developed in each fusion reaction. In the case of deuterium, let us assume that reactions $D(d, n)$ He^3 and $D(d, p)$ T are equally probable. The number of neutrons produced in unit time would then be $N/2$.

If, on the contrary, we assume that focus at the origin is not perfect and the distribution of ions leaving this point with energy eka^2 is isotropic, then the function $\rho(x)$ has a spherical symmetry and is given by

$$\rho(x) = 3/4\pi(A^3 + 3A^2x + 3Ax^2)^{-1}, \quad (13)$$

where

$$A \equiv (a^2 - x^2)^{1/2} \sin \pi/n - x(1 - \cos \pi/n).$$

If n is very large and x is sufficiently small compared to a , we have

$$A = (\pi/n)(a^2 - x^2)^{1/2}.$$

From this we obtain

$$\rho(0) = 3n^3/4\pi^4 a^3. \quad (14)$$

EXPERIMENTAL APPARATUS

It is convenient, for practical considerations, to distinguish two possibilities: (a) we provide for the use of the products (T and He^3) of the primary reaction, (b) we only follow the $D-D$ reaction.

In case (a), the nuclei T and He^3 which are produced close to the origin with energies of about one Mev must reach regions where the potential is of the order of 10^6 volts in order to be focused at this point. This potential represents the minimum voltage that is to be applied to the positive electrodes. Disregarding practical limitations, the dimensions of the chamber are arbitrary. However, we must note the analytical condition $4km < eB^2$. We specify a value of B which is technically feasible and this sets an upper limit for k . Accordingly, since $a = (Vz/k)^{1/2}$, we

shall have a lower limit for a . This value determines the size of the apparatus.

Case (b) is limited to the focusing of deuterium ions only and is of interest, in the main, for experimental purposes. The value of V_z may be considerably reduced, as long as we still have a large enough cross section to permit the D-D reaction to occur.³ The value of a will therefore be reduced and with it the dimensions of the apparatus. The building of this

model may give very useful information on the actual possibility of operating a type (a) reactor.

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Slowing-down of Charged Particles in a Plasma

By G. Boulègue, P. Chanson, R. Combe, M. Félix and P. Strasman *

We shall investigate the case in which the slowing-down of an incident particle P_1 can be explained by binary collisions with the constituents of the plasma, collective oscillations entering only in a weak or negligible manner.¹

EXPRESSION FOR THE SLOWING-DOWN (NON-RELATIVISTIC CALCULATION)

The particle P_1 (mass m_1 , velocity v_1 , kinetic energy E_1 , charge Z_1e) moves in an ensemble of elements P_2 (mass m_2 , charge Z_2e) undergoing Maxwellian agitation (temperature T_2). At a given moment the velocity of one of these particles P_2 is V_2 . We designate by $\mathbf{u}$ the velocity of P_1 relative to P_2 :

$$\mathbf{u} = \mathbf{V}_1 - \mathbf{V}_2$$

When a collision occurs the trajectory of P_1 is deviated by an angle α in the coordinate system of

P_2 . One supposes that the function $\alpha(u, a)$ is known, a being the "collision parameter" (distance of P_2 to the initial relative velocity vector). After the collision the relative velocity becomes $\mathbf{u}'$, with

$$|\mathbf{u}'| = |\mathbf{u}|.$$

Let us call $dn_2(\mathbf{V}_2)$ the density of particles P_2 whose speed is between V_2 and $V_2 + dV_2$ in absolute value and whose velocity vector is in the small element of solid angle $(\Omega, \Omega + d\Omega)$.

At the zero instant all the elements P_2 having velocity V_2 which should interact with P_1 during the unit time following, with collision parameter a and azimuth ϕ (with respect to any origin, e.g., the plane $\mathbf{V}_1, \mathbf{u}$) are contained in a volume $uadad\phi$. Their average number is then

$$dn_2(\mathbf{V}_2)uadad\phi.$$

The momentum conservation in the collision (1)–(2) allows one to write, after a simple calculation:

$$\Delta V_1^2 = V_1^2 - V_1'^2 = \frac{m_2}{m_1 + m_2} \left(V_1^2 - V_2^2 + \frac{m_1 - m_2}{m_1 + m_2} u^2 \right) (1 - \cos \alpha) - \frac{2uV_1m_2}{m_1 + m_2} \sin \beta \sin \alpha \cos \phi,$$

β being the angle between the vectors $\mathbf{V}_1$ and $\mathbf{u}$. $[\Sigma \Delta V_1^2]_{\mathbf{V}_1, \phi}$ is obtained by multiplying by $dn_2(V_2)uadad\phi$. Letting

$$\int_0^R (1 - \cos \alpha) ada = \sigma(u),$$

where R is the interaction radius of the particles, σ is a function of u , obtained from the laws of scattering. Evaluating the integrals over ϕ and a we obtain:

$$[\Sigma \Delta V_1^2]_{\mathbf{V}_1} = dn_2(\mathbf{V}_2) \frac{2\pi m_2}{m_1 + m_2} u \sigma(u) \times \left(V_1^2 - V_2^2 + \frac{m_1 - m_2}{m_1 + m_2} u^2 \right).$$

This represents, but for a numerical coefficient, the mean loss of energy of a particle (1) per unit of time

by collisions with particles (2) having velocities $\mathbf{V}_2$. Let us express $dn_2(\mathbf{V}_2)$ as follows:

$$dn_2(\mathbf{V}_2) = n_2 \mathcal{M}(V_2) dV_2 d\Omega_2 / 4\pi$$

where $\mathcal{M}(V_2)$ represents the Maxwellian distribution. Here n_2 is the total number of particles P_2 per unit volume, θ being the angle $(\mathbf{V}_1, \mathbf{V}_2)$. One obtains after integration over the azimuth:

$$[\Sigma \Delta V_1^2]_{\ell} = \pi \frac{n_2 m_2}{m_1 + m_2} \int \int_{\substack{0 < \theta < \pi \\ 0 < V_2 < \infty}} u \sigma(u) \mathcal{M}(V_2) \times \left(V_1^2 - V_2^2 + \frac{m_1 - m_2}{m_1 + m_2} u^2 \right) \sin \theta d\theta dV_2.$$

The mean loss of energy per unit time is

$$\frac{m_1}{2} [\Sigma \Delta V_1^2]_{\ell};$$

Original language: French.

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and per unit length:

$$-\frac{dE}{dz} = \frac{m_1}{2V_1} [\Sigma \Delta V_1^2]_t.$$

It is convenient to use as variables u and V_2 . One finds:

$$\begin{aligned} -\frac{dE_1}{dz} &= \frac{\pi n_2 m_1 m_2}{2V_1^2(m_1 + m_2)} \int \int u^2 \sigma(u) \frac{\mathcal{M}(V_2)}{V_2} \\ &\times \left[V_1^2 - V_2^2 + \frac{m_1 - m_2}{m_1 + m_2} u^2 \right] dV_2 du \\ &= \frac{\pi n_2 m_1 m_2}{2V_1^2(m_1 + m_2)} \int_0^\infty u^2 \sigma(u) I(u) du, \end{aligned}$$

where

$$I(u) = \int_{|V_1 - u|}^{V_1 + u} \left(V_1^2 - V_2^2 + \frac{m_1 - m_2}{m_1 + m_2} u^2 \right)$$

$$\times \frac{\mathcal{M}(V_2)}{V_2} dV_2.$$

Taking into account $\mathcal{M}(V_2) = A_2 V_2^2 \exp(-\gamma_2^2 V_2^2)$ where $A_2 = (2/\pi)^{1/2} (kT_2/m_2)^{-1/2}$ and $\gamma_2^2 = m_2/2kT_2$, one arrives at the result:

$$\begin{aligned} I(u) &= \frac{A_2 \exp[-\gamma_2^2(V_1^2 + u^2)]}{\gamma_2^4} u^2 \\ &\left\{ \frac{d}{du} \left[\frac{\sinh(2\gamma_2^2 V_1 u)}{u} \right] - 2\gamma_2^2 \frac{m_2}{m_1 + m_2} \sinh(2\gamma_2^2 V_1 u) \right\}. \end{aligned}$$

The calculation of $-dE_1/dz$ is therefore reduced to the evaluation of a single integral, the function $\sigma(u)$ being assumed known.

$I(u)$ can be transformed in various ways of which the most useful for us is the following:

$$I(u) = \frac{A_2 \exp[-\gamma_2^2(V_1^2 + u^2)]}{\gamma_2^4} u^2 \left\{ \frac{d}{du} \left[\frac{\sinh(2\gamma_2^2 V_1 u)}{u} \right] - 2\gamma_2^2 \frac{m_1}{m_1 + m_2} \sinh(2\gamma_2^2 V_1 u) \right\}.$$

$$\int_0^\infty u^2 \sigma(u) I(u) du =$$

$$\begin{aligned} &\frac{A_2 \exp(-\gamma_2^2 V_1^2)}{\gamma_2^4} \int_0^\infty \exp(-\gamma_2^2 u^2) \sinh(2\gamma_2^2 V_1 u) \left[2\gamma_2^2 \frac{m_1}{m_1 + m_2} u^4 \sigma(u) - u^{-1} \frac{d}{du} (u^4 \sigma(u)) \right] du \\ &= \gamma_2^{-4} A_2 \exp(-\gamma_2^2 V_1^2) \int_0^\infty u^{-1} \exp\left(-\gamma_2^2 \frac{m_1}{m_1 + m_2} u^2\right) \sinh(2\gamma_2^2 V_1 u) \frac{d}{du} \left[u^4 \sigma(u) \exp\left(-\gamma_2^2 \frac{m_1}{m_1 + m_2} u^2\right) \right] du. \end{aligned}$$

COULOMB INTERACTION LIMITED BY THE DEBYE LENGTH

Let m be the reduced mass defined by

$$m = \frac{m_1 m_2}{m_1 + m_2};$$

one knows that

$$1 - \cos \alpha = 2 \left[1 + \left(\frac{mu^2 a}{Z_1 Z_2 e^2} \right)^2 \right]^{-1}$$

from which,

$$\begin{aligned} \sigma(u) &= \int_0^D 2 \left[1 + \left(\frac{mu^2 a}{Z_1 Z_2 e^2} \right)^2 \right]^{-1} da \\ &= \left(\frac{Z_1 Z_2 e^2}{mu^2} \right)^2 \ln \left[1 + \left(\frac{mu^2 D}{Z_1 Z_2 e^2} \right)^2 \right]. \end{aligned} \quad (1)$$

Letting D be the Debye length

$$D = \left(\frac{kT_e}{4\pi n_0 e^2} \right)^{1/2}$$

(the subscript e refers to the electrons), one deduces, setting

$$x^2 = \gamma_2^2 \frac{m_1}{m_1 + m_2} u^2,$$

that

$$\begin{aligned} -\frac{dE_1}{dz} &= \frac{(\sqrt{\pi}) n_2}{E_1} (Z_1 Z_2 e^2)^2 \frac{[(m_1 + m_2)m_1]^{1/2}}{m_2} \\ &\times \exp\left(-\frac{m_2 E_1}{m_1 kT_2}\right) \times G. \end{aligned}$$

$$\begin{aligned} G &= \int_0^\infty \exp(-m_2/x^2 m_1) \sinh \beta x \\ &\times \frac{d}{x dx} [e^{-x^2} \ln(1 + x^2 x^4)] dx, \end{aligned}$$

$$\beta = 2 \left[\frac{E_1 m_2}{kT_2 m_1} \frac{m_1 + m_2}{m_1} \right]^{1/2},$$

and

$$\kappa = \frac{2kT_2 D}{Z_1 Z_2 e^2}.$$

If E_1/kT_2 is large, a good approximation is obtained by calculating the term in d/dx for a mean value:

$$x_m = \left[\frac{E_1}{kT_2} \frac{m_2}{m_1 + m_2} \right]^{\frac{1}{2}} / \Phi \left(\left[\frac{E_1}{kT_2} \frac{m_2}{m_1} \right]^{\frac{1}{2}} \right),$$

Φ being the error function

$$\Phi(x) = 2\pi^{-\frac{1}{2}} \int_0^x \exp(-x^2) dx.$$

From this

$$G \approx \sqrt{\pi} \exp \left(\frac{E_1}{kT_2} \frac{m_2}{m_1} \right) \sqrt{\left(\frac{m_1}{m_1 + m_2} \right) \cdot \Phi} \\ \times \left(\left[\frac{E_1}{kT_2} \frac{m_2}{m_1} \right]^{\frac{1}{2}} \right) \left[\ln(1 + \kappa^2 x_m^4) - \frac{2\kappa^2 x_m^2}{1 + \kappa^2 x_m^4} \right]$$

and

$$-\frac{dE_1}{dz} = \frac{\pi n_2}{E_1} \frac{m_1}{m_2} (Z_1 Z_2 e^2)^2 \Phi \\ \times \left(\left[\frac{E_1}{kT_2} \frac{m_2}{m_1} \right]^{\frac{1}{2}} \right) \left[\ln(1 + \kappa^2 x_m^4) - \frac{2\kappa^2 x_m^2}{1 + \kappa^2 x_m^4} \right].$$

This is practically the formula arrived at by the elementary considerations.

On the other hand, for arbitrary values of E_1/kT_2 , the evaluation of G may be rather long; a series expansion of $\sinh \beta x$ is convenient. Some results for a particular case will be given later.

It is interesting to investigate the case of slowing-down of a heavy particle by electrons (m_2/m_1 small), when E_1/kT_2 is small. One then finds (putting $y = x^2$):

$$G \approx -\beta \left(\frac{\beta^2}{6} - \frac{m_2}{m_1} \right) \int_0^\infty e^{-y} \ln(1 + \kappa^2 y^2) dy \\ = -\beta \left(\frac{\beta^2}{6} - \frac{m_2}{m_1} \right) J(\kappa),$$

$$J(\kappa) = 2 \cos \kappa^{-1} \int_{\kappa^{-1}}^\infty x^{-1} \cos x dx \\ + 2 \sin \kappa^{-1} \int_{\kappa^{-1}}^\infty x^{-1} \sin x dx \\ = -2[\cos \kappa^{-1} \text{Ci} \kappa^{-1} + \sin \kappa^{-1} \text{Si} \kappa^{-1}], \quad (2)$$

and finally

$$-\frac{dE_1}{dz} \approx 2\pi^{\frac{1}{2}} n_2 \left(\frac{m_2}{m_1} \right)^{\frac{1}{2}} \left(\frac{2}{3} \frac{E}{kT_2} - 1 \right) \\ \times (E_1 \cdot kT_2)^{-\frac{1}{2}} (Z_1 Z_2 e^2)^2 J(\kappa)$$

One notes that $-dE_1/dz$ is zero for $E_1 = 3kT_2/2$, which is completely reasonable,

TIME OF EQUIPARTITION

We now assume that an ensemble P_1 is slowed down by an ensemble P_2 . The particles (1) have a Maxwellian distribution of velocities at temperature T_1 .

Per cm^3 and per second, the particles (1) lose energy due to interaction with particles (2) that is given by:

$$W_{1,2} = \int_0^\infty V_1(-dE_1/dz) n_1 A_1 \exp \\ \times (-\gamma_1^2 V_1^2) V_1^2 dV_1$$

where A_1 is the coefficient of the Maxwellian distribution for the particles (1). $-dE_1/dz$ is expressed by an integral over u , $W_{1,2}$ by a double integral. These calculations give

$$W_{1,2} = 2(2\pi)^{\frac{1}{2}} n_1 n_2 k^{-3/2} \frac{m_1 m_2}{(m_1 + m_2)^2} \frac{T_1 - T_2}{\left(\frac{T_1}{m_1} + \frac{T_2}{m_2} \right)^{5/2}} \\ \times \int_0^\infty u^5 \sigma(u) \exp \left[-\frac{u^2}{2k \left(\frac{T_1}{m_1} + \frac{T_2}{m_2} \right)} \right] du.$$

If we assume a Coulomb interaction limited to the Debye length, it is necessary to calculate

$$\int_0^\infty \left(\frac{Z_1 Z_2 e^2}{m} \right)^2 u \ln \left[1 + \left(\frac{mD}{Z_1 Z_2 e^2} \right)^2 u^4 \right] \exp \\ \times \left[-\frac{u^2}{2k \left(\frac{T_1}{m_1} + \frac{T_2}{m_2} \right)} \right] du.$$

This integral is derived from that in the previous equation by replacing $\sigma(u)$ by its value given in (1). Letting

$$\xi = \frac{u^2}{2k \left(\frac{T_1}{m_1} + \frac{T_2}{m_2} \right)} \text{ and } g = 2k \left(\frac{T_1}{m_1} + \frac{T_2}{m_2} \right) \frac{mD}{Z_1 Z_2 e^2},$$

one has

$$W_{1,2} = B(T_1 - T_2) \int_0^\infty \ln(1 + G\xi^2) e^{-\xi} d\xi$$

where

$$B = 2 \frac{n_1 n_2}{m_1 m_2} (2\pi/k)^{\frac{1}{2}} \frac{(Z_1 Z_2 e^2)^2}{\left(\frac{T_1}{m_1} + \frac{T_2}{m_2} \right)^{3/2}}$$

and further:

$$W_{1,2} = B(T_1 - T_2) J(g),$$

the function $J(g)$ having been defined in Eq. (2).

When g is large (> 10) one has

$$J(g) \sim 2(\ln g - \ln \gamma) = 2(\ln g - C)$$

where $C = \ln \gamma$ is the Euler constant: $C = 0.5772\dots$. C rapidly becomes negligible in comparison to $\ln g$.

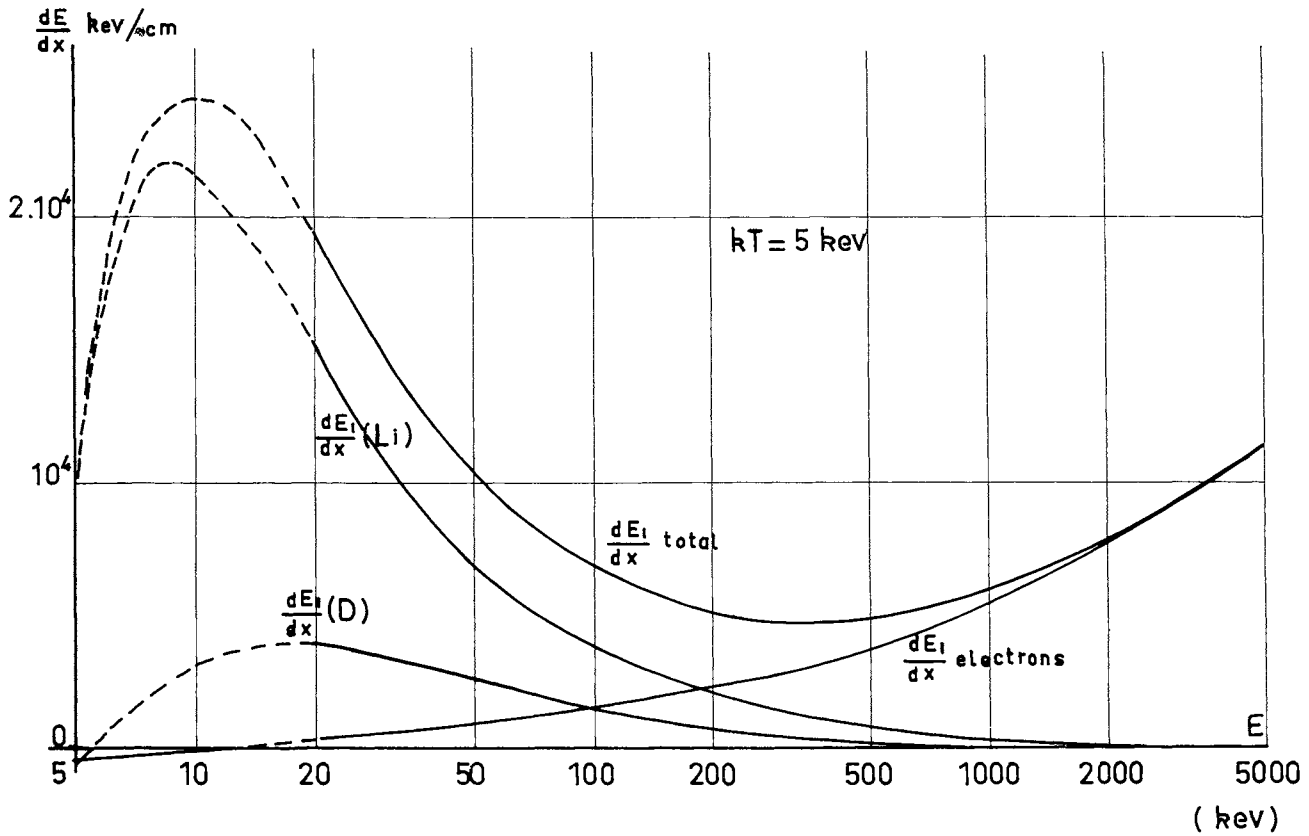


Figure 1. Loss of energy of a tritium nucleus in a plasma of ${}^6\text{Li}$ -D at a temperature of 5 keV, as a fraction of triton energy

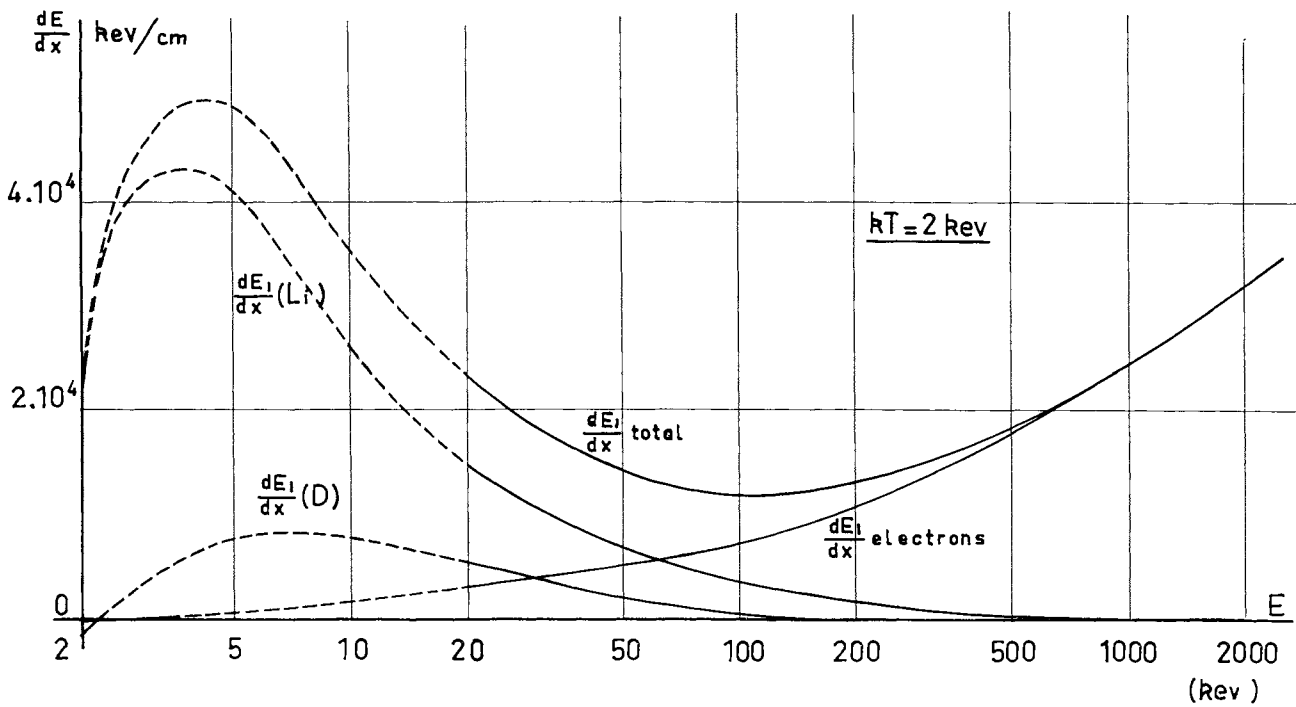


Figure 2. Loss of energy of a tritium nucleus in a plasma of ${}^6\text{Li}$ -D at a temperature of 2 keV, as a function of triton energy

Let us assume, in particular, that the temperatures T_1 , T_2 and T_e are of the same order of magnitude T .

From the equations

$$W_{1,2} = -\frac{3}{2} n_1 k \frac{dT_1}{dt} = \frac{3}{2} n_2 k \frac{dT_2}{dt} = BJ(T_1 - T_2),$$

one easily finds

$$(n_1 + n_2) \frac{dT_1}{dt} + \frac{3}{2} \frac{kn_1 n_2}{BJ} \frac{d^2 T_1}{dt^2} = 0$$

and from an analogous equation for dT_2/dt one has

$$\frac{dT_1}{dt} = C e^{-t/t_{eq}}$$

where t_{eq} is a time for equipartition:

$$\begin{aligned} t_{eq} &= \frac{3}{2} \frac{kn_1 n_2}{BJ(n_1 + n_2)} \\ &= \frac{3}{4(2\pi)^{\frac{1}{2}}} \frac{m_1 m_2}{m_1 + m_2} \frac{\left(\frac{kT_1}{m_1} + \frac{kT_2}{m_2}\right)^{\frac{3}{2}}}{(Z_1 Z_2 e^2)^2} \frac{1}{J(g)} \\ &\approx \frac{3(kT)^{\frac{3}{2}}}{4(2\pi)^{\frac{1}{2}}(n_1 + n_2)} \frac{(m_1 + m_2)^{\frac{3}{2}}}{(m_1 m_2)^{\frac{1}{2}}} \frac{1}{(Z_1 Z_2 e^2)^2 J(g)} \end{aligned}$$

where

$$g \approx \frac{2kTD}{Z_1 Z_2 e^2} = \frac{(kT)^{\frac{3}{2}}}{Z_1 Z_2 e^3 \sqrt{n_e}}.$$

Numerically,

$$\begin{aligned} t_{eq} &= \frac{4.81 \times 10^8 (m_1 + m_2)}{Z_1^2 Z_2^2} \frac{[(m_1 + m_2)c^2]_{\text{kev}} [kT]_{\text{kev}}^{\frac{3}{2}}}{m_1 m_2 [n_1 + n_2]_{\text{cm}^{-3}} J(g)}, \\ g &= \frac{3.27 \times 10^{14} [kT]_{\text{kev}}^{\frac{3}{2}}}{Z_1 Z_2 [n_e]_{\text{cm}^{-3}}^{\frac{1}{2}}}. \end{aligned}$$

Let us compare these results with those of Spitzer:²

$$\begin{aligned} \frac{t_{eq}}{t_{eq}(\text{Spitzer})} &= 2 \ln \Lambda / J(g), \\ g/\Omega &= \frac{2}{3}. \end{aligned}$$

For large values of g and Λ , we have

$$J(g) \approx 2 \ln g - 1.16 = 2 \ln \frac{2}{3} - 1.16 + 2 \ln \Lambda.$$

Our value for the equipartition time approaches that of Spitzer if g is large, but it differs considerably for average or small values of g . We feel, therefore, that our formula is more general than that of Spitzer, which it approaches for large values of Λ .

APPLICATIONS

We shall apply the formulae of a previous section to the problems of the slowing-down of tritium nuclei by

a plasma composed of ${}^6\text{Li}$ nuclei, deuterons and electrons at various temperatures, the densities being

$$n_{\text{Li}} = 0.617 \times 10^{23} / \text{cm}^3$$

$$N_D = 0.617$$

$$n_e = 2.468.$$

Slowing-down by the Electrons

It is especially for this case that our formulae are applicable since the collisions are very frequent and each has only a small effect; the slowing-down is thus quasi-continuous and our expressions can give a good approximation. When the energy E_1 of the triton approaches the thermal energy, the slowing-down factor loses much of its importance and approaches zero for $E_1 = (\frac{3}{2}) kT$.

Slowing-down by the Ions

It is precisely in this zone that the slowing-down by the ions becomes predominant. It is certain that in this case our formulae no longer have the same validity, due to the marked discontinuity of the slowing down. Nevertheless, it seems reasonable that they can represent the average slowing-down.

Figures 1 and 2 show the results of calculations for $kT = 5$ kev and $kT = 2$ kev.

Probability of the Reaction $D(t, n)^4\text{He}$

The probability that such a reaction takes place during the slowing-down of the triton, calculated for the energy range 1 Mev to $4kT$, is expressed by:

$$P = 1 - \exp \left[-n_D \int \frac{\sigma(E_1)(-dE_1)}{(-dE_1/dz)} \right]$$

where $\sigma(E_1)$ is the reaction cross section. One finds:

$kT(\text{kev})$	P
50	2.86×10^{-3}
20	1.77×10^{-2}
10	2.50×10^{-2}
5	3.45×10^{-2}
1	5.38×10^{-2}

Without being completely negligible, this probability stays small. We conclude that the majority of the tritons will be thermalized before producing a thermonuclear reaction.

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Mechanization of Non-linear Calculations in Fusion Reactor Theory

By P. Braffort and M. Chaigne *

From Kurchatov's report¹ and recent British publications,² the physical foundations of fusion reactor operation have gradually been emerging. Before attacking the technological problems presenting themselves to the engineer, it is necessary to specify various quantities roughly, and to indulge in some (inevitably) very approximate calculations for that purpose. The object of our paper is to emphasize the role of automatic computing, in particular analogue computing, in this field. Thus the study of plasmas and the phenomena they involve calls for solution of a large number of differential, integro-differential and partial differential equations which are all but impossible to attack manually. The analogue computation that has afforded a quick and accurate survey of the kinetics and dynamics of a wide variety of fission reactors³ lends itself also to the solution of plasma equations, despite their essentially non-linear character.

For example, the purely dynamic problem of the evolution of a high current discharge in a plasma involves a "resistivity" that depends on the temperature raised to the $3/2$ power. It is therefore necessary to use a generator of functions of the type $y = x^{3/2}$.

Upon entering the space-time field, the conventional use of non-linear elements in analogue computation no longer suffices. But the underlying principle of the analogue method is still applicable (that of generalized negative feedback). At the cost of using certain approximations and complicating the automatic computation somewhat in structure, it is possible in this way to attack the space-time field of partial differential equations or integral equations. We shall offer some suggestions for the application of these methods to problems concerning the physics of fusion reactors.

MATHEMATICAL ASPECTS OF THERMONUCLEAR PROBLEMS

Analysis of thermonuclear phenomena takes place simultaneously on three levels, a fact which we shall

have occasion to use: the molecular and atomic (ionic) level, the macroscopic level, and the nuclear level at which the ultimate energy-producing phenomenon occurs. The various interactions can be described as in Fig. 1.

In this section, we shall exhibit some equations or systems of equations characteristic of various phenomena encountered at all three of these levels, of course without any claim to completeness, and we shall try to bring out certain features they have in common from the point of view of mathematical analysis. In later sections, some of these problems will be taken up and methods of analogue solution suggested.

Nuclear Level

The effect of temperature on the movement of ions appears at the nuclear level and, in turn, affects the probabilities of internuclear reactions, which is to say the effective reaction cross sections and power densities. For DD reactions, the reaction rate R is:⁴

$$R = \frac{1}{2} n^2 \langle \sigma v \rangle_{av} \text{ reactions per cm}^3 \text{ per sec}$$

where n is the density; σ , the effective reaction cross section; and v , the ions' velocity.

With a Maxwellian velocity distribution at temperature T (kev), we have

$$\langle \sigma v \rangle_{av} = 260 \times 10^{-16} T^{-\frac{2}{3}} \exp(-18.76 T^{-\frac{1}{3}});$$

$$(T < 50 \text{ kev}).$$

This relation, we need hardly mention, is typically non-linear.

Atomic Level

On the atomic level, the movement of ions in the plasma is determined by Lorentz forces of the type

$$\mathbf{y} = \sigma[\mathbf{E} + (1/c) \mathbf{v} \times \mathbf{B}]$$

to which correction terms must be added, in particular a term in $\mathbf{y} \times \mathbf{B}$ due to the Hall effect, whence⁵

$$\mathbf{y} = \sigma \left[\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{1}{eNc} \mathbf{y} \times \mathbf{B} + \frac{1}{eN} \nabla p_e \right]$$

where σ = conductivity, $\mathbf{E}$ = electric field, $\mathbf{B}$ = mag-

Original language: French.

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This is a system of three second-order differential equations, for which an analogue can easily be given (Fig. 2). The symbols in the analogue diagrams of this paper are given in Fig. 3.

Since the field, in this instance, is not given in advance and is not independent of time, it is necessary to solve other space-time problems before getting to this stage; we shall therefore defer discussion to a later section. Instead, as an example of a purely dynamic problem, we shall study the evolution of discharge current as a function of time in a fully ionized gas enclosed in a toroidal chamber. The problem reduces to the coupling of two electric circuits, one of which is the gas in the toroidal chamber.¹³ The system of equations to be solved is therefore:

$$\begin{aligned}v_1 &= R_1 i_1 + L_1 di_1/dt + M di_2/dt, \\v_2 &= R_2 i_2 + L_2 di_2/dt + M di_1/dt, \\dv_1/dt &= -i_1/C_1 \quad dv_2/dt = -i_2/C_2\end{aligned}$$

where the circuit diagram is shown in Fig. 4. From this, by introducing the damping coefficients and the characteristic frequencies of these circuits,

$$\begin{aligned}a_1 &= R_1/2L_1 & a_2 &= R_2/2L_2 \\ \omega_1 &= 1/(L_1 C_1)^{1/2} & \omega_2 &= 1/(L_2 C_2)^{1/2},\end{aligned}$$

we obtain the following system of equations

$$\begin{aligned}d^2 i_1/dt^2 + 2a_1 di_1/dt + \omega_1^2 i_1 &= -(M/L_1) d^2 i_2/dt^2, \\d^2 i_2/dt^2 + 2a_2 di_2/dt + \omega_2^2 i_2 &= -(M/L_2) d^2 i_1/dt^2.\end{aligned}$$

The difficulty of such an investigation lies in the fact that the resistance of the gas is a function of the temperature and varies with time.

Supposing that the inductance L_2 is constant, using the formula given by Spitzer¹⁴ to compute R_2 , and neglecting radiation losses, the term a_2 is finally given by

$$a_2 = 2.3 \times 10^4 \left(\int i_2^2 dt \right)^{-3/4}.$$

Hence we have to solve the integro-differential system

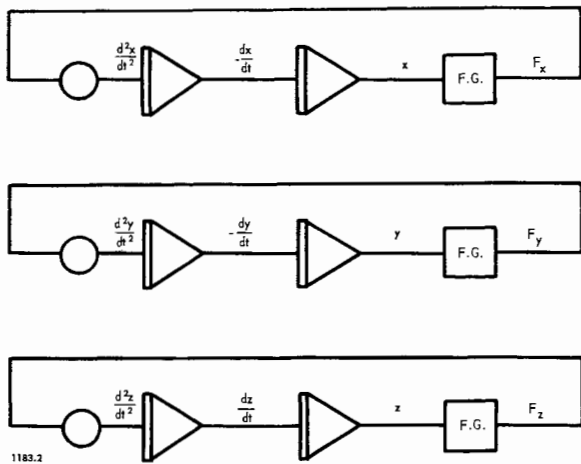


Figure 2. Block diagram of analogue for investigating motion of a charged particle in a magnetic field

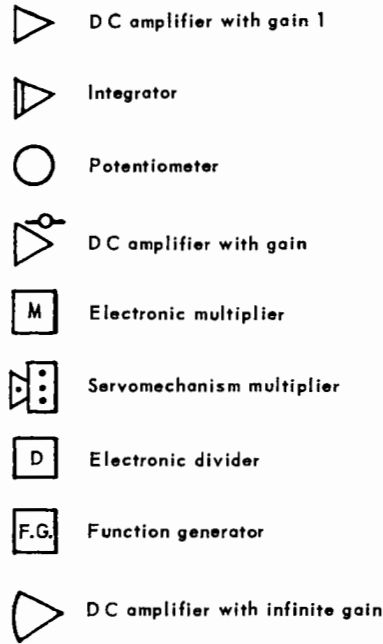


Figure 3. Analogue symbols

$$\begin{aligned}\frac{d^2 i_1}{dt^2} &= -\frac{2a_1 di_1}{dt - \omega_1^2 i_1} - k \left(\frac{L_2}{L_1} \right)^{1/2} \frac{d^2 i_2}{dt^2}, \\ \frac{d^2 i_2}{dt^2} &= -\frac{4.6 \times 10^4}{(i_2^2 dt)^{3/4}} \frac{di_2}{dt} - k \left(\frac{L_1}{L_2} \right)^{1/2} \frac{d^2 i_1}{dt^2}.\end{aligned}$$

The coupling coefficient k is defined by $k^2 = M^2/L_1 L_2$.

We immediately see the non-linear character of such a system. Also, the duration of the physical phenomenon is very short, the investigation extending over only 100 to 200 μsec . On the other hand, the intensity of the secondary current is very high and may reach a maximum of 10^5 amp. For analogue solution of this system, we shall therefore expand the time scale so as to study the phenomenon in about 2 minutes.

The $(i_2^2 dt)^{3/4}$ term will be given by a servo-mechanism function generator in which an $x^{3/4}$ curve is introduced and which has a rotational speed governed by time.

We suggest two analogues (Figs. 5a and 5b), the choice depending on technical considerations (whether a divider is available or not).

This method of calculation will quite readily yield by direct recording the movement of discharge current for very different values of the coefficients, these being allowed to vary by a factor of 10. As to the accuracy of the results obtained, this depends on the precision of the recorder used.

A Purely Spatial Problem

In this section, we shall investigate the ionic density distribution $n = f(r)$ in a fully ionized cylindrical plasma in an axial electric field E_z and an axial magnetic field B_z , as a function of the distance r from the axis of the cylinder, as shown in Fig. 6.

In practice, the study of such a distribution requires knowledge of the behavior of the longitudinal magnetic

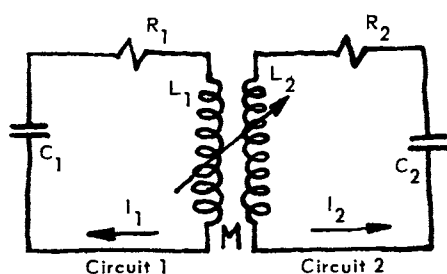


Figure 4. Equivalent circuit for a gas discharge

field B_z and the circular magnetic field B_θ as a function of the distance r from the axis of the cylinder. These two fields are determined by the following two differential equations

$$\frac{dB_{zs}}{dr_s} = 4\pi \frac{B_{\theta s} B_{zs}}{B_{\theta s}^2 + B_{zs}^2},$$

$$\frac{dB_{\theta s}}{dr_s} + \frac{B_{\theta s}}{r_s} = 4\pi \frac{2B_{zs}^2 + B_{\theta s}^2}{B_{\theta s}^2 + B_{zs}^2}.$$

In these two equations, the unknowns are the normalized dimensionless variables

$$B_{\theta s} = B_\theta/B_{z0}, \quad B_{zs} = B_z/B_{z0}, \quad \text{and} \quad r_s = r E_z \sigma_\perp / B_{z0}$$

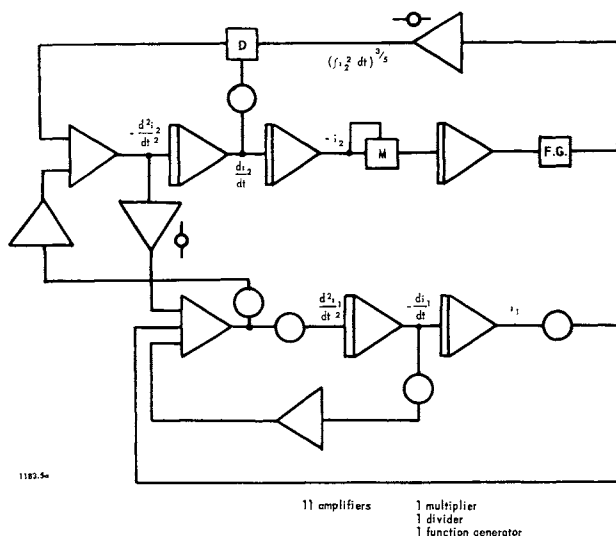
where B_{z0} is the initial value of the longitudinal magnetic field B_z , or in other words, the value of B_z for $r = 0$; B_θ and B_z are functions of r ; and $\sigma_\perp$ is the perpendicular conductivity of the plasma.¹⁵ Computation with these dimensionless variables will yield results that can be used regardless of the initial conditions.

To this system we may add the equation

$$dn_s/dr_s = -B_{\theta s} dr_s$$

giving the ionic density distribution in terms of the dimensionless variable

$$n_s = 2nkT/B_{z0}^2,$$



in which n = ionic density, k = Boltzmann's constant, and T = absolute temperature of plasma.

The problem so stated is to be solved in a stationary case. Note that, in this instance, the plasma is symmetrical with respect to two perpendicular planes passing through the axis of the cylinder. We can therefore convert the spatial problem to a plane problem and confine ourselves to calculation along a radius.

To handle this problem simply on our machines, we shall take time as the analogue of distance from the axis of the cylinder. Thus all quantities that are functions of distance from the axis will become functions of time. In the present case, the movement of these various quantities over unit "normalized distance" will correspond to the behavior of the corresponding quantities during 10 seconds. Fair precision is thus obtainable by this method.

For this problem we also suggest an analogue diagram (Fig. 7), according as dividers are or are not available. Note, in passing, the simplification of this calculation by use of an "electronic operator", serving to perform an operation of the form AB/C . Elimination of a considerable number of amplifiers (several with high gain) and avoidance of division by means of a reciprocal operator substantially improve the accuracy of the calculation, which is limited by the precision of the recorder.

Analogue computation will also permit the handling of the inverse problem; that is, to find the value of a parameter (E_z for example) required for a given value of the ionic density n at a distance r from the axis.¹⁶ To solve this problem, we can proceed by successive approximation, systematically manipulating all the potentiometers in which the parameter in question is concerned. The procedure will then be to perform

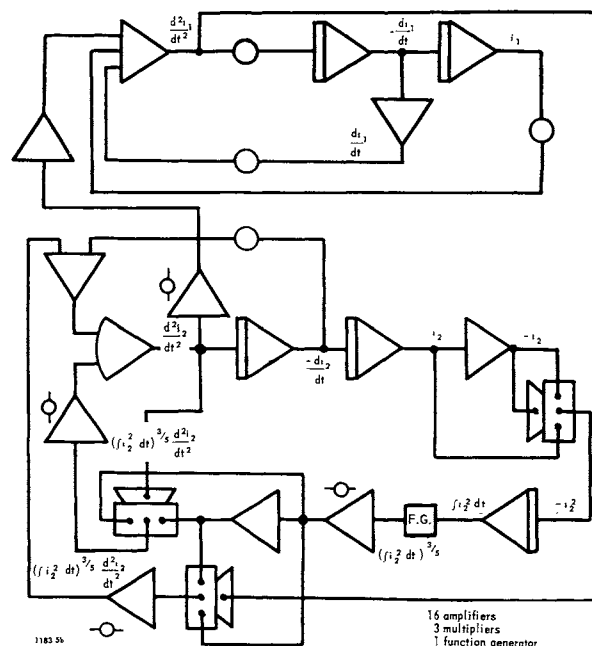


Figure 5. Analogue diagrams for the study of a discharge current: (a) when a divider is available, (b) without a divider

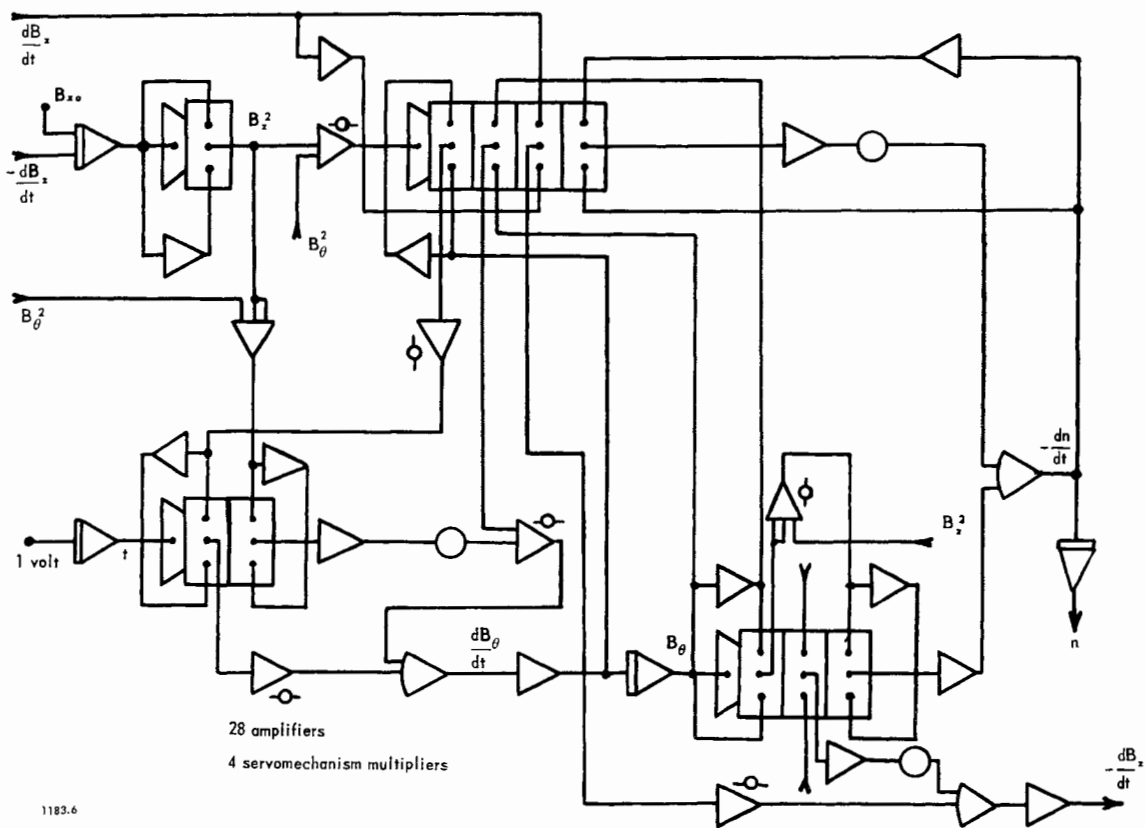


Figure 6. Analogue diagram for the determination of the ionic density distribution in an axial electric and magnetic field vs. the distance from the axis

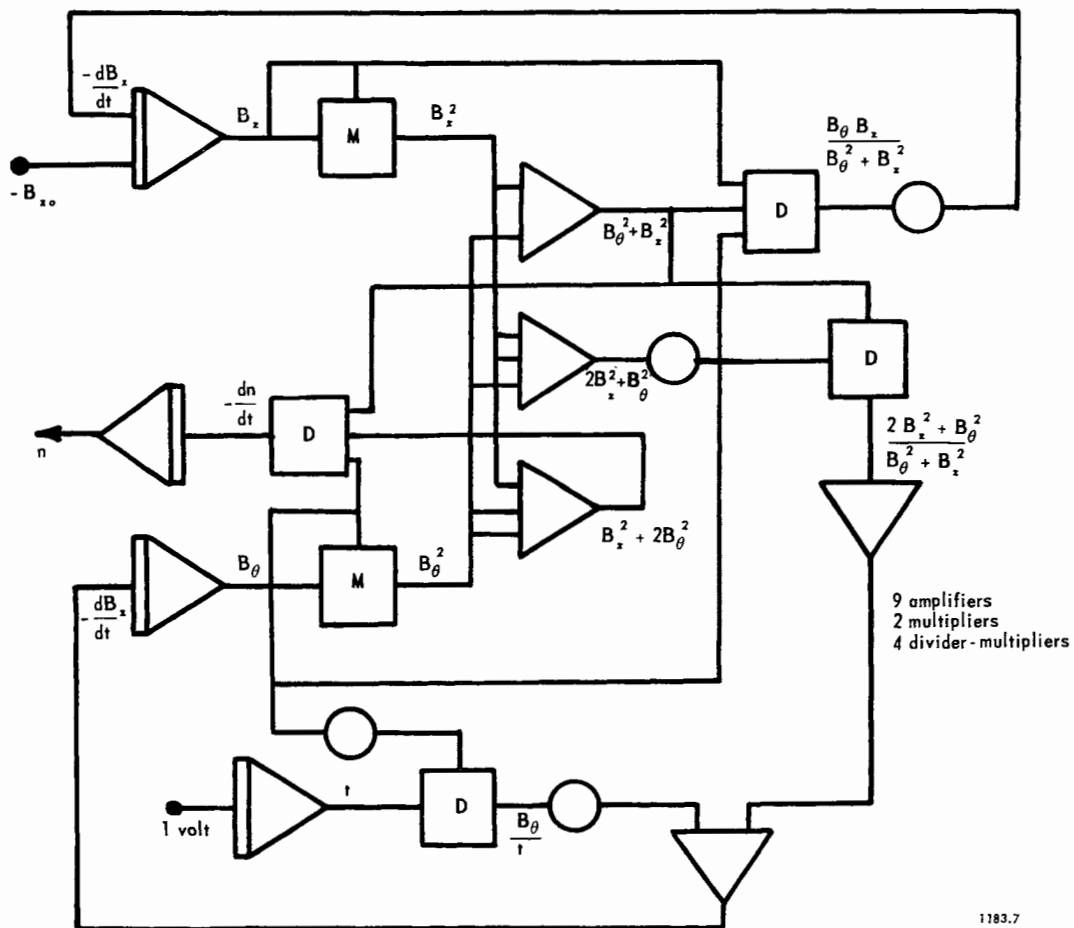


Figure 7. Analogue diagram for investigation of ionic density distribution

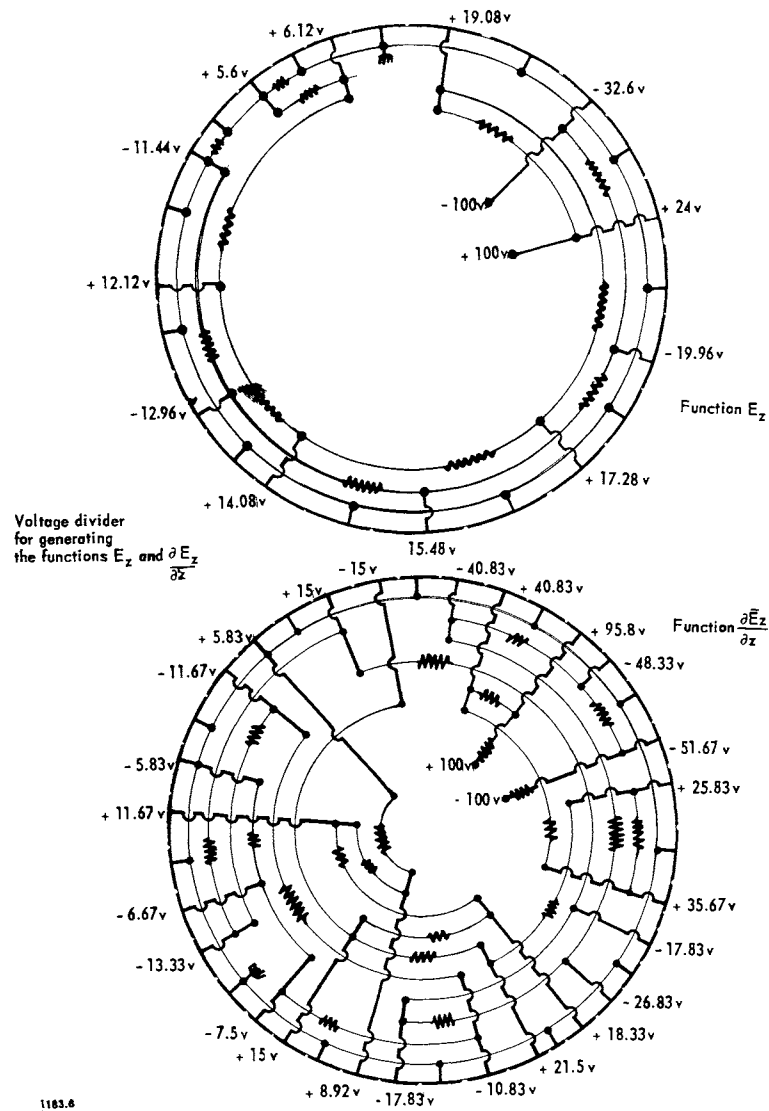


Figure 8. Distribution of potentials on the generator to obtain the function E_z and $\partial E_z / \partial z$

the computation, in each case, for a time t corresponding to the distance r_0 , so as to obtain a smaller and smaller error relative to the fixed value of n .

A Simple Space-time Problem

We here attack the problem of motion of a particle in an electromagnetic field. Let us take the case of a cylindrical geometry. Using the classical notation, the equations giving us the position z of the particle along the axis of the cylinder, the deflection r from that axis and the velocity dz/dt (and hence the energy) are

$$m \frac{d^2 z}{dt^2} = e E_z(z) \cos(\omega t + \phi)$$

$$E r(z) = - \frac{r \partial E_z(z)}{2 \partial z}$$

$$m \frac{d^2 r}{dt^2} = e E_r(z) \cos(\omega t + \phi).$$

As we said before, in such a problem the difficult part is to provide a function generator that will obtain the values of E_z and dE_z/dz at any moment. Since the analytical expressions do not lend themselves readily to computation, we have adopted approximate representations for $E_z(z)$ and $\frac{\partial E_z(z)}{\partial z}$.¹⁷ To generate

these functions, an apparatus has been devised, as shown diagrammatically and pictorially in Figs. 8 and 9. The two "Plexiglas" discs measure about a meter in circumference.

Each zone is divided into several sectors with conductive coating, marked with the potentials supplied by a network of resistors. The over-all analogue diagram is shown in Fig. 10.

Obviously such a device does not permit very great precision and, moreover, applies only to discontinuous geometries. Accordingly, a field simulator¹⁸ of more general usefulness has been devised. This field simulator automatically provides the solutions

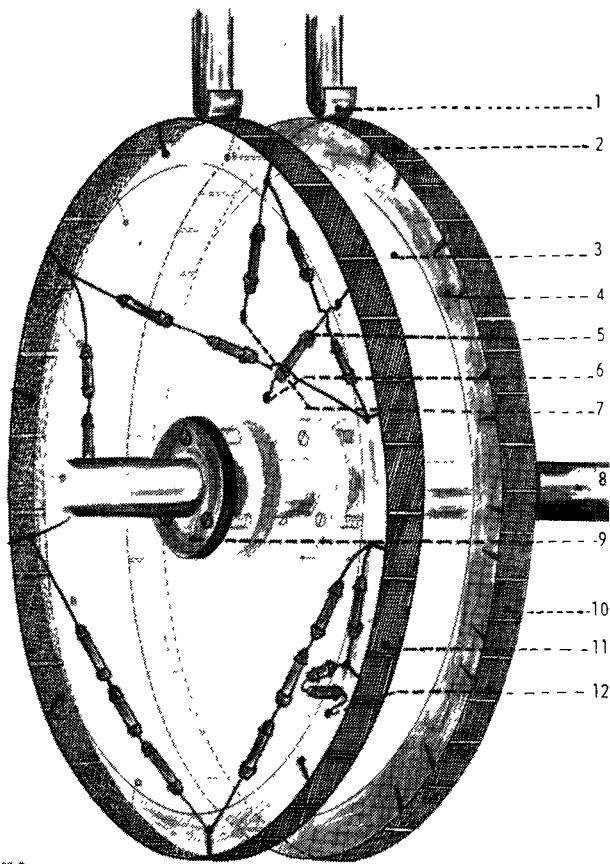


Figure 9. Pictorial sketch of disc function generator (servomotor not shown). 1 = Pick-up contacts; 2 = Silvered sectors with 1 mm separation; 3 = "Plexiglas"; 4 = Dull finish "Plexiglas"; 5 = 1% resistors; 6 = + 80 volts; 7 = - 80 volts; 8 = Shaft connected to motor; 9 = Metal disc attaching wheel to shaft; 10 = Second disc with 36 sectors; 11 = First disc with 28 sectors; 12 = Ground

of the potential equation in a two-dimensional space. It can also give the solutions of partial differential equations of various kinds and do this for any boundary conditions, even those dependent on time.

The principle of this field simulator is the following: Let (x, y) be a plane space within which arbitrary areas $S_1, S_2, \dots, S_n$ are subjected to potentials $V_1, V_2, \dots, V_n$. These potentials may vary with time. For any moving point M with coordinates $x(t)$ and $y(t)$, the values $\partial V/\partial x$ and $\partial V/\partial y$ are required.

For this purpose, let us make a small-scale model of the space in question (see Fig. 11).

We draw the outlines of the areas $S_1, S_2, \dots, S_n$, cover them with conductive coating and bring them to potentials $k_1 V_1, k_2 V_2, \dots, k_n V_n$. Then, by the familiar network method, we construct a grid consisting of a large number of resistors and provided at each intersection with a sliding contact (Fig. 12). The grid is such that: (1) each intersection represents a point M of the space; (2) the potential of each intersection is proportional to the potential $V(x, y)$ of the corresponding point M ; and (3) the contacts on areas $S_1, S_2, \dots, S_n$ supply and impress the poten-

tials $k_1 V_1, k_2 V_2, \dots, k_n V_n$ on suitable intersections, thus setting up boundary conditions.

Now the potential difference between two successive intersections in the x direction is proportional to $\partial V/\partial x$ and the potential difference between two successive intersections in the y direction is proportional to $\partial V/\partial y$.

Hence, if we require a potential reader to follow the $x(t)$ and $y(t)$ law, it will at any time deliver two signals, one proportional to $\partial V/\partial x$ and the other to $\partial V/\partial y$, where $V = V(x(t), y(t))$.

SOLUTIONS OF SPACE-TIME PROBLEMS IN THE GENERAL CASE

Though the investigation of the individual movements of charged particles in the plasma is very interesting, particularly for the study of various limiting cases, it will not yield an over-all conception of the phenomenon because the logical "loop" relating the phenomena diagrammed in Fig. 1 is not closed. Hence it is necessary to attack the macroscopic problem of the plasma.

The underlying phenomena, as we have seen, are the following: individual movements of ions in the field between two impacts; impacts which cause ion diffusion; and generation of fields by the motion of the ions. Expressed in mathematical form, this leads to two systems of equations: (a) diffusion equations, derived from an equation of the Boltzmann type, and (b) field equations derived from Maxwell's equations.¹⁹

We are therefore confronted with a system of partial differential equations so complex that the method described above is not directly applicable. Numerical calculation could of course be used, but here again the complexity of the system to be solved would render the programming very extensive and perhaps beyond the memory capacity of ordinary machines.

Now there is an approximate method of calculation currently used in numerical computation to avoid excessive memory requirements—the Monte Carlo method. The principle is familiar; to compute an integral of the form $\int f(x)g(x)dx$, we regard $f(x)$ as a probability density, find a random variable x having

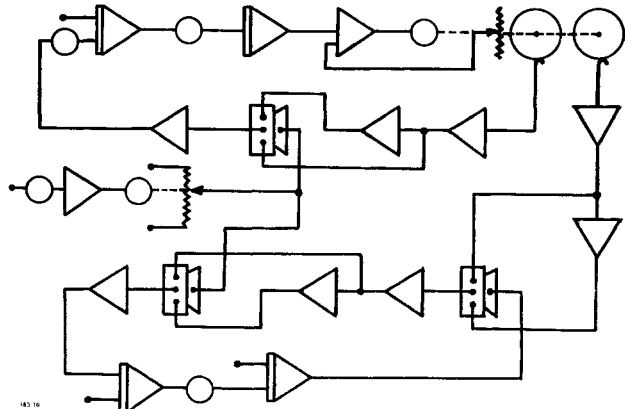


Figure 10. Analogue diagram for an investigation of the motion of an ion in an electromagnetic field

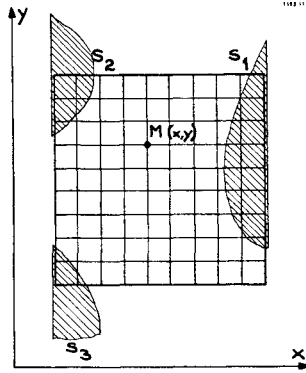


Figure 11. Scale model as discussed in text

a density of this type, and take means over randomly chosen populations of values of x , or rather $g(x)$.

This method is especially convenient for solving diffusion equations.²⁰ An interesting fact may here be noted: the transition from an equation of the type

$$\frac{\partial P}{\partial t} = K \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right)$$

to an equation of the type

$$P(m, n, s+1) - P(m, n, s) = \frac{1}{4} \{ [P(m+1, n, s) - 2P(m, n, s) + P(m-1, n, s)] + [P(m, n+1, s) - 2P(m, n, s) + P(m, n-1, s)] \},$$

defined over a two-dimensional grid, is equivalent to the transition from the macroscopic transport equation to the microscopic equation describing the "random walk" of individual molecules. (This is only natural; for the inverse transition was precisely that used by Chapman-Kolmogoroff in setting up their diffusion equations.)

This tells us what route to follow, a route which we shall here describe only in very broad outline. Instead of taking as random phenomenon a discontinuous Brownian movement with transition probabilities given by a table of random numbers, we shall make direct use of simulation of the motion and collision of individual particles, of which we shall take the means. Each individual calculation will be performed as follows. The motion of an ion will be integrated in the manner outlined in the previous section, but: (a) the initial values for position and velocity will be given by a noise generator; and (b) the duration of the computation will likewise be determined by chance with the aid of a noise generator (with mean value corresponding to the mean-free path).

The results of the several calculations will be placed in a memory circuit and averaged periodically; they will be used to compute the macroscopic velocity and current according to the formulas:²¹

$$\mathbf{v} = (n_i m_i \mathbf{v}_i + n_e m_e \mathbf{v}_e) / \rho$$

$$\mathbf{j} = (n_i Z \mathbf{v}_i - n_e \mathbf{v}_e) e / c.$$

The current $\mathbf{j}$ will, in turn, be put into the device in order to solve Maxwell's equations.

Certain technical problems arise at this point. Provision must be made for memory storage of certain results and for their delivery at definite times. We have constructed an apparatus, meeting these requirements, which will be described elsewhere.²² Also, it will be necessary to make individual calculations on a time scale differing from the scale used for the simulator taken as a whole. We have replaced the space averages by time averages, in effect, by making use of the ergodicity of the phenomena considered, and these averages taken successively must not too greatly affect the behavior of the whole. Here again, multiple-time simulation techniques have been developed.²³

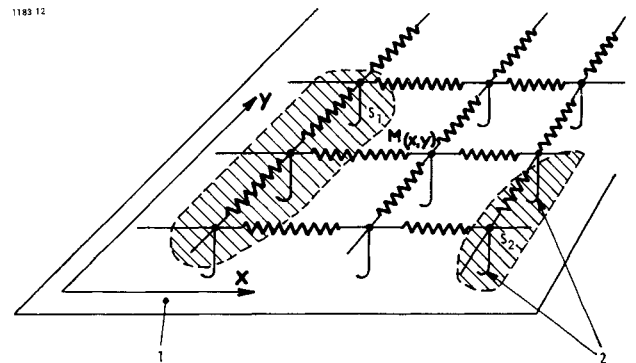
Finally, the phenomena connected with the electromagnetic field take place in a three-dimensional geometry. This leads to the use of not one but two field simulators of the type previously described, suitably coupled together. The general diagram of such a simulator is given in Fig. 13; a more detailed description will be published elsewhere.

Our proposed device has the advantage of being based on simulation of the elementary physical phenomena themselves and thereby avoids simplifications whose macroscopic effect is difficult to ascertain in such a highly non-linear field. Its disadvantage is that it cannot function in real time and hence does not correspond to real thermonuclear engines. It appears to us that this disadvantage should not prevent our machine from finding a wide field of application, even outside the domain of plasma physics.

ROLE OF NON-LINEARITIES IN THE OPERATION OF A THERMONUCLEAR REACTOR

Though the solution of engineering problems involved in the harnessing of thermonuclear energy is far from complete, it is not impossible to investigate immediately some of the dynamic problems presenting themselves in this field.

Schultz,²⁴ taking his departure especially from Post's work,⁴ has proposed a system of dynamic equations expressing the balance of energy and the numerical balance of particles within a reactor, respectively. These equations are as follows:

Figure 12. Grid of resistors for field simulator: (1) Paper supporting the conducting surfaces $S_1 \dots S_n$; (2) Pick-up contacts

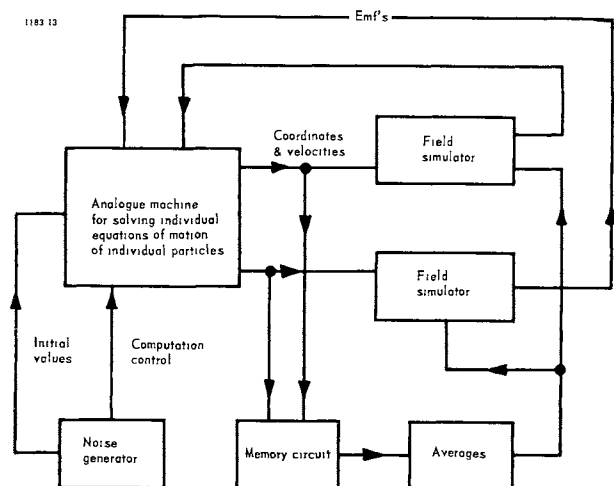


Figure 13. Over-all diagram of a plasma simulator

$$K_1 \frac{d}{dt} nT = K_2 n^2 \alpha(T) - K_B n^2 T^{\frac{1}{2}} - K_D nT^2$$

$$\frac{dn}{dt} = -\frac{2}{3} \frac{K_d}{k} nT - \frac{1}{2} n^2 \alpha(T) + S,$$

where $K_1 = \frac{3}{2} K$ and $K_2 = \frac{1}{2} W$; W is the energy liberated by a reaction; $\alpha(T)$ is a function of temperature, depending on S , the injection term; and K_B, K_D are coefficients. The meaning of the various terms and the detailed analysis of these two equations will be found in the work of Schultz.

Schultz then puts:

$$n = n_s + \Delta n(t)$$

$$T = T_0 + \Delta T(t)$$

$$S = S_0 + \Delta S(t).$$

The equations then assume the form

$$E \frac{d\Delta T}{dt} + F \frac{d\Delta n}{dt} = A\Delta T + B\Delta n$$

$$\frac{d\Delta n}{dt} = C\Delta T + D\Delta n + \Delta S.$$

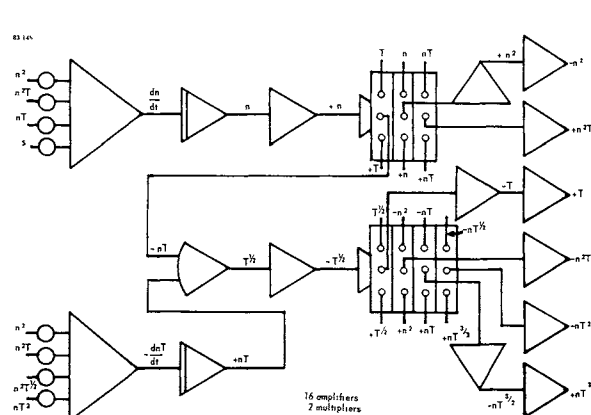
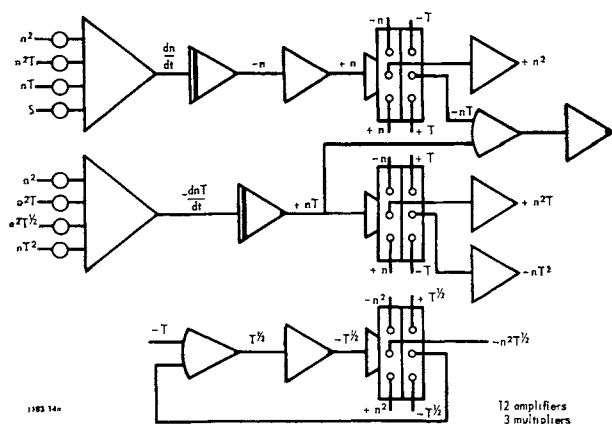


Figure 14. Analogue diagram for investigating the dynamics of a thermonuclear reactor: (a) with servomechanism multipliers, (b) with electronic multipliers

The source S here represents the supply of fuel to the reactor. It is then possible to compute the Laplace transforms of this linear system and all the transformation factors, $S \rightarrow T$ and $S \rightarrow n$, and then find the conditions of stability by using the familiar criteria of servo theory. As these equations are highly non-linear, it may be wondered whether Schultz's linearization may not pass over some regions of instability. We have therefore undertaken some attempts to simulate the non-linear equations. Construction of the analogue circuit presents some difficult problems when it is desired to minimize the number of amplifiers and multipliers. Figures 14a and 14b show two equally possible set-ups, to be adopted according to technical considerations (whether servomechanism multipliers or electronic multipliers are available).

For some values of the coefficients, instabilities are actually observed.

Of course, the formulation proposed by Schultz involves drastic simplifications whose implications are difficult to assess. But, in any case, the foregoing inquiry has the merit of emphasizing the danger of linearization in this field. It may be inferred that simulation is here absolutely indispensable.

CONCLUSION

We have presented a group of methods affording a considerable extension of the field of application of analogue computing to theoretical studies concerning fusion reactors. In all the cases considered, of course, methods of numerical computation are possible, but seem to be avoidable because:

(1) The precision required is not high, considering that the physical quantities involved in the equations are not accurately known; and (2) at the present stage of inquiry, it is essential to vary all parameters within very wide limits, which is especially convenient by the analogue method.

At any event, the methods described permit one to avoid linear approximations, thereby pointing directly to certain points of difficulty in the systems studied and preparing the way for more thorough investigation in the future.

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Engineering Aspects of a Thermonuclear Power Plant

By L. Geller, R. W. Kupp and S. Koslov *

The optimistic picture of unlimited fusion power in the near future for all nations of the world is tacitly based upon the assumption that a D-D fusion reactor is possible. A T-D reactor, though having numerous advantages, e.g., a lower ignition temperature and a relatively small fraction of bremsstrahlung radiation, among others, does not seem to be economically within the reach of any but the largest nations, since the inventory of tritium required is extremely expensive and difficult to come by. Of course, once a T-D machine is in operation, or even a D-D machine for that matter, it is presumably possible to breed sufficient tritium to maintain the inventory required. However, the very difficult neutron balance problems and material corrosion problems associated with this tritium breeding on the one hand, and the attractive possibility of using only naturally occurring materials on the other make it of considerable importance to pursue the development of a D-D reactor.

The plasma containment problem, common to all fusion reactors, is assumed in the present paper to have been solved. It is absolutely necessary that this phase of fusion reactor development be satisfactorily completed before any subsequent work can be done; and since the present paper is a gross economics-engineering evaluation of the D-D reactor it is quite reasonable to assume that such a reactor can be built. The results derived below are limited by our present knowledge concerning steady state containment of a plasma so that a large solenoidal magnet surrounding the reactor chamber is assumed to be an essential constituent of the power generator. Future related discoveries could play a large part in altering the conclusions derived here.

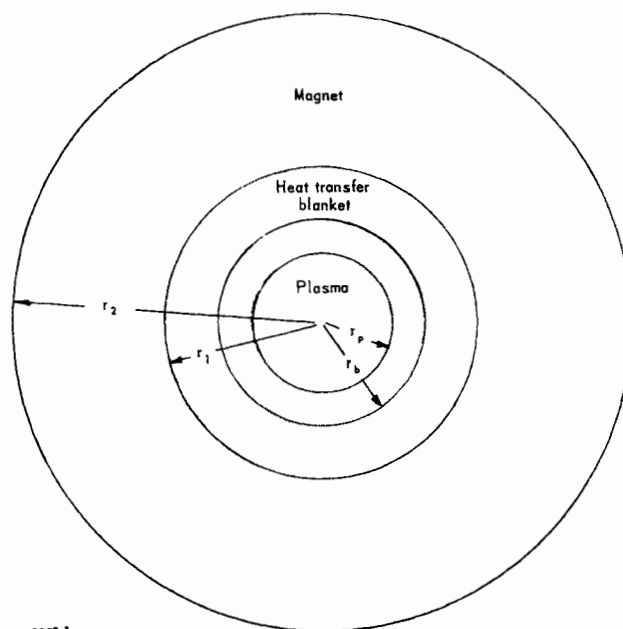
A cross section of the kind of machine considered is shown in Fig. 1. The reacting plasma in the center is surrounded by the "bremsstrahlung shield", followed by the main coolant annulus and then by the confining magnet coils. The bremsstrahlung shield is a thin-walled water-cooled jacket capable of removing the enormous power brought away by the bremsstrahlung radiation and the ions in their collisions with the wall. The main coolant annulus carries water which absorbs and carries away the energy brought out by the neutrons. In operation, the plasma ions

are continuously being purged and replenished. The axial geometry of the reactor is left unspecified so that both toroidal and rectilinear machines are included in the analysis. All calculations are carried out per unit length of solenoid structure and the total length is chosen so that the gross electric power produced is specified. For definiteness, the gross electrical power is taken here to be 1000 Mw plus the ohmic loss in the magnets. The saleable electric power will then be that portion of the 1000 Mw not used by pumps and other plant auxiliary requirements.

PLASMA

The plasma is assumed to be of uniform density across the tube and to occupy a volume with a radius less than the radius of the inner tube wall. This plasma radius, r_p , can then be made to correspond to a real distribution, although for present purposes the uniform assumption is satisfactory.

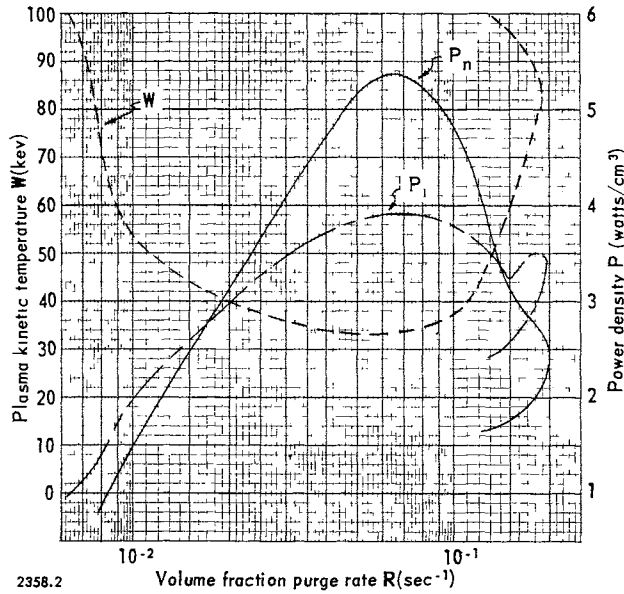
The temperature of the plasma was the first important variable to be determined, and fortunately,



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Figure 1. Typical cross section of a fusion reactor

* Vitro Engineering Company.



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Figure 2. Stable operating powers and temperatures vs. purge rate

within wide limits, this is independent of all structural parameters and so may be considered separately. For a given purge rate R and magnetic field B the steady-state plasma temperature can be determined by establishing a balance of the particle and magnetic pressures, of each ionic species, and of the ionic energy within the reactor chamber.

At pressure balance, the plasma density is related to the magnetic field by

$$nW = 2.49 \times 10^7 B^2,$$

where n = total number of particles (electrons and ions) cm^3 of plasma, W = temperature in kev and B = magnetic field strength in gauss.

In order to keep the number of variables to a minimum, a fixed value of B equal to 50,000 gauss has been used in the reactor design evaluated. This value was chosen because it is the largest field strength that appears to be attainable using existing technology.

For an input of pure D, the reactions considered are (The numbers in parentheses are the energies in Mev associated with each particle.):

- (a) $D + D \rightarrow T (1.0) + p (3.0)$.
- (b) $D + D \rightarrow \text{He}^3 (0.8) + n (2.45)$.
- (c) $D + T \rightarrow \text{He}^4 (3.5) + n (14.1)$.
- (d) $D + \text{He}^3 \rightarrow \text{He}^4 (3.6) + p (14.7)$.

At particle balance, the ionic densities, n_i , are given by

$$\frac{n_{\text{He}^3}}{n_D} = \frac{\frac{1}{2} \langle \sigma v \rangle_b (n_D/R)}{1 + \langle \sigma v \rangle_d (n_D/R)}$$

$$\frac{n_T}{n_D} = \frac{\frac{1}{2} \langle \sigma v \rangle_a (n_D/R)}{1 + \langle \sigma v \rangle_c (n_D/R)}$$

$$\frac{n_\alpha}{n_D} = \frac{n_D}{R} \left[\langle \sigma v \rangle_e \left(\frac{n_T}{n_D} \right) + \langle \sigma v \rangle_d \left(\frac{n_{\text{He}^3}}{n_D} \right) \right]$$

$$\frac{n_p}{n_D} = \frac{n_D}{R} \left[\frac{1}{2} \langle \sigma v \rangle_a + \langle \sigma v \rangle_d \left(\frac{n_{\text{He}^3}}{n_D} \right) \right].$$

The average reaction coefficients were obtained as functions of temperature by fitting a formula of the type

$$(\sigma v)_i = W^{-2/3} \exp (A_i + B_i W^{-1/3})$$

to the data of Thompson.¹ The actual numbers used in the calculations are given in Table 1.

For a power balance, it is necessary that the fusion power P_i carried by the ions be sufficient to account for the power radiated away via bremsstrahlung, P_b , and swept away via the purge system, P_f :

$$P_i = P_b + P_f.$$

The explicit forms for these powers in watts/cm^3 of plasma volume are

$$P_i = 1.6 \times 10^{-13} (2 \langle \sigma v \rangle_a n_D^2 + 0.4 \langle \sigma v \rangle_b n_D^2 + 3.5 \langle \sigma v \rangle_c n_D n_T + 18.3 \langle \sigma v \rangle_d n_D n_{\text{He}^3})$$

$$P_b = 4.84 \times 10^{-31} W^{\frac{1}{2}} n_e \times (n_D + n_p + n_T + 4 n_\alpha + 4 n_{\text{He}^3})$$

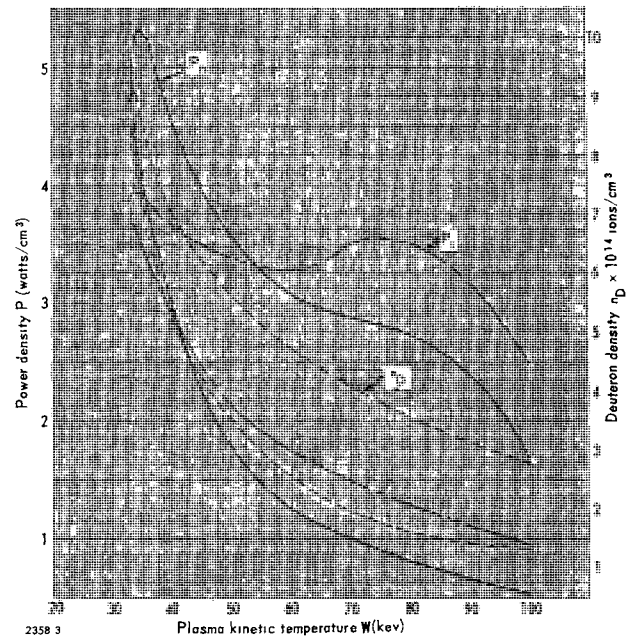
and

$$P_f = 2.4 \times 10^{-16} nWR$$

where $n_e = n_D + n_p + n_T + 2 n_\alpha + 2 n_{\text{He}^3}$ is the number of electrons per unit plasma volume. The neutrons which are produced in fusion reactions are assumed to leave the chamber with their nascent energy, all of which is deposited in the coolant annulus. The rate at which neutrons carry out energy is

$$P_n = 1.6 \times 10^{-13} (1.23 \langle \sigma v \rangle_b n_D^2 + 14.1 \langle \sigma v \rangle_c n_D n_T).$$

The equations listed above were solved for the quantities W , P_i , P_b , P_f , P_n and the ionic densities,



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Figure 3. Stable operating powers and deuteron density vs. plasma temperature

Table 1. Temperature Variation of Reaction Coefficients

Reaction	Temperature, kev						
	10	20	30	40	60	80	100
(a) $D(D, p)T$. . .	0.61	2.82	5.67	8.63	14.2	23.1	23.2
(b) $D(D, n)He^3$. . .	0.57	2.84	5.89	9.23	15.7	25.9	26.4
(c) $D(T, n)\alpha$. . .	105.0	260.0	387.0	486.0	630.0	880.0	796.0
(d) $D(He^3, p)\alpha$. . .	0.07	4.78	12.5	34.4	120.0	314.0	442.0

all as functions of the sweep rate, R . Some of these quantities are presented in Figs. 2 and 3, which show a number of surprising results. In the first place, it may be seen that for each admissible plasma temperature there are two stable sweep rates, and the curves of Fig. 2 indicate that the reverse is also true. Correspondingly, there are two sets of P_i , P_n and W_D for each temperature, as indicated in Fig. 3. It may be observed that, as the temperature increases above the minimum value of 33 kev, these quantities decrease except for the slight initial rise on the upper branch. The operating temperature giving the largest total steady power generation rate is the lowest rather than the highest possible temperature. Another significant point is that at this lowest temperature the ratio of P_i to P_n is a minimum and, for a given total power, represents the lowest bremsstrahlung radiation. For these reasons, the reactor temperature is chosen to be 33 kev.

It is important to note that this choice of temperature is permissible only if other considerations do not place restrictions on the sweep rate R . At a temperature of ~ 33 kev, there is a relatively wide range of possible sweep rates permitted because of the rather broad minimum in the W - R curve of Fig. 2, running from about 0.04/sec. to 0.08/sec. However, mechanical limitations on removal of ions (or, more likely, ionic diffusion rates to walls as determined by the specific containment scheme), tacitly assumed to be accounted for by adjustment of R , may require that R lie in some other range of values. The choice of $W = 33$ as the operating plasma temperature is perhaps only opportune, but within the limits of our present knowledge is justifiable. At this temperature, as may be seen from the curves of Fig. 3, the neutron and ionic powers are 5.3 and 3.8 watts/cm³ of plasma, respectively.

HEAT TRANSFER

With the temperature thus suitably chosen, the heat removal problems will be considered. The bremsstrahlung shield surrounding the plasma is designed to transfer to the power generating equipment the large amount of heat carried away by the electron bremsstrahlung radiation and also the energy carried by the ions swept out of the reaction volume. The inclusion of this power, P_i , of magnitude comparable with the actual bremsstrahlung radiation, is somewhat arbitrary but conditioned by the feeling that some fraction of the "sweeping out" of ions will actually be a loss to the walls. In any event, it is a conservative assumption since any energy physi-

cally swept out of the chamber will reduce the load on the bremsstrahlung shield, and insofar as the over-all heat balance is concerned could conceivably be partially recovered.

The bremsstrahlung shield will have to be fabricated out of thin-walled tubing joined together in a laminated cylindrical envelope. Water pumped through this tubing will carry away the heat deposited in the walls. No appreciable amount of neutron energy will be deposited in the bremsstrahlung shield which is considered to be essentially transparent to these particles.

Experience with fission reactors indicates that there is a limiting heat flux above which heat is difficult to remove by conductive heat transfer to a flowing coolant. This limit is not well defined, of course, but a value of 5×10^5 BTU/hr ft² seems as high as would be justifiable at the present time. Ingenious schemes for convoluting or otherwise artificing the bremsstrahlung shield may elevate this limit somewhat, but will not change the main conclusions derived below. These are based on the assumption that 5×10^5 BUT/hr ft² is the allowed rate for heat deposited in the bremsstrahlung shield, i. e., P_b and P_i . Letting r_b denote the radius of the latter, the condition that has to be met is

$$2\pi r_b \cdot \frac{5 \times 10^5 \text{ (BTU/hr ft}^2\text{)}}{3.413 \text{ (BTU/hr watts)}}$$

$$= 3.8 \text{ (watts/cm}^3 \text{ plasma)} \times (30.48)^3 \times \pi r_p^2$$

or

$$r_p^2 = 2.72 r_b.$$

As previously noted, the neutrons give up their energy to the coolant water in the annular blanket surrounding the bremsstrahlung shield. The thickness of this coolant blanket should be sufficient to remove essentially all the neutron kinetic energy and also to provide adequate radiation shielding for the surrounding magnets. (On the other hand, it should not be so thick that it introduces pumping or heat transfer problems.) Preliminary calculations indicate that a thickness of 3 ft is reasonable, and it is this value that is consistently employed. Rather wide variations in this value have only a slight effect on the efficiency of the system so that its exact value is not critical.

It has often been suggested that a considerable increase in the energy carried away by the main coolant water could be obtained by dissolving in the water a substance, which is a strong neutron absorber and which yields a high energy capture reaction product. There certainly are several substances

which have these properties, but unfortunately they generally bring with them severe corrosion problems. Alternately, a liquid metal coolant could be used. For this reason, no "energy multiplier" has been assumed dissolved in the coolant stream and credit for only the 2.23 Mev capture gamma of the (n, p) reaction is taken. By far the larger number of these will be captured in the coolant water directly, and so this is a justifiable assumption. Using the numbers derived from the solution of the balance equations, the gain in energy due to this source turns out to be 1.6 w/cm³ of plasma.

The total power, per foot of reactor, carried away by the main coolant water will thus be $6.18 \times 10^5 r_p^2$ or $1.68 \times 10^6 r_b$ w/ft. Combining this with the power carried away by the bremsstrahlung shield, the gross thermal power to the steam plant is seen to be

$$P_{th} = 2.62 r_b L \text{ Mw}$$

where L (ft) is the axial length of the reactor. This thermal power is converted to electrical power at some efficiency γ . For a large plant similar to the type utilized in fission reactor technology naturally adaptable to a fusion reactor, a value of $\gamma = 0.28$ is attainable, so that the total electrical power at the generators will be:

$$P_e = 0.731 r_b L \text{ Mw.}$$

MAGNET POWER

Of the power, a large part P_m will be needed to maintain the confining magnetic field. A smaller fraction will be needed for other plant devices. The remainder is available for sale. Considerations involving the problems associated with distributing power from large conventional central stations indicate that 1000 Mw of net electrical power is the most that can be economically handled, and this value has been used in determining the parameters of the reactor. For simplicity, it has been assumed that the energy requirements of the pumps and other plant machinery can be ignored so that all electrical power generated goes either to the magnets or is available for sale; in symbols,

$$P_e = P_m + P_{saleable} = P_m + 10^9.$$

The magnet power is determined once the outer magnet radius is fixed. Using the common form for the ohmic loss in a solenoid, it will be:

$$P_m = 1.05 \times 10^{-4} \left(\frac{\rho_t}{\rho_{20}} \right) B^2 \left(\frac{r_2 + r_1}{r_2 - r_1} \right) \frac{L}{1 - f},$$

where

ρ_t = resistivity of copper (assumed coil material) at temperature t ;

B = magnetic field in gauss = 50,000 gauss;

r_1 = inner radius of magnet = $r_b + 3$ ft;

r_2 = outer radius of magnet in feet;

L = axial length of reactor in feet;

f = "free-flow" fraction of the solenoid cross-sectional area required for magnetic cooling. It is assumed here to be equal to 0.4.

The operating temperature of the magnet coils for maximum efficiency has been studied to some extent. The coils could be run hot so that heat could be recovered, although at the cost of an increased ohmic loss. Alternatively, the coils might be refrigerated to extremely low temperatures. Using simple considerations and available cost data, the cheapest scheme turns out to be the operation of the magnets at a temperature somewhat above ambient, around 60°C. For the sake of illustration, the costs relative to 60°C of removing the heat from the magnet coils at various operation temperatures are shown, by the following:

Temp. °C	Heat removed Mw	Extra cost, M\$
+ 200	1640	48
60	1160	—
— 20	850	57
— 200	116	61

Inserting the known numbers in the formula for P_m gives:

$$P_m = 5 \times 10^5 \frac{r_2 + r_b + 3}{r_2 - r_b - 3} L \text{ watts,}$$

so that the relationship connecting P_e and P_m may be used to determine the axial length of the reactor as a function of r_2 and r_b :

$$L = 1370 \left/ \left(r_b - 0.683 \frac{r_2 + r_b + 3}{r_2 - r_b - 3} \right) \right.$$

REACTOR SIZE

In establishing a reasonable set of parameters for a typical machine, a number of limiting factors have to be generally considered. The maximum diameter of the magnet coil winding, r_2 , should not be too large, although it will undoubtedly be much larger than usually occurs in industrial devices. In order not to be too restrictive in this regard, values of r_p up to 28 ft were included in the calculations. Another limitation on the outer coil radius is imposed by geometrical and confinement requirements. For example, if the magnet coil is in the form of a simple toroid, then its axial length, L , must exceed its minor radius, r_2 , by at least a factor of 2π . In general it seems evident that some restriction relating the length and radius of the magnetic coil will be necessary. The restriction used in this paper is that $L > 6\pi r_2$, which implies that if the magnet is in the form of a simple toroid, its major diameter is at least three times as large as its minor diameter.

Using the formulas and restrictions developed above, it is now possible to make a practical parametric study of fusion reactors with the bremsstrahlung shield radius, r_b , and the outer magnet radius, r_2 , as the independent variables. As a "figure of merit" for the system, it seems natural to choose its relative total cost obtained as the sum of relative capital

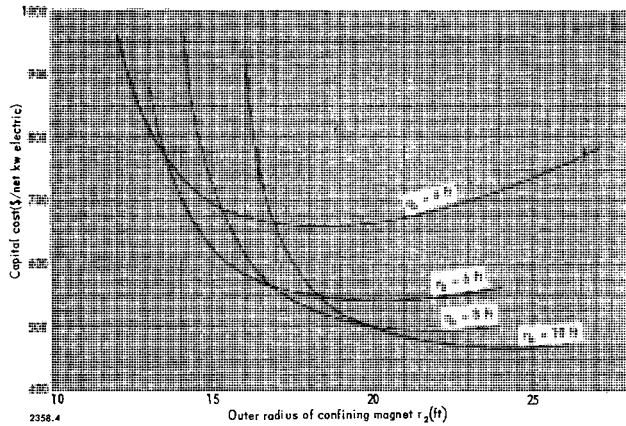


Figure 4. Capital cost vs. outer radius of confining magnet, for specified values of r_b .

costs of the conventional power generating machinery and of the copper in the magnet coils. Other costs have been ignored or implicitly factored into one of the above. To determine the relative total cost, it is assumed that the capital costs appropriate for a fission power plant, including the reactor, steam generating equipment and electrical system, may also be used in the present case of a fusion reactor to cover the same items. The capital cost assumed for a fission power plant is \$300/kw_e. The cost of fabricated copper is taken to be \$1/lb. It is clear that, since the system generates the electricity needed for its own magnets, less electrical generating capacity will be required if the current supplied to the magnets could be reduced by increasing their size and thus reducing their resistance. This would require more copper and could not be carried too far before the cost of the copper becomes greater than that of the generating capacity saved.

With the figures assumed above, the cost of the plant exclusive of the magnets will be $0.3P_e = 2.19 \times 10^5 r_b L$ dollars, and the cost of the magnets will be $555(1 - f)\pi(r_2^2 - r_1^2)L$ or $1050 L(r_2^2 - r_1^2)$ dollars, using a density of 555 lb/ft³ for copper. The sum of these quantities was computed as a function of r_2 for a number of values of r_b . The results, normalized to a net electrical power of 1000 Mw, are presented in Fig. 4. It may be seen that for each fixed value of r_b , there is a corresponding value of r_2 which minimizes the capital cost. These minima decrease as r_b increases and without other restrictions an extremely large value of r_b and also r_2 would seem to be indicated. However, it was noted previously that physical considerations restrict the length of the reactor in accordance with the equation $L > 6\pi r_2$. In Fig. 5 are drawn curves of L versus r_2 , for a fixed value of r_b and also the line $L = 6\pi r_2$.

Only those design values lying to the left of this straight line are admissible. The corresponding maximum admissible values of r_2 are indicated on the curves of Fig. 4 as small vertical bars. It may be seen that a large portion of the possible designs are

excluded and that the optimum design corresponds to the parameters

$$r_b = 6 \text{ ft.}$$

$$r_2 = 18 \text{ ft.}$$

$$\text{Capital cost} = \$550/\text{Net kw}_e.$$

$$L = 350 \text{ ft.}$$

The mechanical design of the plant will perhaps be as critical a factor in assessing the ultimate feasibility of the concept as the plasma-magnet heat balance. The capital cost of the D-D machine is certainly higher than either a conventional or fission power plant, and the design complexity and tolerance for error more difficult. Even assuming a virtually perfect initial construction for machine components, the thermal stress, corrosion in the heat transfer and magnet piping and radiation damage will certainly require replacement. Although we are not faced with a fission product inventory problem, the intense neutron flux impinging on the heat transfer system will result in activation considerably above tolerance levels for repair.

The over-all machine design must visualize a sectionally demountable system. The machine in either a toroidal or linear geometry is of essentially constant construction except perhaps for end sections in a linear machine. Assuming a toroidal geometry, a study of possible designs would seem to indicate that the entire machine can be divided into sections of the toroid. Each section can be fastened together insofar as the vacuum envelope is concerned by using the access space permitted between the magnet coil sections. It will be recalled that the heat transfer blanket piping is an integral part of the vacuum envelope. Therefore, each toroid section will consist of

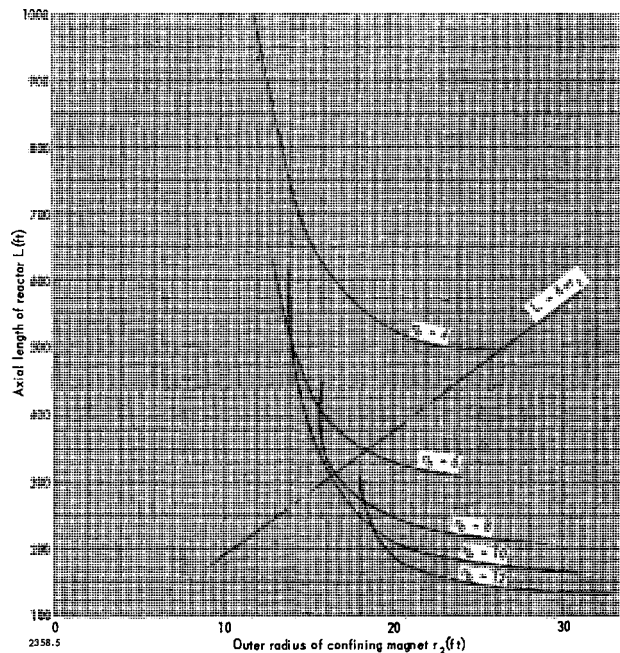


Figure 5. Axial length of reactor vs. outer radius of confining magnet, for specified values of r_b .

a separate coolant loop. The access pipes for this coolant loop will emerge in the allowed space between the magnet sections and be suitably paralleled externally.

Recalling that the use of relatively small diameter (because of the bremsstrahlung absorption characteristics) pressure piping in the interior appears to be necessary, the use of relatively short coolant flow lengths to a larger manifold would be desirable in any event. One of the really critical design points will be the elimination of hot spots inside the heat transfer blanket.

A possible choice for the magnet coils would be to wind them with square pipe with the interior walls stainless steel clad for corrosion resistance. The over-all coil-space factor allowing for the magnet coolant flow and the access spaces between coil sections can be reasonably estimated at about 0.6. Each section can now be disconnected after shutdown by a semi-remote system and moved radially outward from the machine for replacement or repair.

CONCLUSION

The simple treatment given above indicates that a D-D reactor may be a relatively efficient power producer, even under the severe limitations imposed. This conclusion is, of course, based upon optimistic capital cost assumptions which at the present time cannot be evaluated because of lack of engineering data. Many of the practical design problems have been glossed over or completely ignored. Among these may be mentioned the very difficult problem of part replacement; the design of the ion purge system; the ignition heating system; the details of the bremsstrahlung shield and vacuum chamber wall; and the radiation damage incurred. The possibility of developing a fusion reactor will certainly depend upon the successful solution of these problems, but it is hoped that these will be dealt with as the state of the art advances.

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Thermonuclear Reaction Rate in a Discharge

By H. Grad *

The electrical conductivity of a plasma is found by computing the deviation from Maxwellian of the electron distribution produced by an externally applied electric field. This problem has been solved for the Boltzmann equation using the Chapman-Enskog theory by Landshoff¹ and for the Fokker-Planck equation by Spitzer *et al.*² In principle, the same methods could be used to compute the ion distribution and from it the thermonuclear output. However, for this application, the high-velocity tail of the ion distribution is of crucial importance and this cannot be evaluated by the conventional methods. The reason lies in the method of linearization employed. The electric field $\mathbf{E}$ is assumed to be small, and the resulting deviation of the distribution f from the Maxwellian $f^{(0)}$ is also small. The "forcing" term $\mathbf{E} \cdot \partial f / \partial \mathbf{\xi}$ is replaced by

$$\mathbf{E} \cdot \partial f^{(0)} / \partial \mathbf{\xi} = \mathbf{E} \cdot \mathbf{\xi} f^{(0)} / RT$$

(see Eq. (12)). This is a small perturbation of a Maxwellian $f^{(0)}$ only if $\mathbf{E} \cdot \mathbf{\xi}$ is small, not when $\mathbf{E}$ alone is small. Furthermore, the collision term, which is a quadratic functional of f in either the Boltzmann or the Fokker-Planck equation, is usually linearized according to the scheme $f = f^{(0)} + f^{(1)}$, so that,

$$Q(f, f) \rightarrow Q(f^{(0)}, f^{(0)}) + Q(f^{(0)}, f^{(1)}) + Q(f^{(1)}, f^{(0)}),$$

assuming that $f^{(1)} \ll f^{(0)}$. We shall find the perturbation $f^{(1)}$ to be much larger than $f^{(0)}$ in the tail of the distribution even though $f^{(1)}$ is always small in an absolute sense.

We shall be able to avoid these difficulties by making use of certain special features of the Fokker-Planck equation and by introducing an asymptotic expansion in the electron-ion mass ratio.

THE FOKKER-PLANCK EQUATION

The two equations ($i = +$ and $-$) can be written in the form³

$$\frac{e^i}{m^i} E_r \frac{\partial f^i}{\partial \xi_r} = \frac{\partial^2}{\partial \xi_r \partial \xi_s} (\frac{1}{2} b_{rs}^i f^i) - \frac{\partial}{\partial \xi_r} (a_r^i f^i) \equiv J^i. \quad (1)$$

The summation convention is used for tensor subscripts; time and space variation have been dropped; $f^i(\xi)$ is the number density in velocity space; $e^i = \pm e$, the electronic charge. The dynamical friction coefficient a_r^i and dispersion coefficient b_{rs}^i are given by

$$a_r^i(\xi) = L^i \sum_j [(m^i + m^j)/m^j] A_r^j(\xi) \quad (2)$$

$$b_{rs}^i(\xi) = L^i \sum_j B_{rs}^j(\xi) \quad (3)$$

where

$$L^i = (e^4 / 4\pi \kappa^2 m^{i2}) \ln \lambda \quad (4)$$

$$\lambda = 12\pi (\kappa kT)^{\frac{3}{2}} / (2n)^{\frac{1}{2}} e^3 \quad (5)$$

$$A_r^j(\xi) = \int (V_r / V^3) f^j(\eta) d\eta \quad (6)$$

$$B_{rs}^j(\xi) = \int [(V^2 \delta_{rs} - V_r V_s) / V^3] f^j(\eta) d\eta (V_r = \eta_r - \xi_r). \quad (7)$$

Rationalized mks units are used; κ is the permittivity of free space; λ is the ratio of the Debye length to the mean distance of closest approach in a Coulomb encounter; n is the ion density. Other convenient forms for the collision term are

$$J^i = \frac{\partial}{\partial \xi_r} (\frac{1}{2} b_{rs}^i \frac{\partial f^i}{\partial \xi_s} - Q_r^i f^i) \quad (8)$$

$$J^i = \frac{1}{2} b_{rs}^i \frac{\partial^2 f^i}{\partial \xi_r \partial \xi_s} + P_r^i \frac{\partial f^i}{\partial \xi_r} - \frac{\partial Q_r^i}{\partial \xi_r} f^i, \quad (9)$$

where

$$P_r^i = L^i \sum_j [(m^j - m^i)/m^j] A_r^j \quad (10)$$

$$Q_r^i = L^i \sum_j (m^i / m^j) A_r^j. \quad (11)$$

The coefficients A and B can be evaluated explicitly for a Maxwellian f ,

$$f^{(0)} = n(2\pi RT)^{-3/2} \exp(-\xi^2 / 2RT). \quad (12)$$

In terms of

$$x = \xi(RT)^{-\frac{1}{2}} = \xi/c \quad (13)$$

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we have

$$A_r^{(0)} = - (2/\pi)^{1/2} (n/RT) (x_r/x^2) F(x) \quad (14)$$

$$B_{rs}^{(0)} = (2/\pi)^{1/2} (n/\sqrt{RT}) \{ (\delta_{rs} x - x_r x_s/x^3) \\ \times \int_0^x (\exp - \frac{1}{2} y^2) dy - F(x) [\delta_{rs}/x^2 - 3x_r x_s/x^4] \} \quad (15)$$

where

$$F(x) = x^{-1} \int_0^x \exp(-\frac{1}{2} y^2) dy - \exp - \frac{1}{2} x^2 \quad (16)$$

$A_r^{(0)}$ is directed radially inward. $B_{rs}^{(0)}$ is not isotropic and can be written

$$B_{rs}^{(0)} = B_1 x_r x_s/x^2 + B_2 (\delta_{rs} - x_r x_s/x^2). \quad (17)$$

B_1 measures the dispersion in the radial direction and B_2 measures the tangential dispersion on a sphere $x = \text{constant}$.

The dependence of $|A^{(0)}|$, B_1 and B_2 on x is shown in Fig. 1. For small x we have the expansions

$$\begin{cases} A_r^{(0)} = - \frac{1}{3} (2/\pi)^{1/2} (n/RT) x_r (1 - \frac{3}{10} x^2 + \dots) \\ B_1 = + \frac{2}{3} (2/\pi)^{1/2} (n/\sqrt{RT}) (1 - \frac{3}{10} x^2 + \dots) \\ B_2 = + \frac{2}{3} (2/\pi)^{1/2} (n/\sqrt{RT}) (1 - \frac{1}{10} x^2 + \dots) \end{cases} \quad (18)$$

and for large x ,

$$\begin{aligned} A_r^{(0)} &\sim - (n/RT) (x_r/x^3) \\ B_1 &\sim 2n/(\sqrt{RT})^{1/2} / x^3 \\ B_2 &\sim [n/(\sqrt{RT})^{1/2}] (1/x - 1/x^3) \end{aligned} \quad (19)$$

the terms neglected are exponentially small in this case.

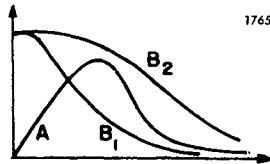


Figure 1.

The coefficients a_r and b_{rs} have a simple intuitive interpretation. A particle which is known to have the velocity ξ_r at time zero "diffuses" in the interval dt into a normal distribution peaked about $\xi_r + a_r dt$ with second moments $b_{rs} dt$. The equilibrium distribution $j^{(0)}$ is the result of a competition between a directed frictional force, a_r , tending to bring each particle to rest and a random dispersive "force" b_{rs} directed outward. In equilibrium, the logarithmic derivative of j (which is $-\xi_r/RT$) is given by the ratio of Q to b_1 (b_1 and b_2 are defined as in Eq. (17)).

Many interesting properties can be deduced by inspection of the formulas for a_r and b_{rs} . For example, the ion friction coefficient a^+ has a local minimum for a speed somewhere between the mean ion speed C_+ and the mean electron speed C_- ; it is dominated by ion-ion collisions for low ion velocities and by ion-electron collisions for high. Although the transverse component of the dispersion, b_2^+ , is ion dominated for

all velocities, the radial component b_1^+ becomes electron dominated above a certain speed between C_+ and C_- . One conclusion we draw is that the relaxation time for that part of the ion distribution which lies between C_+ and C_- is in the order of the mean ion-electron relaxation time, which is about $(m_+/m_-)^{1/2}$ slower than the relaxation time for the bulk of the ion distribution.

LINEARIZATION OF THE EQUATIONS

For the main part of the distribution, a solution of the form

$$j(\xi) = j^{(0)}(\xi) + \mathbf{E} \cdot \xi \phi(\xi^2/2) \quad (20)$$

is valid. Let us assume, tentatively, that such a solution has been obtained for both the ion and the electron distributions. Since $A_r(\xi)$ and $B_{rs}(\xi)$ are given as integrals over j , they can be computed from (20) and the results are valid for *all values* of ξ . It is assumed (later verified) that the tail of the distribution is exponentially small even though it may differ markedly from $j^{(0)}$. If we insert the known functions $A_r(\xi)$, $B_{rs}(\xi)$ into the Fokker-Planck equation, the collision term becomes a *linear* second-order differential operator on j . The complete Fokker-Planck equation is now linear (one need not tamper with the term $\mathbf{E} \cdot \partial j / \partial \xi$), and it is uniformly valid for all values of ξ . Note that the equation is homogeneous; $\mathbf{E}$ does not multiply an inhomogeneous term, it is a parameter in the differential equation. Because of this we can no longer expect the solution to be linear in $\mathbf{E}$.

We shall find that the deviation of the ion distribution from Maxwellian is extremely small except for the tail. Consequently, we can take the Maxwellian values $A^{(0)}$, $B^{(0)}$ for the ion contributions to a^+ and b^+ . For the electron coefficients we write

$$\begin{aligned} A_r &= A_r^{(0)} + A_r^{(1)} \\ B_{rs} &= B_{rs}^{(0)} + B_{rs}^{(1)}, \end{aligned} \quad (21)$$

where

$$\begin{aligned} A_r^{(1)} &= E_k \int (V_r/V^3) \eta_k \phi(\frac{1}{2} \eta^2) d\eta \\ B_{rs}^{(1)} &= E_k \int [(V^2 \delta_{rs} - V_r V_s)/V^3] \eta_k \phi(\frac{1}{2} \eta^2) d\eta. \end{aligned} \quad (22)$$

We are interested only in that part of the ion distribution for which $\xi \ll C_-$. For the electron coefficients (22) we use an expansion about small ξ , obtaining

$$\begin{cases} A_r^{(1)} = (4/3) \pi \phi_1 E_r + \dots \\ B_{rs}^{(1)} = (8/15) \pi \phi_1 (\mathbf{E} \cdot \xi \delta_{rs} + E_r \xi_s + E_s \xi_r) + \dots \end{cases} \quad (23)$$

where

$$\phi_1 = \int_0^\infty \phi(x) dx. \quad (24)$$

The value of ϕ_1 can be found without solving for the entire electron distribution function $\phi(\frac{1}{2} \xi^2)$ by inte-

grating the ion Fokker-Planck equation after multiplying through by ξ_r . The result is

$$\int [a_r^+(\xi) + (e/m_+)E_r]f^+(\xi)d\xi = 0. \quad (25)$$

Using the identity

$$\int A_r^i(\xi)f^i(\xi)d\xi = 0 \quad (26)$$

for A^+ and the approximation (23) for A^- , we find

$$\phi_1 = -3em_-/4\pi L_+ m_+^2 = 3\kappa^2 m_-/e^3 \ln \lambda. \quad (27)$$

Surprisingly, the only fact that we need to know about the perturbed electron distribution in order to compute the ion distribution is the single parameter ϕ_1 which represents the mean frictional force between ions and electrons. This comes out of the formal analysis of the next section, but it can also be explained qualitatively.

The very slight deviation from Maxwellian of the ion distribution is easily explained. To a first approximation, the ion distribution can be considered to be a δ function on the scale of electron velocities. The electric field accelerates the ion stream relative to the electrons until the mean frictional force balances the applied field. However, the individual ions all see essentially the same value of electron friction; consequently, for each ion the electric field is balanced by the electron friction and, to this order, there is no de-Maxwellizing effect. To the next order of approximation, we consider the electron friction as a linear function over the range of ion velocities instead of a constant. To the same order, the dispersion is still a constant since its slope is zero at zero velocity. It is easily verified that a linear variation of friction combined with a constant dispersion is compatible with a Maxwellian ion distribution. We conclude that only the extreme tail of the ion distribution, where these approximations are not valid, will exhibit any deviation from Maxwellian. It is now also clear why to a certain order of approximation the perturbation of the electron friction and dispersion coefficients can be computed from the single parameter ϕ_1 , which selects the point on the ion friction and dispersion curve around which the ion distribution lies.

ASYMPTOTIC EVALUATION OF THE DISTRIBUTION FUNCTION

We are interested in ion speeds in the range $C_+ \ll \xi \ll c_-$. To make this statement precise we introduce the expansion parameter ε :

$$m_-/m_+ = \varepsilon^6, \quad C_+/C_- = \varepsilon^3. \quad (28)$$

For deuterons, $m_+/m_- \sim 3600$, $C_-/C_+ \sim 60$, and $\varepsilon \sim \frac{1}{4}$. Actually, there are two small parameters, E and ε , but they will eventually be related. For the ion coefficients we use the high-velocity approximation (19) and for the electron coefficients the low-velocity approximation (18) and (23). First we transform to spherical coordinates, taking $\mathbf{E}$ as the polar axis,

$$\mathbf{E} \cdot \boldsymbol{\xi} = Er \cos \theta, \quad \xi^2 = r^2, \quad (29)$$

obtaining

$$\begin{aligned} & \frac{1}{2} b_1 \partial^2 / \partial r^2 - b_3 E \sin \theta \partial^2 / \partial r \partial \theta + (b_2 / 2 r^2) \partial^2 / \partial \theta^2 \\ & + [b_2 / r + r p_1 + E \cos \theta (p_2 - e/m)] \partial f / \partial r \\ & + [\frac{1}{2} b_2 \cot \theta + E \sin \theta (b_3 - b_2 + e/m)] (1/r) \partial f / \partial \theta \\ & - [3q_1 + r(\partial q_1 / \partial r) + E \cos \theta (4r q_2 + r^2 (\partial q^2 / \partial r) \\ & + \partial q_3 / \partial r)] f = 0, \end{aligned} \quad (30)$$

where we have set

$$\left. \begin{aligned} b_{rs} &= b_1 \xi_r \xi_s / \xi^2 + b_2 (\delta_{rs} - \xi_r \xi_s / \xi^2), \\ &+ b_3 (E_r \xi_s + E_s \xi_r - 2 \mathbf{E} \cdot \boldsymbol{\xi} \xi_r \xi_s / \xi^2) \\ P_r &= p_1 \xi_r + p_2 E_r, \\ Q_r &= q_1 \xi_r + q_2 \mathbf{E} \cdot \boldsymbol{\xi} \xi_r + q_3 E_r. \end{aligned} \right\} \quad (31)$$

The coefficients b_i , p_i , q_i are functions of r alone. To the order of approximation taken,

$$\left. \begin{aligned} b_1^+ &= (nL^+/C_+) \{ 2/x^3 + \frac{2}{3} (2/\pi)^{\frac{1}{2}} \varepsilon^3 \\ &+ (8\pi C_+^2 \phi_1 / 5n) \mathbf{E} \cdot \mathbf{x} \}, \\ b_2^+ &= (nL^+/C_+) \{ 1/x + \frac{2}{3} (2/\pi)^{\frac{1}{2}} \varepsilon^3 \\ &+ (8\pi C_+^2 \phi_1 / 15n) \mathbf{E} \cdot \mathbf{x} \}, \\ b_3^+ &= (8/15) \pi \phi_1 L^+, \\ q_1^+ &= - (nL^+/C_+^3) \{ 1/x^3 + \frac{1}{3} (2/\pi)^{\frac{1}{2}} \varepsilon^3 \}, \\ q_2^+ &= 0, \\ q_3^+ &= \frac{4}{3} \pi \phi_1 L^+ / \varepsilon^6 = -e/m, \\ p_1^+ &= \frac{1}{3} (2/\pi)^{\frac{1}{2}} \varepsilon^3 nL^+ / C_+^3, \\ p_2^+ &= -\frac{4}{3} \pi \phi_1 L^+ / \varepsilon^6 = e/m, \end{aligned} \right\} \quad (32)$$

where

$$\mathbf{x} = \boldsymbol{\xi} / C_+. \quad (33)$$

Factoring out the Maxwellian function

$$f = f^{(0)} g, \quad (34)$$

we obtain

$$\begin{aligned} & \partial^2 g / \partial x^2 - x \partial g / \partial x - \varepsilon^3 \mathcal{E} x^3 \cos \theta g + \frac{2}{3} \varepsilon^3 \mathcal{E} \sin \theta \partial^2 g / \partial x \partial \theta \\ & + \frac{2}{3} (\pi/2)^{\frac{1}{2}} (x\varepsilon)^{-3} \partial^2 g / \partial \theta^2 + \cot \theta (\partial g / \partial \theta) = 0 \end{aligned} \quad (35)$$

in terms of the dimensionless electric field

$$\begin{aligned} \mathcal{E} &= - (3/5) (2\pi)^{\frac{3}{2}} \phi_1 C_-^2 E / n \\ &= (9/5) (2\pi)^{\frac{3}{2}} \kappa^2 k T E / n e^3 \ln \lambda, \end{aligned} \quad (36)$$

This equation is to be solved subject to the restrictions $1 \ll x \ll 1/\varepsilon^3$ and $\mathcal{E} \ll 1$. If $x \sim 1/\varepsilon$ or smaller, it is easy to see that g does not depart appreciably from unity. In the more interesting range, for $x \gg 1/\varepsilon$ (e.g., $x \sim 1/\varepsilon^2$), the dominant terms in the equation are $\partial g / \partial x$ and g ; the solution is

$$g \sim \exp \{ -\frac{2}{3} \varepsilon^3 x^3 \mathcal{E} \cos \theta \}, \quad x \sim 1/\varepsilon^2. \quad (37)$$

The dimensionless electric field $\mathcal{E}$ is essentially the energy gain of a particle in the electric field over a mean-free-path compared to the mean thermal energy kT . If this quantity is large then no linear

theory is adequate. The exponent in g has order $\mathcal{E}/\varepsilon^3$, say $1/\varepsilon^2$ if $\mathcal{E} \sim \varepsilon$, and this can be large, producing an enormous perturbation of the Maxwellian distribution. On the other hand, the exponent in the Maxwellian has order $1/\varepsilon^4$ for $x \sim 1/\varepsilon^2$, so the distribution function f itself is very small in this range.

One of the most interesting features of the formula (37) is the minus sign in the exponent. This implies that the ion distribution function is depressed below the Maxwellian in the direction of $\mathbf{E}$ and is enhanced in the opposite direction. Of course, since we have already observed that to lowest order, the forcing term $\mathbf{E}$ is exactly cancelled by the increase in electron friction and it is no longer a matter of simple intuition which direction this higher-order effect will take. As it turns out, the fact is that the electron dispersion coefficient decreases in the direction of $\mathbf{E}$ (the ion distribution has been shifted in the same direction as $\mathbf{E}$ from the maximum of b), that produces the effect.

THERMONUCLEAR OUTPUT

Since we have computed only the dominant term, i.e., the exponent, of the distribution function, we carry out the thermonuclear computation to the same order. If we write

$$\sigma \sim \exp \mu(\frac{1}{2}mV^2), \quad m = m_+ \quad (38)$$

for the D-D cross-section (V is the relative velocity as in Eq. (6)), the thermonuclear output is given by

$$\nu \sim \int \exp \{ \mu(\frac{1}{2}mV^2) \} f^{(0)}(\xi) g(\xi) f^{(0)}(\xi_1) g(\xi_1) d\xi d\xi_1. \quad (39)$$

With the exception of μ , the integrand is the exponential of a polynomial in ξ and ξ_1 . We introduce the variables $V = \xi_1 - \xi$, $W = \xi_1 + \xi$ and find that the integrand has a maximum at $W^{(1)}$ and $V^{(1)}$ given by

$$W^{(1)} \sim -\frac{1}{2}\mathcal{E}(V^{(0)})^2 \varepsilon^3 / C_+ \\ \mu'(\frac{1}{2}m(V^{(1)})^2) = (1/2kT)[1 - \frac{1}{2}\mathcal{E}^2(V^{(0)})^2 \varepsilon^6 / C_+^2]; \quad (40)$$

$V^{(0)}$ is the maximizing value for $\mathcal{E} = 0$. We approxi-

mate the integral by the maximum value of the integrand to obtain

$$\nu = \nu^{(0)} \exp \{ [\frac{1}{4}\mathcal{E}(V^{(0)})^2 \varepsilon^3 / C_+^2] \}; \quad (41)$$

$\nu^{(0)}$ is the Maxwell-averaged thermonuclear output with $\mathcal{E} = 0$. It is convenient to convert $\theta = kT$ into kev, n into ions/cm³ and E into v/cm. For temperatures which are not too high, we have ⁴

$$m(V^{(0)})^2/kT \sim 12.4/\theta^{\frac{1}{2}} \quad (42)$$

from which we obtain

$$\nu = \nu^{(0)} \exp \{ [0.9 \times 10^{15} E \theta^{\frac{1}{2}}]^2 / n \ln \lambda \} \quad (43)$$

$$W^{(1)}/C_+ = 1.5 \times 10^{15} E \theta^{\frac{1}{2}} / n \ln \lambda. \quad (44)$$

A representative set of values for which the exponent in (43) has order unity might be $n \sim 10^{14}$ ions/cm³, $E \sim 1$ v/cm and $\theta \sim 0.5$ kev; $\ln \lambda$ is about 14. The most interesting point about the formula for ν is the extremely sensitive dependence on n , E and θ . The transition from a negligible correction to an enormous one requires a small change in the parameters. A specific feature is the emphasis on low density; combined with the factor n^2 which is in $\nu^{(0)}$, we find that ν has a minimum as a function of n when the exponent is unity and then ν rises very rapidly. For example, in a discharge of varying density it is likely that E would be approximately uniform and the thermonuclear output might then be concentrated in the lower density regions. However, one should recall the fact that the appropriate relaxation time—i.e., the time required to build up the tail of the distribution—is, for one thing, long (comparable to the ion-electron relaxation time) and, for another, inversely proportional to the density.

Formula (44) is also of interest since $W^{(1)}$ measures the mean shift in the center of mass of the reacting deuterons. The energy spectrum of the neutrons emitted has a shift of the order of the thermal energy when the dimensionless term has order unity. In the example above the shift is about 0.5 kev. We recall that this shift should be observed in the direction *opposite* to $\mathbf{E}$.

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Transient Temperature Variations During the Self-heating of a Plasma by Thermonuclear Reactions*

By Howard D. Greyber†

To discuss the energy balance in a fully ionized gas with very low Z nuclei and a temperature in the region of 10–1000 keV, it is necessary to consider the temperatures of the ion and electron gases separately. This is because the energy transfer mechanism between ions and electrons becomes, in this region, smaller than the characteristic energy sources and sinks for the two gases.

If, in addition, the radiation produced escapes more or less freely from our system,‡ the radiation present is not in equilibrium with the matter (and, in fact, the concept of radiation temperature is not applicable). If we state that the energy source for our fully ionized gas is the thermonuclear reactions among its constituents, then the combination of the aforesaid conditions,

$$T_n \neq T_e$$

$$T_n, T_e \gg \text{"radiation temperature,"}$$

where T_n = nuclear or ion temperature, T_e = electron temperature, is usually described as "nonequilibrium thermonuclear burning".

Such a process is probably at least a rough approximation to the thermonuclear burning contemplated in some form of CTR.§ Due to the extremely low density of the plasma, the system is essentially "open" to radiation. Also, if a 50%D, 50%T plasma is contemplated, for temperatures above 10 keV, T_n will be appreciably different from T_e .

The motivation for this work arose from an observation by Rosenbluth that in a different but related physical situation, T_e could exceed T_n during transient heating. We have undertaken to trace the transient temperatures to be expected in an idealized physical situation that still bears some resemblance to what one envisions for the CTR.

If we assume a system completely open to neutrons and radiation and closed as far as loss of charged particles or any other form of energy drain other than neutrons and radiation is concerned, we may write for a constant volume system (magnetic field increasing in such a way as to keep constant volume)

$$(1.6 \times 10^{-9}) 3/2 n \dot{T}_n = a P_{DD} + b P_{DT} - P_{i-e} \quad (1a)$$

$$(1.6 \times 10^{-9}) 3/2 n \dot{T}_e = (1-a) P_{DD} + (1-b) P_{DT} + P_{i-e} - P_B \quad (1b)$$

where T = temperature in keV, n = number of particles per cm^3 and P_{DD} = energy production rate from DD reaction

$$= (n_D^2/2) \langle \sigma_{DD} v \rangle (2.4 + 3.5/2) \times (1.6 \times 10^{-6}) \text{ ergs sec}^{-1} \text{ cm}^{-3}. \quad (2)$$

Only deposition of charged particle energy is counted. One-half of a DT reaction is added for each DD reaction. We assume no depletion and instantaneous deposition.

$$P_{DT} = \text{energy production rate from DT reaction}$$

$$= n_D n_T \langle \sigma_{DT} v \rangle (3.5) \times (1.6 \times 10^{-6}) \text{ ergs sec}^{-1} \text{ cm}^{-3} \quad (3)$$

where n_D = number of deuterons per cm^3 and n_T = number of tritons per cm^3 .

The coefficients a and b indicate the fraction of nuclear energy deposited to the nuclear and electron gases. The expression for a is identical with that used by Fermi and Ulam of Los Alamos:

$$a = 0.3 + 0.005 T_e$$

$$1-a = 0.7 - 0.005 T_e$$

$$b = 0.01 T_e$$

$$1-b = 1 - 0.01 T_e; a, b \text{ are always kept } \leq 1,$$

P_{i-e} = energy transfer rate from nuclear gas to electron gas

$$= 1.244 \times 10^{-21} n^2 \frac{T_i - T_e}{T_e^{3/2}} \text{ ergs sec}^{-1} \text{ cm}^{-3},$$

P_B = energy production rate in bremsstrahlung radiation

$$= 5.38 \times 10^{-24} n^2 T_e^{1/2} \text{ ergs sec}^{-1} \text{ cm}^{-3}.$$

* Work done at University of California Radiation Laboratory, Livermore, California.

† Now at General Electric Co., Schenectady, N. Y.

‡ Dimensions "reasonable"; i.e., 0.1 to 100 meters; the density is low.

§ Controlled Thermonuclear Reactor—any geometry.

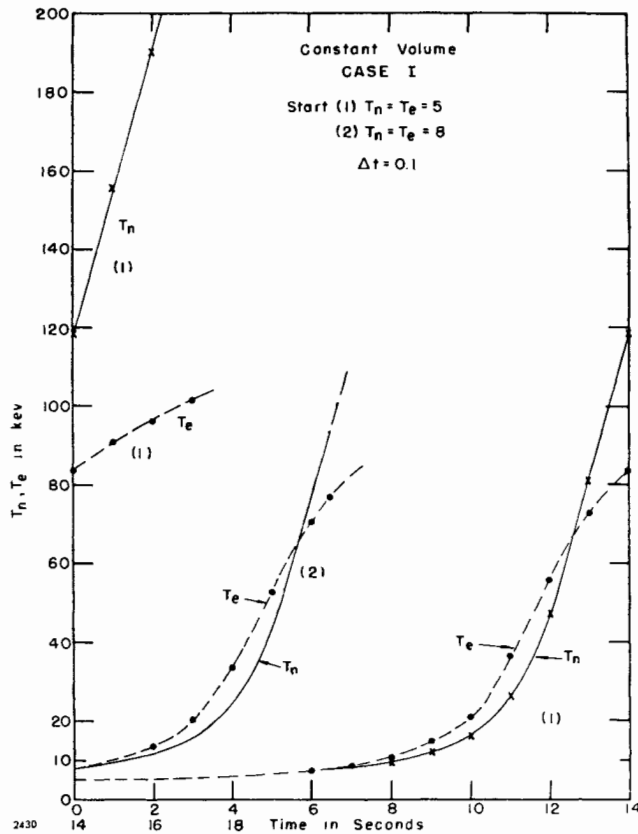


Figure 1

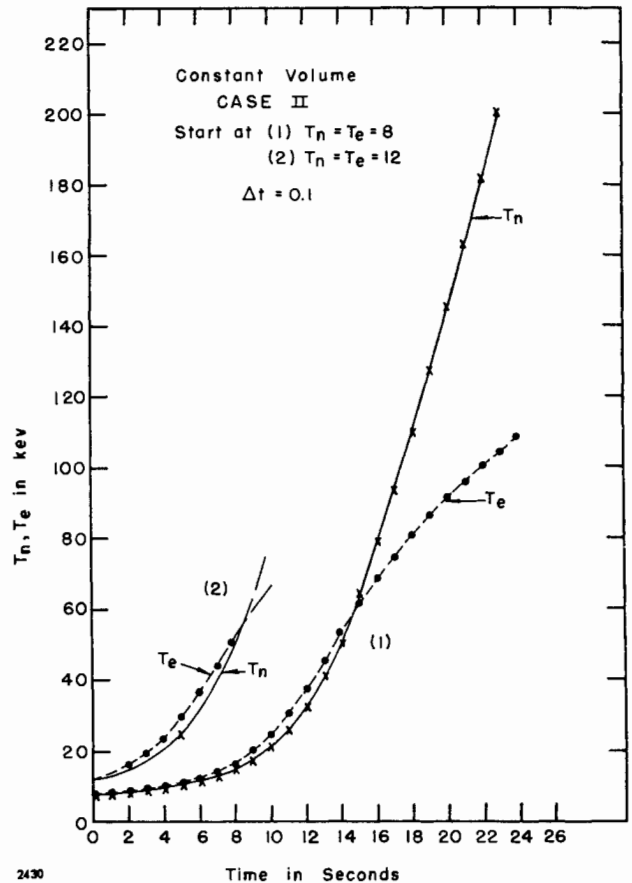


Figure 2

We have done the calculations for $n = 10^{14}$ but it is clear that one may scale the results simply for number density. The time scale will vary inversely with density; hence for $n = 10^{15}$, divide times on accompanying graphs by 10, etc.

The integrations were carried out on the IBM Card Program Calculator (CPC), and the results are shown in Figs. 1, 2 and 3. Three cases were run: (I) 50% T and 50% D; (II) 10% T, 90% D; and (III) 1% T, 99% D. Some cases were run for two different starting temperatures.

A check calculation was run with $\Delta t = 0.01, 0.1$ and 1.0 second and the comparison showed that the interval of $\Delta t = 0.1$ used gives graphs quite close to those for $\Delta t = 0.01$, and hence, are fairly accurate. However, a much larger source of error is our assumption of instantaneous deposition of energy by the fast charged reaction products. A rough estimate of the time for a 3.5-Mev He^4 nucleus (or a 1-Mev T) to slow down to zero energy in a deuterium-tritium mixture at $n = 10^{14}$ is given by $\tau = 0.4(T_e^{1/2})$, where T_e is in keV and τ is in seconds. Thus for $T_e = 25$ keV, $\tau = 2.0$ seconds. One can infer how the curves would look were this factor correctly introduced into the calculation. First we qualitatively note that a fast charged particle spends most of its time losing energy to the electron gas (only near the end of its range) and rapidly loses the remainder to the nuclear gas; (1) the T_e curve will exceed the T_n curve by a larger amount in the range below 70 keV

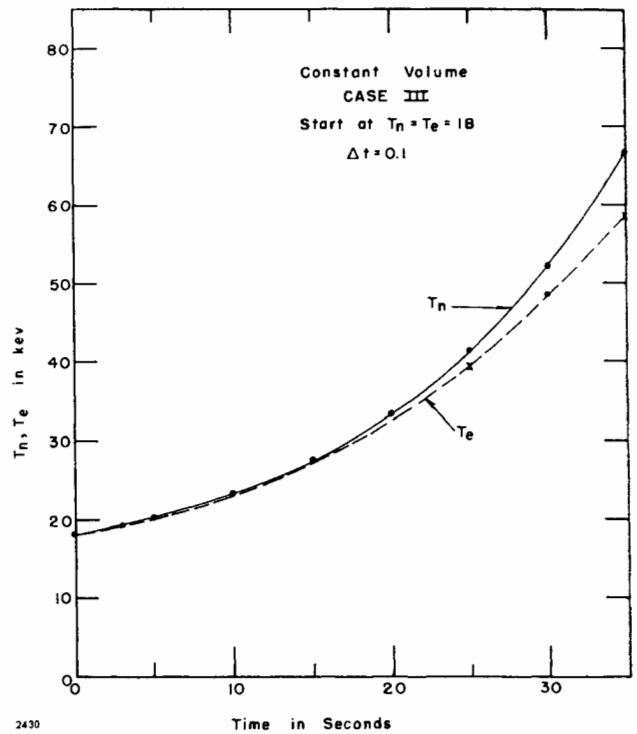


Figure 3

than calculated, (2) the cross-over points where $T_e = T_n$ will be at higher values than calculated. Also the time scales for the rise of temperature with time will be increased; i.e., the time to reach $T_n = 100$ kev may be greater than calculated by perhaps as much as a factor of 2.

The results of the calculation, although rough, give us an idea of the time scales involved in the self-heating process. Of course, other losses than those considered here, such as energy lost in particles diffusing to the walls, energy lost in particles leaking through magnetic mirrors, or useful output such as work against moving mirrors, or particles exciting to a diverter, etc., would all tend to lengthen the time for a given temperature rise. It is clear that our Eqs. (1a) and (1b) could easily be generalized to include extra terms such as above mentioned or energy input terms such as compression or magnetic pumping while burning, if simple expressions are available. The time-dependent energy deposition feature, although complicated, could also be added.

We derived the equations for the case of a plasma at constant pressure while burning (constant magnetic field—experimentally obtained by “crow-barring” the magnet coils) and integrated them also on the CPC. Unfortunately, there appears to have been a numerical error in these calculations and no graphs are given for these cases.

One qualitative result of this work is that it would be unsafe to assume that the nuclear temperature is always equal to or greater than electron temperature in a discussion of plasma instabilities for a plasma with tritium concentration equal to or greater than 10 %.

ACKNOWLEDGEMENTS

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Fusion Chain Reaction

By M. Gryziński *

CHAIN REACTION WITH CHARGED PARTICLES

With the discovery of the fission chain reaction with neutrons, the possibility of obtaining a chain reaction with charged particles was abandoned because of the small efficiency of charged particles in nuclear reactions. In the most advantageous case, T (D, n) He⁴, the efficiency attained is only $\sim 5 \times 10^{-3}$ reactions per 14 Mev deuteron. However, no note was taken of the fact that the efficiency depends on the physical conditions and in some cases may be greatly increased. This is especially true for nuclei of small charge, where the factor of Coulomb barrier penetration is not too high. The development of the chain reaction with charged particles is therefore possible only for light nuclei, where the release of nuclear energy is due to the process of fusion. Only highly exoenergetic reactions of large cross sections may lead to the fusion chain reaction; these are the same reactions as those which are involved in thermonuclear reactions.¹

FORMULATION OF THE PROBLEM

The mechanism of a fusion chain reaction, which is due to *in statu nascendi* reactions, is as follows. In an exoergic reaction $A + B$, in which weakly bound groups of nucleons of nuclei A and B form strongly bound groups of reaction products, we obtain particles having kinetic energy. Part of their kinetic energy is transferred in elastic collisions directly to the A and B nuclei of the medium. The recoiling A and B nuclei, in the process of slowing down to thermal energy, have some probability of leading again to the reaction $A + B$. Under normal physical conditions the dissipation of the energy of the charged particles in collisions with electrons is so large that their range, L , in the medium is much smaller than the mean-free path, λ , with respect to the nuclear process. Therefore, only a small fraction of recoil nuclei lead again to the $A + B$ reaction. The development of an avalanche is possible when the sum of ranges of the recoil nuclei $\sum_i L_i$ is comparable to λ . Since $\sum_i L_i \simeq E_Q (dE/dx)^{-1}$ and $\lambda \simeq (N\langle\sigma\rangle)^{-1}$, where E_Q denotes the kinetic energy released in the $A + B$ reaction, dE/dx the average energy losses of recoil nuclei per unit path length, N the density of reacting nuclei of the

medium, $\langle\sigma\rangle$ the mean cross section for the $A + B$ reaction, we can write

$$NE_Q\langle\sigma\rangle/(dE/dx) \sim 1. \quad (1)$$

If we assume $\langle\sigma\rangle \sim 10^{-24}$ cm², $E_Q \sim 10$ Mev, we find that the atomic stopping power or stopping factor $\langle dE/dx \rangle/N = \langle\sigma\rangle E_Q$ would be 10^{-17} ev atom⁻¹ cm². Under normal physical conditions it is about a thousand times higher,^{2, 3} and we are far from satisfying the criterion (1).

The main idea of the problem involves the dependence of the atomic stopping factor on the physical conditions.

Under normal physical conditions, in the moderate energy range, the most important losses of energy of heavy charged particles are due to scattering from electrons. They are about four thousand times higher than the energy losses in all other processes.³ The atomic stopping factor due to scattering from electrons was discussed in detail by the author,⁴ and, according to Eq. (6) of Ref. 4, for a particle ξ having a velocity V_ξ and a charge $Z_\xi e$, it is

$$\sigma \left(\frac{dE}{dx} \right)_{el}^{sc} = \frac{4\pi e^4}{mV_\xi^2} Z_\xi^2 \int_0^\infty f(V_e); G[(V_e)] dV_e \quad (2)$$

where $f(V_e)$ is the momentum distribution of electrons in the medium and G the universal stopping power function given by Eq. (8) of Ref. 4. There it was shown that these losses depend mainly on the velocity distribution of the electrons, especially in the case $V_\xi \leq V_e$. In the limiting case $V_\xi \ll V_e$, the asymptotic value of G becomes $\frac{2}{3}(V_\xi/V_e)^3$ whereupon we have

$$\sigma \left(\frac{dE}{dx} \right)_{el}^{sc} \sim \frac{1}{V_e^3}. \quad (3)$$

Hence we see that the energy losses connected with the scattering from electrons decrease very strongly with their velocity. The electron momentum distribution can be shifted into higher velocities by a considerable rise of temperature or by increasing the density up to the strong degeneracy of the electron gas. In this way we can decrease the atomic stopping factor so that the condition (1) is fulfilled.

Atomic Stopping Factor

As mentioned above, the main energy losses of charged particles are due to scattering from electrons,

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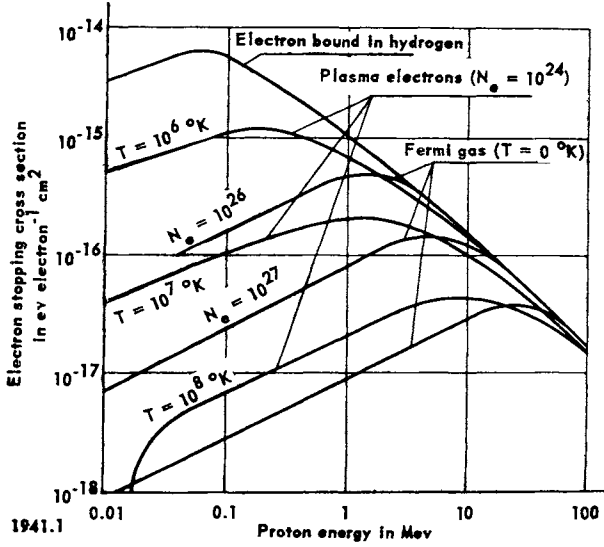


Figure 1. Theoretical calculations of the stopping power of electrons bound in hydrogen, electrons of plasma at different temperatures, and electrons of a Fermi gas

which scattering depends on the state of the medium.

To evaluate the stopping factor of plasma electrons⁵ we have to use the Maxwellian momentum distribution in Eq. (2). We obtain an approximate dependence on the temperature of the plasma if we make the substitution $f(V_e) = \delta(\langle V_e \rangle - V_e)$, where $\langle V_e \rangle = (8kT/\pi m)^{1/2}$ is the mean thermal velocity of electrons. In the case of interest, $V_\xi \ll V_e$, we have

$$\sigma \left(\frac{dE}{dx} \right)_{\text{plasma electrons}}^{\text{sc}} \simeq - \frac{8\pi e^4}{3m} \left(\frac{\pi m}{8kT} \right)^{1/2} Z_\xi^2 V_\xi. \quad (4)$$

Similarly, taking into account the momentum distribution of a Fermi gas, we obtain the stopping factor of Fermi gas electrons (cf. Eq. (18), Ref. 4; also Ref. 6)

$$\sigma \left(\frac{dE}{dx} \right)_{\text{Fermi electrons}}^{\text{sc}} \simeq - \frac{1}{N_e} \left(\frac{4}{3\pi} \right) \times \left(\frac{2e^4 m^2}{\pi^3} \right) Z_\xi^2 V_\xi \ln \left(\frac{V_{\text{max}} \pi}{e^2} \right) \quad (5)$$

where $V_{\text{max}} = (3\pi^2)^{1/3} (\hbar/m) N_e^{1/3}$, N_e the number of electrons per cm^3 .

The results of exact computations, where for the maximum impact parameter we have put $D_{\text{max}} = N_e^{-1/3}$ (Appendix 1), are plotted in Fig. 1 for various temperatures and densities.

A decrease in the electron scattering losses increases the role of energy losses connected with the interaction with the nuclei of a medium.

The contribution to the atomic stopping factor due to elastic scattering from nuclei of mass m_A and charge $Z_A e$ is:

$$\sigma \left(\frac{dE}{dx} \right)_{\text{nuc}}^{\text{elastic sc}} \simeq - \frac{4\pi e^4}{m_A V_\xi^2} (Z_\xi Z_A)^2 \ln \left(\frac{\mu_{\xi A} V_\xi^2}{Z_\xi Z_A e^2 N^{1/3}} \right) - \frac{1}{2} K_{\xi A} E_\xi \sigma_{\xi A}^{\text{sc}} (\sigma_\xi) \quad (6)$$

where $\mu_{\xi A}$ is the reduced mass, $K_{\xi A} = 4m_\xi m_A / (m_\xi + m_A)^2$ and $\sigma_{\xi A}^{\text{sc}}$ the elastic nuclear scattering

cross section of particle ξ from the nuclei A , and N the number of nuclei per cm^3 . The first term in Eq. (6) represents the Coulomb scattering, and the second the nuclear scattering, which we assumed isotropic in the center-of-mass system.

The stopping factor due to inelastic collisions with nuclei is:

$$\sigma \left(\frac{dE}{dx} \right)_{\text{nuc}}^{\text{inel sc}} = \sum_i \Delta E_i \sigma_{\xi A}^{\text{inel sc}} (E_\xi). \quad (7)$$

The sum is taken over all channels with the excitation energies ΔE_i .

The energy losses connected with the bremsstrahlung⁷ of heavy charged particles are very low in comparison with the losses given above; therefore, they can be safely neglected.

Finally, the stopping factor of nucleus A and its Z_A electrons is:

$$\sigma \left(\frac{dE}{dx} \right) = Z_A \sigma \left(\frac{dE}{dx} \right)_{\text{el}}^{\text{sc}} + \sigma \left(\frac{dE}{dx} \right)_{\text{nuc}}^{\text{el sc}} + \sigma \left(\frac{dE}{dx} \right)_{\text{nuc}}^{\text{inel sc}}. \quad (8)$$

The total atomic stopping factor of hydrogen plasma† for protons and the relative contribution of their components in various conditions are plotted in Fig. 2.

Evaluation of Multiplication Factor

To determine the exact conditions for the development of an avalanche, we shall examine an infinite homogeneous medium formed by a mixture of two

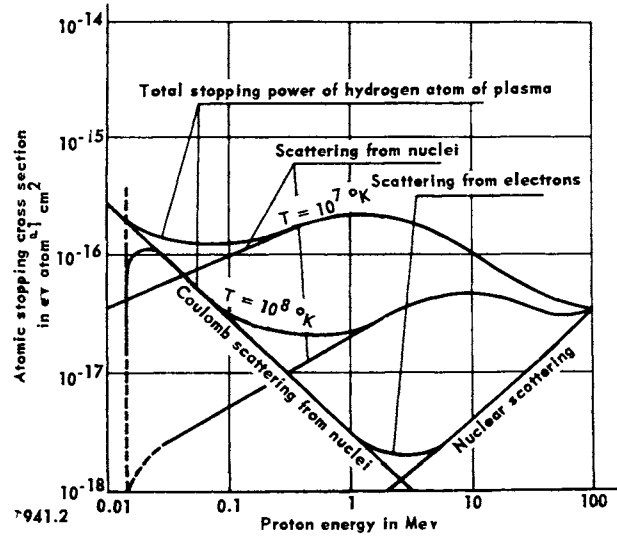


Figure 2. The stopping power of hydrogen plasma and the relative contribution of its components

† As shown above, the stopping factor of hydrogen plasma under the conditions existing in the sun, $\sim 2 \times 10^7$ °K, is about one hundred times lower than that of hydrogen under normal physical conditions. Therefore, Bethe's calculations⁸ of the efficiency of reactions *in statu nascendi* in the sun (with the assumption that the energy losses are approximately the same in both cases) are not valid.

kinds of nuclei, A and B, which can initiate the exoergic reaction. We denote by N_A and N_B the number densities of reacting particles, by σ_{AB}^ξ the laboratory cross section for the reaction A + B (the bombarding particle is denoted by the first lower index) with the emission of the particle ξ . The particles, of high kinetic energy, obtained from this reaction produce a certain number of recoil nuclei.

If $f_\xi(E_\xi^0)$ is the energy distribution of the ξ particles obtained from each reaction A + B, then the number of ξ particles in the energy interval, E_ξ^0 to $E_\xi^0 + dE_\xi^0$, is $f_\xi(E_\xi^0)dE_\xi^0$. Since the major part of the reaction A + B in the avalanche occurs in the moderate energy range (100 — 500 keV) and since the reaction A + B is strongly exoenergetic, we have assumed that this distribution is independent of the energy of the entrance channel. If in the result of reaction A + B we obtain two particles, the function $f_\xi(E_\xi^0)$ is the $\delta(E_\xi^0 - E_\xi)$ function. Owing to the destruction of particles ξ on interaction with the A and B nuclei, the initial number $f_\xi(E_\xi^0)dE_\xi^0$ of particles along the path x drops to the value $q(E_\xi^0, x)f_\xi(E_\xi^0)dE_\xi^0$ where

$$q(E_\xi^0, x) = \exp - \int_0^x (N_A \sigma_{\xi A} + N_B \sigma_{\xi B}) dx \quad (9)$$

and $\sigma_{\xi A}(\sigma_{\xi B})$ is the total reaction cross section of the particle ξ with the nucleus A(B). Taking into account that the energy of particle ξ on the path x drops from E_ξ^0 to E_ξ because of energy losses, we can write the last expression in terms of E_ξ

$$q(E_\xi^0, E_\xi) = \exp - \int_{E_\xi^0}^{E_\xi} \frac{N_A \sigma_{\xi A} + N_B \sigma_{\xi B}}{(dE_\xi/dx)} dE_\xi \quad (10)$$

where (dE_ξ/dx) denotes the loss of energy of the particle per unit path length. Introduce

$$\sigma_{\xi A}^{sc}(E_\xi, E_A) dE_A,$$

the cross section for the production of recoil nuclei A of energy E_A to $E_A + dE_A$ by the particle ξ with energy E_ξ . Then the number of the recoil A nuclei, with energy interval E_A to $E_A + dE_A$ produced by the particles ξ from the reaction A + B along their paths, is

$$g_{\xi A}(E_A) dE_A = dE_A \int_{E_\xi^0}^{E_A/K_{A\xi}} f_\xi(E_\xi^0) dE_\xi^0 \int_{E_\xi^0}^{E_A/K_{A\xi}} N_A \sigma_{\xi A}^{sc} \times (E_\xi, E_A) q(E_\xi^0, E_\xi) \frac{dE_\xi}{(dE_\xi/dx)} \quad (11)$$

$\sigma_{\xi A}^{sc}(E_\xi, E_A)$ is given by the scattering differential cross section $\sigma_{\xi A}^{sc}(E_\xi, \theta)$ and the relation between the angle of scattering and loss of energy in the collision.⁹

Summing up all the products of the A + B reaction we obtain:

$$g_A^{(1)}(E_A) = \sum_\xi g_{\xi A}(E_A). \quad (12)$$

We can write a similar expression for the energy distribution of recoil nuclei B. As a result of the elastic scattering of the first generation of A and B nuclei we obtain the second generation of recoil nuclei of the medium. With the help of the above considera-

tions we can write the energy distribution for the n th generation of recoil nuclei A:

$$g_A^{(n)}(E_A) = \int g_A^{(n-1)}(E'_A) dE'_A \int_{E'_A}^{E_A} N_A \sigma_{AA}^{sc}(E''_A, E_A) \times q(E'_A, E''_A) \frac{dE''_A}{(dE''_A/dx)} + \int g_B^{(n-1)}(E'_B) \times dE'_B \int_{E'_B}^{E_A/K_{AB}} N_B \sigma_{BA}^{sc}(E''_B, E_A) q(E'_B, E''_B) \frac{dE''_B}{(dE''_B/dx)}. \quad (13)$$

If we add the energy distributions of all generations we obtain the energy distribution of the whole cascade initiated by the particles from the reaction A + B:

$$G_A(E_A) = \sum_n g_A^{(n)}(E_A). \quad (14)$$

Having obtained the distributions $G_A(E_A)$ and, in a similar way, $G_B(E_B)$, we can give the number of A + B reactions in the slowing-down process of the cascade initiated by the particles from the one reaction A + B:

$$k = \int G_A(E_A) dE_A \int_{E_A}^0 N_B \sigma_{AB}(E'_A) q \times (E_A, E'_A) \frac{dE'_A}{(dE'_A/dx)} + \int G_B(E_B) dE_B \int_{E_B}^0 N_A \sigma_{BA}(E'_B) q \times (E_B, E'_B) \frac{dE'_B}{(dE'_B/dx)}. \quad (15)$$

If the reaction A + B has only one exoergic channel the number k is the multiplication factor for the given medium. The condition for the development of the avalanche, therefore, is $k > 1$.

In the numerical calculations, as long as the slowing down process of the products of reaction A + B and recoil nuclei is due to scattering from electrons, we can take into consideration only the first generation of recoil nuclei. Then from Eqs. (4), (5) and (15) we have:

a. in the case of charged products of the reaction A + B

$$k \sim \left[\sigma \left(\frac{dE}{dx} \right)_{el}^{sc} \right]^{-2} \sim \begin{cases} T^3 & \text{for plasma} \\ N_e^{-2} & \text{for degenerate medium} \end{cases} \quad (16)$$

b. in the case of neutrons

$$k \sim \left[\sigma \left(\frac{dE}{dx} \right)_{el}^{sc} \right]^{-1} \sim \begin{cases} T^{3/2} & \text{for plasma} \\ N_e & \text{for degenerate medium} \end{cases} \quad (17)$$

Critical Mass

All our present considerations concern the conditions for the development of fusion chain reactions in infinite media. In a finite medium the conditions are different and a critical mass exists as in a fission chain reaction.

In the first approximation, the critical mass can be estimated very easily if we consider that the mean-free path λ with respect to the elastic scattering of the particles taking part in the reaction has to be comparable with the dimension L of the system. If we denote by N the number of nuclei of a medium in a

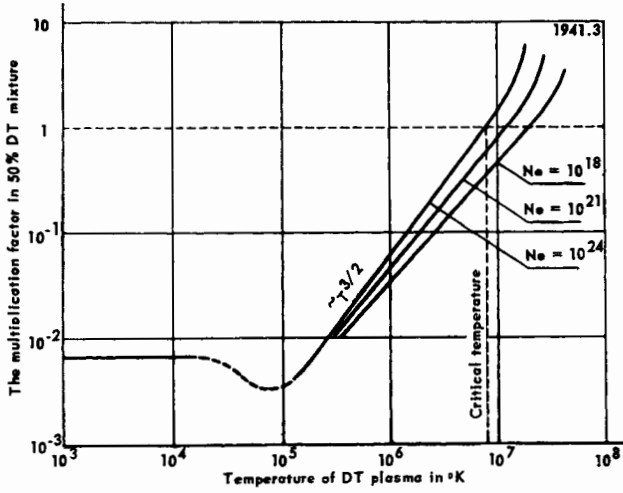


Figure 3. The multiplication factor in 50% D-T plasma as a function of temperature

unit volume, by m the mass of a nucleus in grams, and by $\langle\sigma\rangle^{\text{sc}}$ the cross section for elasting scattering, we obtain $m_{\text{cr}} \approx m_1 N_A^3 = m_1 (\langle\sigma\rangle^{\text{sc}})^{-3} N^{-2}$. Taking into account that $\langle\sigma\rangle^{\text{sc}} \approx 10^{-24} \text{ cm}^2$ and $m_1 \approx 10^{-24} \text{ g}$, the critical mass in grams is

$$m_{\text{cr}} \approx 10^{48}/N^2. \quad (18)$$

We see that the critical mass is very strongly dependent on the density of a medium. For densities $N = 10^{28}, 10^{24}, 10^{20}$ nuclei/cm³, the critical masses are $10^{-8}, 10^0, 10^8$ grams respectively.

NUMERICAL CALCULATIONS

Now, to illustrate the theory given above we shall determine the conditions for the development of a fusion chain reaction in a D-T mixture.

As a result of reaction $D + T$ we obtain alphas and neutrons with energies $\sim 3.5 \text{ Mev}$ and $\sim 14.1 \text{ Mev}$ respectively. Because of the much greater initial energy and much lower energy losses for the neutrons, most recoil nuclei D and T result from scattering of neutrons; therefore, according to Eq. (12), $g_D^{(1)}(E_D) \approx g_{nD}(E_D)$ and $g_T^{(1)}(E_T) \approx g_{nT}(E_T)$. Taking into account the fact that the absorption of fast and intermediate neutrons in the D-T medium is negligibly small we have $q_n \approx 1$. Assuming the scattering of neutrons from D and T nuclei isotropic in the center-of-mass system, we obtain

$$\begin{aligned} \sigma_{nD}^{\text{sc}}(E_n, E_D) &\approx \frac{\sigma_{nD}^{\text{sc}}(E_n)}{K_{nD}E_n}, \\ \sigma_{nT}^{\text{sc}}(E_n, E_T) &\approx \frac{\sigma_{nT}^{\text{sc}}(E_n)}{K_{nT}E_n}. \end{aligned} \quad (19)$$

Because, in the first approximation, the slowing down of neutrons is due to elastic scattering from D and T nuclei

$$dE_n/dx \approx \frac{1}{2} K_{nD} E_n \sigma_{nD}^{\text{sc}} N_D + \frac{1}{2} K_{nT} E_n \sigma_{nT}^{\text{sc}} N_T \quad (20)$$

the energy distributions of the first generation of recoil nuclei are, respectively:

$$g_{nD}(E_D) \approx \int_{14.1}^{E_D/K_{nD}} \frac{dE_n}{(K_{nD}E_n)^2} \times \frac{1}{1 + \sigma_{nT}^{\text{sc}} K_{nT} N_T / \sigma_{nD}^{\text{sc}} K_{nD} N_D} \quad (21a)$$

$$g_{nT}(E_T) \approx \int_{14.1}^{E_T/K_{nT}} \frac{dE_n}{(K_{nT}E_n)^2} \times \frac{1}{1 + \sigma_{nD}^{\text{sc}} K_{nD} N_D / \sigma_{nT}^{\text{sc}} K_{nT} N_T}. \quad (21b)$$

Since the energy losses of deuterons and tritons up to 10^7 °K, in the case of a plasma, and up to 10^3 g/cc , in the case of a degenerate medium, are due to scattering from electrons we can write

$$\begin{aligned} k &\approx \frac{N_D}{N_D + N_T} \int g_{nD}(E_D) dE_D \int_{E_D}^0 \frac{\sigma_{DT}}{\sigma \left(\frac{dE'_D}{dx} \right)_{\text{el}}^{\text{sc}}} dE'_D \\ &+ \frac{N_T}{N_D + N_T} \int g_{nT}(E_T) dE_T \int \frac{\sigma_{TD}}{\sigma \left(\frac{dE'_T}{dx} \right)_{\text{el}}^{\text{sd}}} dE'_T \end{aligned} \quad (22)$$

where $\sigma \left(\frac{dE_D}{dx} \right)_{\text{el}}^{\text{sc}}$ and $\sigma \left(\frac{dE_T}{dx} \right)_{\text{el}}^{\text{sc}}$ are given by Eq. (2).

The value of the multiplication factor obtained by the numerical calculations for a 50% D-T mixture under various conditions are plotted in Figs. 3 and 4. The cross sections we have put equal to the geometrical cross section and σ_{DT} is taken from Bame and Perry.¹⁰

CONCLUSIONS

The role of the *in statu nascendi* reactions in the release of nuclear energy depends on the physical conditions and, in the case of high temperature or high density, they are decisive. If we denote by E_{th} the energy released in a unit volume in the thermo-

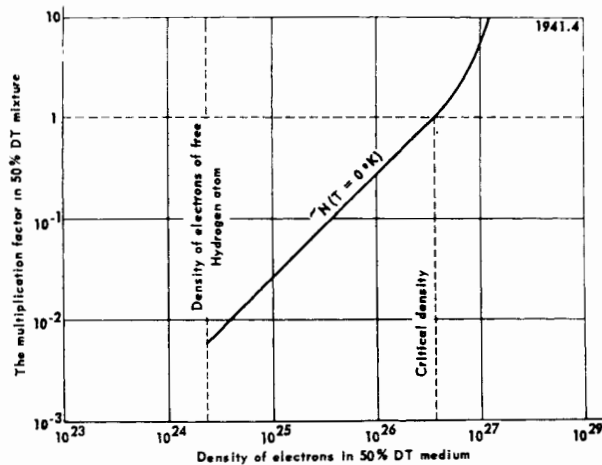


Figure 4. The multiplication factor in 50% D-T medium as a function of density for $T = 0^\circ\text{K}$

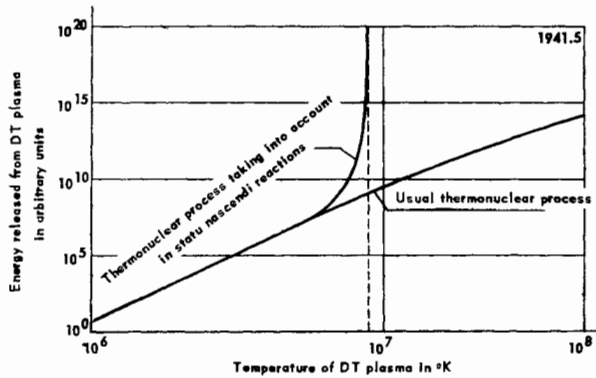


Figure 5. The release of energy in exoergic mixture for the usual thermonuclear process, and for a process taking into account *in statu nascendi* reactions

nuclear process, then the energy released in a unit volume with respect to the *in statu nascendi* reactions is

$$E_{\text{tot}} = E_{\text{th}} / (1 - k(T, N)),$$

where k is the multiplication factor for the given medium. The factor k depends on the temperature of the medium—or, more accurately, on the temperature of its electrons—and on its density. As the multiplication factor approaches unity, the process of energy release has an avalanche character and the entire nuclear energy of an exoergic mixture is released instantaneously. The stationary state for a slow release of energy does not exist above the critical temperature or above the critical density; even at a temperature of absolute zero the exoergic mixture is explosive.

A plot of E_{th} (Ref. 11) and E_{tot} as a function of temperature for 50% D-T mixture is given in Fig. 5.

APPENDIX

As was pointed out previously,⁴ the maximum impact parameter is, in general, a function of the velocities of interacting particles, their masses and charges, as well as of the external fields. In each problem this parameter must be determined separately.

In the case of electrons bound in atoms, or Fermi gas electrons, the determination of the maximum impact parameter does not present any difficulty, but in the case of plasma electrons it is the subject of many discussions. According to Cowling,¹² Chandrasekhar¹³ and others, it is suitable to put the maximum impact parameter equal to the mean distance between the ions, but according to Landau,¹⁴ Cohen, Spitzer and Routly¹⁵ and others, it must be equal to the Debye radius.

From Eq. (2) it follows at once that in the limiting case $V_{\xi} \ll V_e$ the atomic stopping cross section is independent of the assumed value of D_{max} .

To determine the maximum impact parameter in the second limiting case, $V_e \ll V_{\xi}$, we must take into account the fact that the charged particles of the plasma are interacting with each other. Consider two particles with charges $+Ze$ and $-Ze$ and the distance r between them: the Coulomb force between them is $(Ze/r)^2$. The transfer of momentum to such a binary system from a particle ξ is negligibly small when the force of interaction between the particle ξ and each particle of the system is less than the force of internal interaction, or: $Z^2(e/r)^2 \gtrsim ZZ_{\xi}(e/D)^2$. Taking into account the mean value of distance between charged particles in the plasma we finally obtain $D_{\text{max}} \approx N^{-1/2}$.

The assumption that the maximum impact parameter is equal to the Debye radius will slightly change the numerical results owing to the logarithmic dependence on D_{max} .

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Theory of the Energy Development in a Thermonuclear Plasma of Deuterium and of Deuterium-Tritium

By Mrs. D. Magnac-Valette, E. Lacombe and P. Cuer *

We have studied the evolution of a thermonuclear mixture for concentrations of 10^{18} and 10^{17} ions per cm^3 and for temperatures of $(116 \text{ and } 1160) \times 10^6 \text{K}$, taking into account the contribution from secondary reactions. It is assumed that no deuterium or tritium are supplied during the evolution time. The temperature is assumed constant and the pinching of the charged species perfect. Neutrons leave the mixture without causing any secondary reactions because of their long mean free path.

We denote the different concentrations of

$$\begin{array}{cccccc} \text{D} & \text{He}^4 & \text{n} & \text{p} & \text{T} & \text{He}^4 \\ \text{by } x & y & z & u & v & w \end{array}$$

and the coefficients $\langle \sigma v \rangle$ for the different possible reactions are tabulated below (Table 1).

Table 1

Reactions	$\langle \sigma v \rangle$
$\text{D} + \text{D} \rightarrow \text{p} + \text{T} + 4.03 \text{ Mev}$	
$\text{D} + \text{D} \rightarrow \text{n} + \text{He}^3 + 3.25 \text{ Mev}$	
$\text{D} + \text{T} \rightarrow \text{He}^4 + \text{n} + 17.58 \text{ Mev}$	
$\text{D} + \text{He}^3 \rightarrow \text{He}^4 + \text{p} + 18.34 \text{ Mev}$	
$\text{T} + \text{T} \rightarrow \text{He}^4 + 2\text{n} + 11.32 \text{ Mev}$	

The evolution of the mixture proceeds according to the following differential equations

$$\begin{aligned} \dot{x} &= -\beta x^2 - \gamma x^2 - \alpha vx - \delta xy \\ \dot{y} &= \frac{1}{2}\beta x^2 - \delta xy \\ \dot{z} &= \frac{1}{2}\beta x^2 + \alpha vx + \epsilon v^2 \\ \dot{n} &= \frac{1}{2}\gamma x^2 + \delta xy \\ \dot{v} &= \frac{1}{2}\gamma x^2 + \alpha vx - \epsilon v^2 \\ \dot{w} &= \alpha vx + xy + \frac{1}{2}\epsilon v^2. \end{aligned}$$

This set of equations can be reduced since u , w , z represent inactive reaction products which are functions of x , y , v and because the conservation of mass and charge give us two other equations

$$\begin{aligned} 2\dot{x} + 3\dot{y} + \dot{z} + \dot{u} + 3\dot{v} + 4\dot{w} &= 0 \\ \dot{x} + 2\dot{y} + \dot{u} + \dot{v} + 2\dot{w} &= 0. \end{aligned}$$

Integration of these equations yields the total nuclear power in the plasma, $P_t (\text{Mev/sec cm}^3)$, and the power carried off by the neutrons, P_n . The calculations were made with the aid of an electronic computer. The initial concentration is unimportant since a scaling of the concentrations changes nothing if the inverse scaling is carried out on the time variable. The initial conditions which have been considered are given in Table 2. It is assumed, moreover, that the thermalizing time is negligible in comparison with the mean life of the particles.

Table 2. Initial Conditions

Case	$T(^{\circ}\text{K})$	$x_0(\text{cm}^{-3})$	$v_0(\text{cm}^{-3})$
A	116×10^6	10^{18}	0
B	1160×10^6	10^{17}	0
C	116×10^6	10^{18}	10^{17}
D	175×10^6	10^{18}	0

The calculations were performed using a successive approximation method in which the mesh size was selected such that the error was always less than 10^{-6} . The results are summarized below and in Figs. 1 and 2 for the four cases given in Table 2.

I. On the curves marked A ($T = 116 \times 10^6 \text{K}$, $x_0 = 10^{18} \text{ cm}^{-3}$), we note that the primary reactions $\text{D}(\text{D}, \text{p})\text{He}^3$ and the secondary reactions $\text{D} + \text{T}$ make equivalent contributions to P_t , but that the secondary reaction $\text{D} + \text{He}^3$ does not exceed that of $\text{D}(\text{D}, \text{n})$ until after $t = 0.61 \text{ sec.}$; P_n being always greater than P_t . The threshold, as determined from the radiated power lost in the plasma and calculated according to Post,^{1, 2} is not surpassed. Therefore, it seems that a thermonuclear reactor cannot supply energy even at this high temperature.

II. We have performed the calculation for $T = 175 \times 10^6 \text{K}$ in order to compare our results with those of Lawson.³ According to Lawson the threshold is surpassed at this temperature for $P_t = P_b$. However, our numerical results do not agree with this conclusion. We performed the calculation several times and restricted ourselves to the evolution during the first 35 msec.

* Particle Physics Laboratory, Strasbourg.

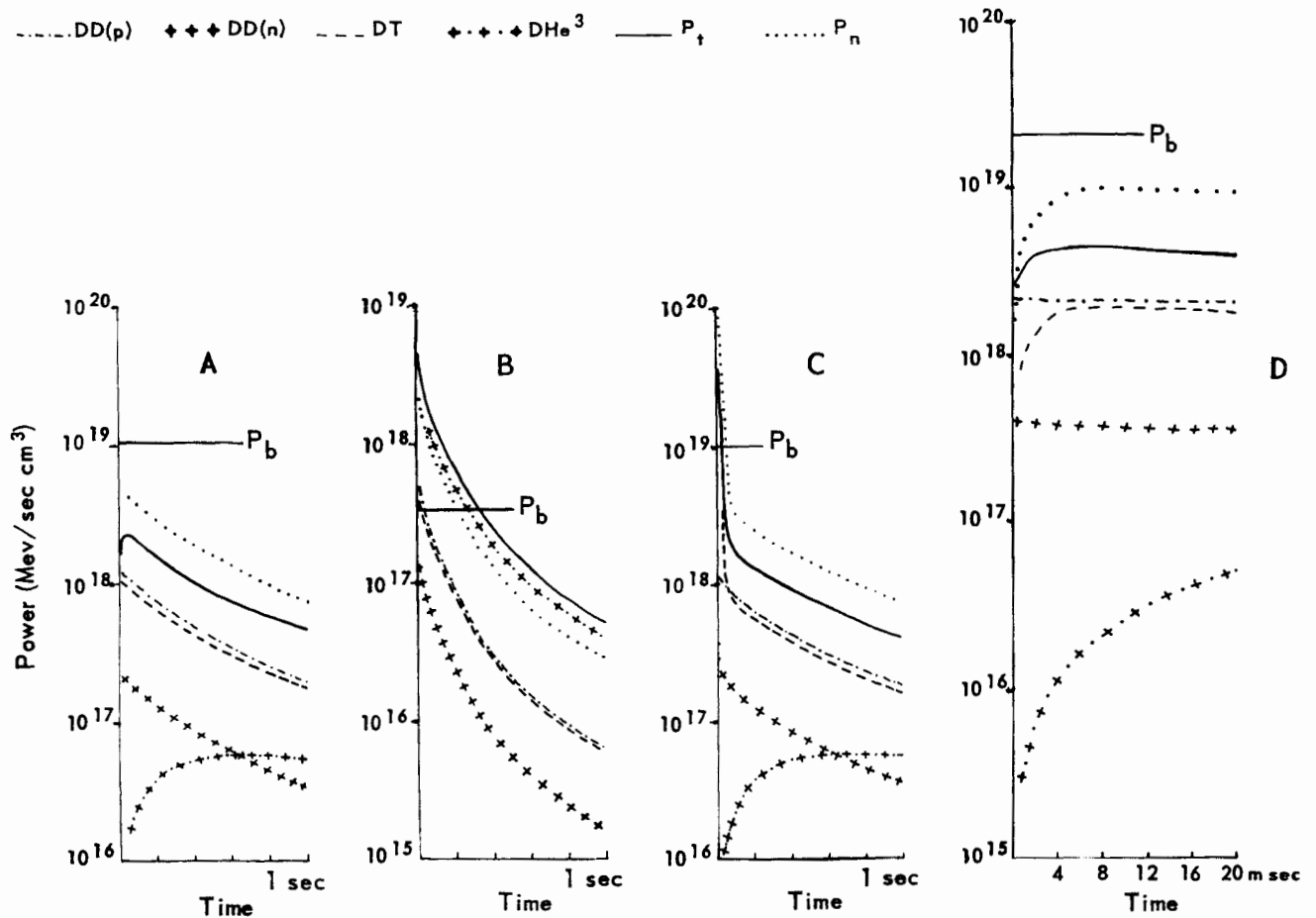


Figure 1

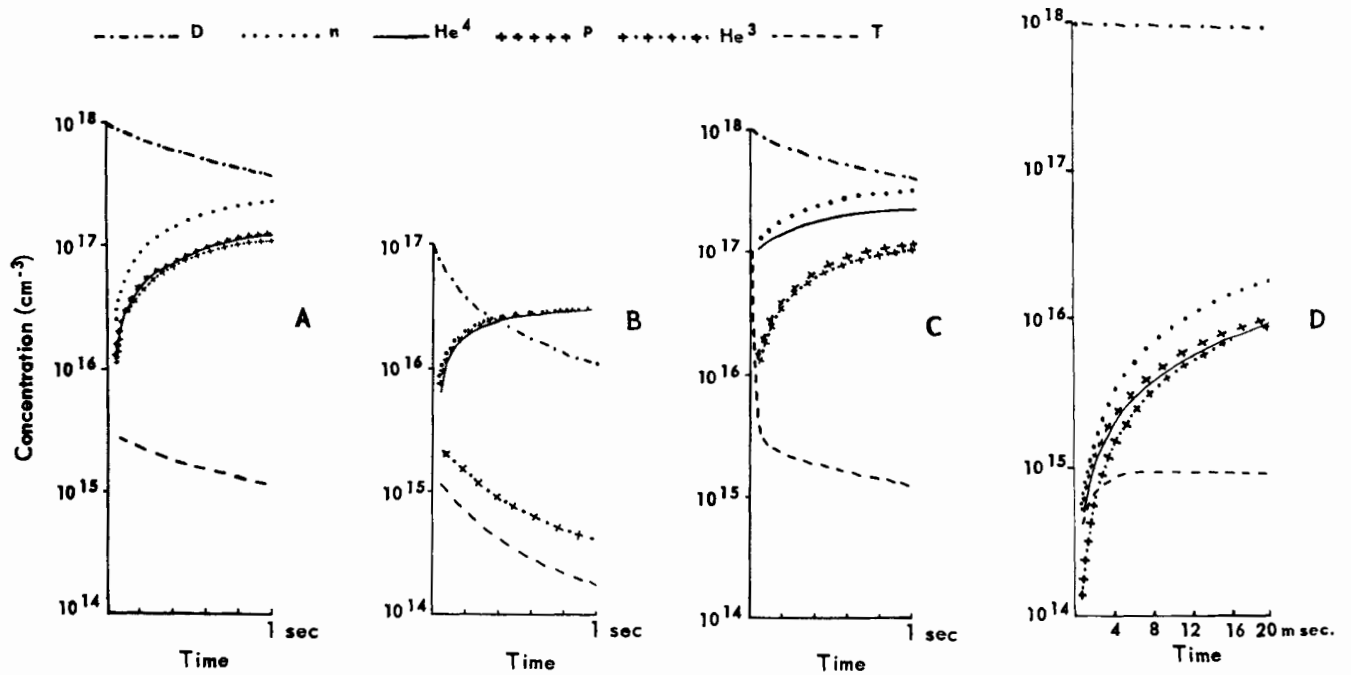


Figure 2

Moreover, according to Ruark⁴ the threshold is displaced to somewhere around 400×10^6 K. Generally, it can be said that at these temperatures it is the triton which reacts and not helium 3, the temperature still being too low.

III. We also investigated the possibility of surpassing the threshold P_b by adding to the deuterium an amount of tritium determined from the initial conditions by making $y_0/x_0 = 0.1$ as shown in example C ($T = 116 \times 10^6$ K, $x_0 = 10^{18}$ cm³, $v_0 = 10^{17}$ cm³). The reaction $T + T$ is cited although it does not play an important role. While the tritium is being used up, the threshold is surpassed at 14 msec, as is shown rather badly on the power graphs but clearly on the tables of numerical results. Afterwards, the development is similar to that in case A. It would appear, therefore, that the reinjection of tritium would be interesting from all points of view since the charge ($Z = 1$) of this reaction is low and, therefore, the radiation product losses are barely increased.

IV. Finally we studied a pure deuterium plasma in case B ($T = 1160 \times 10^6$ K and $x_0 = 10^{17}$); the threshold temperature lies between A and C. Here the threshold is surpassed because of the $D D(p)$ and $D T$ reactions during the first 6 msec and also due to the $D He^3$ reaction which becomes paramount and plays the role of a relay which prolongs the life

(of the reactor) to 0.31 sec. This can be explained through the contribution to the 430 Mev resonance by a significant portion of the Maxwell distribution and because a large fraction of the reaction energy (18.34 Mev) is completely utilized (no emission of neutrons). However, the introduction of He^3 into the plasma would be less interesting than the addition of tritium because $Z = 2$ and the radiation losses would be substantially increased.

In conclusion, we note that one must be cautious in the calculation which we have done here with the help of Post's formula. If we use the results of Thomson and Redhead,⁵ P_b is doubled and this modifies the life of the reactor but does not alter the general conclusions.

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Record of Proceedings of Session A-5

Theoretical Aspects of Plasma Physics

WEDNESDAY MORNING, 3 SEPTEMBER 1958

Chairman: Mr. I. E. Tamm (USSR)

Vice-Chairman: Mr. L. Simons (Finland)

Scientific Secretaries: Messrs. A. Salaam and A. Sabry

PROGRAMME

- P/365 Stability of a plasma in static equilibrium M. D. Kruskal and C. R. Oberman
(Presented by Mr. M. D. Kruskal, who also included material from paper P/349 in this volume.)
- P/2214 Dynamics of a rarefied plasma in a magnetic field R. Z. Sagdeev *et al.*
(Presented by Mr. R. Z. Sagdeev.)
- P/347 Stability and heating in the pinch effect M. N. Rosenbluth
(In his oral presentation of this paper, the author also included material from papers P/354, P/1861 and P/376 in this volume.)
- P/33 Stability of a constricted gas discharge R. T. Tayler

DISCUSSION

- P/374 Hydromagnetic shock waves in high-temperature plasmas C. S. Gardner *et al.*
(Presented by Mr. H. Grad.)
- P/1 Hydromagnetic disturbances of large amplitude in a plasma J. E. Allen
- P/1307 Plasma oscillations in a stationary external magnetic field S. Ciulli and M. Micu
(Presented by Mr. S. Ciulli.)
- P/2213 Plasma radiation in a magnetic field..... B. A. Trubnikov and V. S. Kudryavtsev
(Presented by Mr. H. A. Leontovich.)
- P/2215 Absorption of high-frequency electromagnetic energy in a high temperature plasma R. Z. Sagdeev and V. D. Shafranov
(Presented by Mr. V. D. Shafranov.)
- P/2300 High frequency plasma oscillations A. Akhiezer *et al.*
(Presented by Mr. Y. B. Fainberg.)
- P/2292 Runaway electrons H. Dreicer

DISCUSSION

Mr. I. E. TAMM (USSR): The first projects for creating controlled thermonuclear reactions that arose in the various countries were relatively simple in principle. However the approximate theory on which these projects were based has proven to be completely inadequate in practice. We have been obliged to fall back on calculations of a vast number of corrections and effects of the second or third order of small-

ness. Most of the work to be reported in this session is devoted to the solution of this kind of problem, without which further progress is impossible.

DISCUSSION OF P/365, P/2214, P/347 AND P/33

Mr. TAMM (USSR): There is now some time for discussion. Since questions have not been forthcoming, and since there are close relations between

these papers and the ones following—e.g. Dr. Rosenbluth's, and those of Shafranov and others—we shall add this time to the next discussion.

DISCUSSION OF P/347, P/1, P/1307, P/2213, P/2215, P/2300 AND P/2292

Mr. P. HUBERT (France): Dr. Rosenbluth, has the United States practically realized a stabilized pinch in which the magnetic field outside is opposite to the magnetic field inside?

Mr. M. N. ROSENBLUTH (USA): Experiments on this are just beginning. I believe that we have had difficulties for other reasons, probably due to the unstable plasma oscillations. I should mention that the instabilities which are cured by having the external reverse longitudinal field are actually very slow instabilities, and therefore one has to have a quite well stabilized plasma for these instabilities even to show up; therefore we have not yet been able to cure them.

Mr. S. CIULLI (Romania): I should like to ask Dr. Rosenbluth, what is the order of magnitude of the oscillations of the electric field caused by the ion-electron separation?

Mr. ROSENBLUTH (USA): I do not quite understand the question. If you mean the unstable plasma oscillations, this would seem to depend in great detail on the distribution functions of the different particles. In particular, if there is any tendency to form beams of electrons one gets very rapid instabilities.

Mr. B. KURŞUNOĞLU (Turkey): Mr. Leontovich, I should like to ask how you calculated the bremsstrahlung. Have you used the Born approximation? What kind of plasma potential did you use?

Mr. TAMM (USSR): The main point of the paper of Mr. Leontovich, which I should like to reformulate, is the calculation of the intensity of the magnetic radiation. One must take account not only of magnetic radiation at the gyro-magnetic frequency, but at the higher harmonics as well. If the magnetic field is of such a magnitude that the magnetic pressure is equal to the pressure of the gas, magnetic radiation at the higher harmonics becomes most important, making the dimensions of any working reactor of the order of tens of thousands of meters.

Mr. F. CAP (Austria): Do you see from your theory, Mr. Leontovich, any possibilities of reducing the radiation loss of a plasma?

Mr. A. LEONTOVICH (USSR): Yes, for example, by reflection, evidently.

Mr. R. F. POST (USA): I should like to ask Mr. Leontovich whether he has calculated the effect of the reflections from the walls suggested by Budker. Also, is the absorption coefficient calculated by detailed balance or another method?

Mr. LEONTOVICH (USSR): We have estimated the reduction in magnetic-radiation losses that can be achieved with reflecting apparatus. It is hardly convenient to reproduce these estimates here, since

they depend on the practical matter of what the coefficients of reflection can be for the materials and operating conditions. This depends on the temperature as well as on the materials. The results are sensitive to the value of the reflection coefficient. In contrast to what appears at first glance, the appraisal shows that while the losses can be decreased, it is hardly possible to gain an order of magnitude.

The non-uniformity of the magnetic field leads to a broadening of all the absorption curves. Therefore, it is obvious that in the principal range of radiation, the absorption curve is rather monotone, without sharp maxima.

It should be noted that the radiation losses are material only if the magnetic field penetrates the plasma. For example, if the magnetic field is that of a current with a pronounced skin-effect, the radiation output will be small.

The numerical calculations expressed by the graphs relate only to the special cases we indicated. It is difficult to give a general formula.

Mr. S. A. COLGATE (USA): Dr. Allen, you state no assumption of electron adiabaticity but assume a drift $\mathbf{v} = \mathbf{E} \times \mathbf{B}$. Does not this drift apply only for adiabatic electrons?

Mr. J. E. ALLEN (UK): No such assumptions were made. The equations used were Newton's equation of motion for the individual electrons. In fact, the electron motion is not adiabatic, in the sense that the energy is not proportional to the magnetic field, for example. Nor are the electrons tied to lines of force; rather, they move with respect to the lines of force.

Mr. COLGATE (USA): But then $\mathbf{E} \times \mathbf{B}$ is not the drift velocity, unless the electrons are unaccelerated.

Mr. ALLEN (UK): We have not used any concept of drift velocity at all.

Mr. POST (USA): I should be grateful, Dr. Fainberg, if you would please discuss whether there is a threshold of particle density for the 10^9 -fold enhanced Cherenkov radiation?

Mr. S. M. FAIBERG (USSR): If it is a matter of electron oscillations, then there really is a threshold, since for densities of order 10^{13} or 10^{14} cm^{-3} , say, the degree of division of the waves is very small. The dimensions of a bunch must be of the order of a millimeter and less to fulfil the coherency conditions. Therefore it is necessary to make very short bunches. However, this difficulty can be evaded by going to longer wavelengths, to ion-acoustic oscillations or other ion oscillations.

Mr. COLGATE (USA): Mr. Sagdeev, have you considered the possibility of non-adiabatic electrons in extremely strong shock transitions?

Mr. R. Z. SAGDEEV (USSR): I have considered the limiting case in which the shock width and the wavelengths greatly exceed the Larmor radius of the electrons, since the possibility of a shock wave is easiest to understand in this case.

Mr. B. D. NAG CHAUDHURI (India): Can Mr. Shafranov give us any information on the Cherenkov

absorption coefficient as a function of the frequency of the incident electromagnetic radiation?

Mr. V. D. SHAFRANOV (USSR): The Cherenkov absorption of electromagnetic waves by electrons is considered in my article in the *J. Exptl. Theor. Phys.*, No. 6, 1958. The Cherenkov absorption of magneto-

hydrodynamic waves by ions was considered by Braginskii and Kazantsev in *Fizika Plazmy i Problema Upravlyaemykh Termoyadernykh Reaktsii*, 4, 24 (Moscow, 1958). This question was also considered at the Kharkov Institute by Stepanov. It is discussed in the paper of Fainberg in these Proceedings.

Session A-6

EXPERIMENTAL ASPECTS OF PLASMA PHYSICS

LIST OF PAPERS

	<i>Page</i>
P/360 Ion cyclotron resonance T. H. Stix and R. W. Palladino	282
P/2211 Ion cyclotron oscillations and new injection methods L. V. Dubovoi <i>et al.</i>	288
P/2536 Electron cyclotron resonance, plasma wave guides and plasmoids K. D. Sinelnikov <i>et al.</i>	292
P/344 The Oak Ridge thermonuclear experiment C. F. Barnett <i>et al.</i>	298
P/1790 Intense gaseous discharges J. S. Luce	305
P/2396 Effect of an applied pressure gradient on a magnetically collimated arc R. V. Neidigh and C. H. Weaver	315
P/2383 Theoretical and experimental discussion of Ixion, a possible thermonuclear device K. Boyer <i>et al.</i>	319
P/1708 Liquid conductor model of instabilities in a pinched discharge A. Dattner <i>et al.</i>	325
P/345 Magnetic compression of shock preheated deuterium A. C. Kolb	328
P/355 Acceleration of plasma into vacuum J. Marshall	341
P/2301 High current pulse discharges A. M. Andrianov <i>et al.</i>	348
P/1460 Studies of high-current gas discharges at high rates of build up.....S. C. Curran <i>et al.</i>	365
P/2302 Development of a powerful discharge in deuterium V. S. Komelkov <i>et al.</i>	374
P/2504 Plasma motion in powerful discharges V. S. Komelkov	382

Ion Cyclotron Resonance

By T. H. Stix and R. W. Palladino *

Calculations and experiments have been made on the ion cyclotron resonance phenomenon in a fully ionized gas. One is especially interested in this for directly heating the ions in a plasma. In ion cyclotron resonance (Fig. 1), an ion is in a magnetic field and a transverse electric field which oscillates at the ion cyclotron frequency. The ion velocity changes direction every half period, but so does the applied electric field and they remain in phase with each other. Energy is fed into the ion motion, and the ion will move in a growing spiral orbit.

If now we try to induce such a transverse electric field in a plasma, one asks whether some electron skin effect might prevent the penetration of the applied electric field into the plasma. However, the electrons are tied to the magnetic lines of force and they cannot carry a skin current. Thus, it turns out that if the plasma density is sufficiently low, an applied electric field will penetrate completely into the plasma and,

at resonance, each ion will move around a line of force in a tiny growing spiral orbit. In this manner, a low density plasma can be heated very effectively.

At higher densities, say 10^{13} ions/cm³, this single particle picture is no longer valid. Ion currents induce an electric field inside the plasma which has a decelerating effect on the ions, and heating a moderately dense plasma at the ion cyclotron frequency

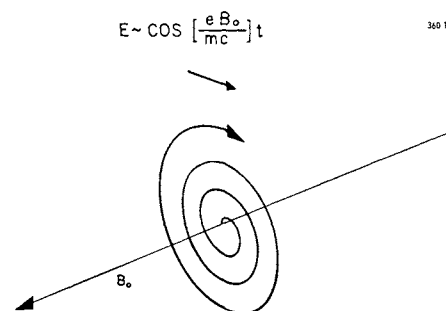


Figure 1. Acceleration of an ion at the ion cyclotron frequency

* Project Matterhorn, Princeton University, Princeton, New Jersey.

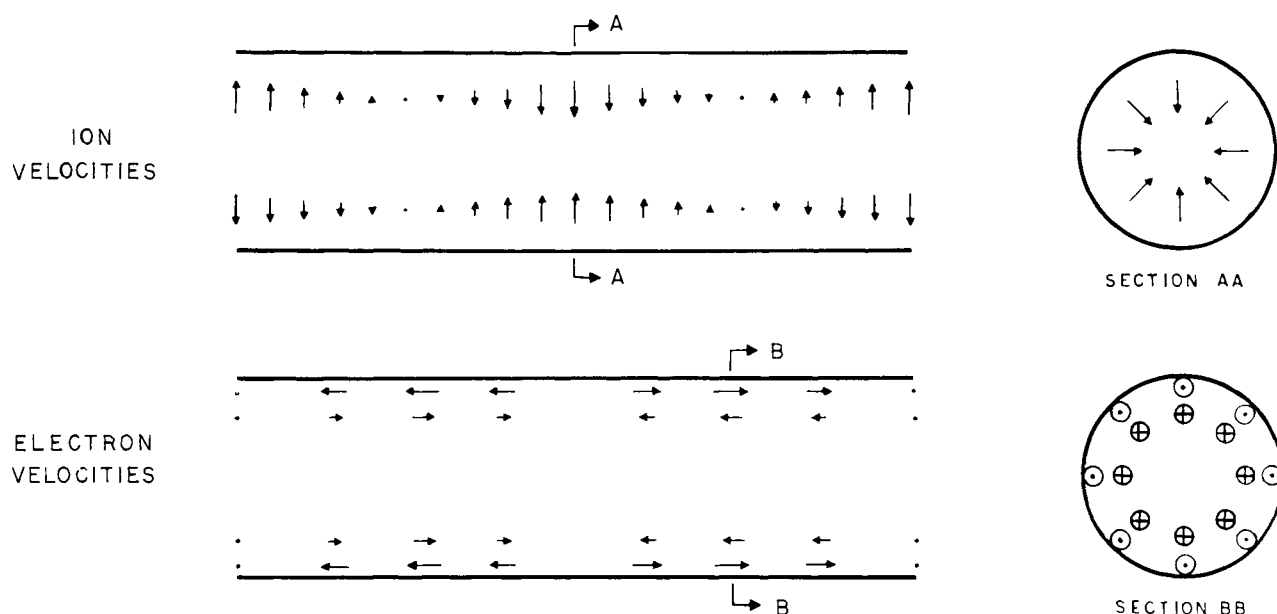
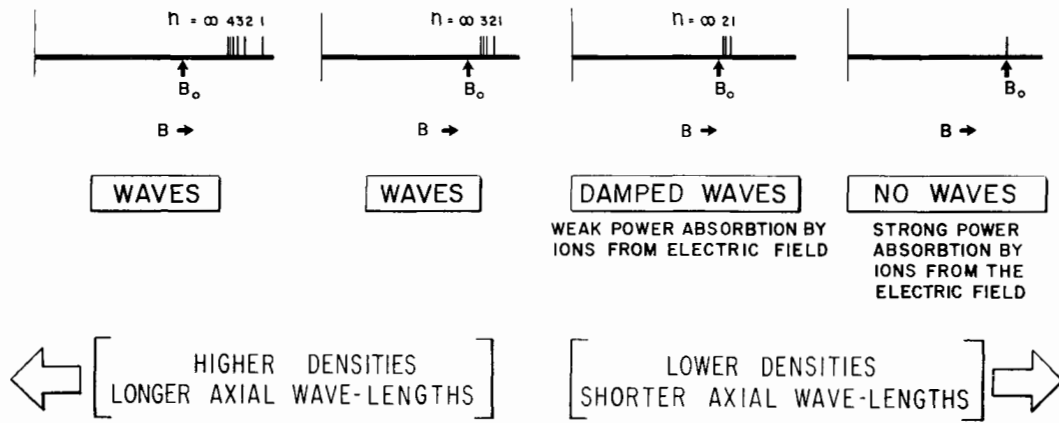


Figure 2. Ion cyclotron waves in a cylindrical plasma of uniform density surrounded by a vacuum in an axial magnetic field



$$2\pi f = \frac{eB_0}{mc}$$

n = radial mode number

Figure 3. Spectra for ion cyclotron waves

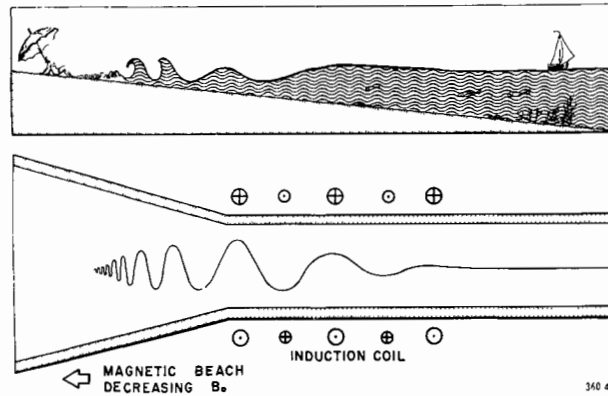


Figure 4. A "magnetic beach" for the thermalization of ion cyclotron waves

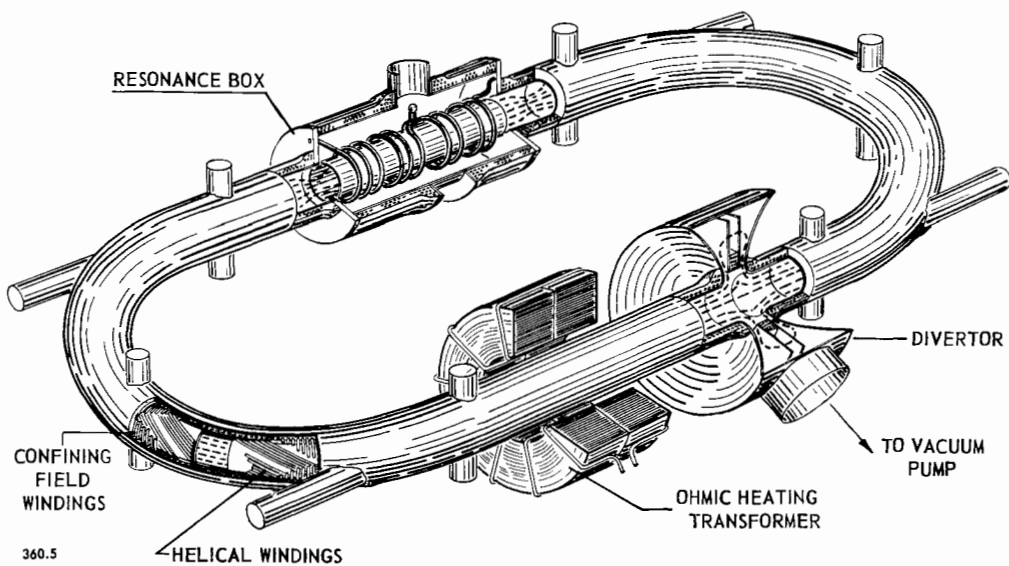


Figure 5. B-65 stellarator

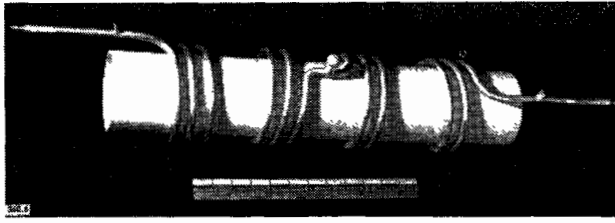


Figure 6. Induction coil

becomes very inefficient. However, if one keeps the frequency fixed, but goes to a slightly stronger magnetic field, one finds the existence of natural oscillations of the plasma. These natural oscillations, which we call "ion cyclotron waves", are in fact the short-wave, low-density limit for the extraordinary hydro-magnetic waves described by Alfvén and Åström. Figure 2 illustrates some of the standing wave patterns for these "ion cyclotron waves"; a side view and a cross section of a cylindrical plasma at the same instant of time are shown. The plasma is assumed to have a uniform density, to be surrounded by vacuum and to be in a strong axial magnetic field. Individual ions move in circles around the lines of force, but the phases and amplitudes of the ion velocities vary sinusoidally both in time and in space.

The ion flow has a non-zero divergence, which ordinarily would produce a large space charge and would result in large electric fields. However, since the ion flow pattern is periodic in the axial direction, the electrons are able to flow *along* the lines of force and neutralize the ion space charge. This axial flow of electrons is also illustrated in Fig. 2. Later on, in the

experimental work, we will find evidence that the flow of this neutralizing electron current along the lines of force heats the gas.

Figure 3 shows the spectra for ion cyclotron waves. The resonances are indicated as lines and are plotted against magnetic field strength. We look first at the picture denoted by "no waves". Here we have the lowest densities and, as a consequence, single particle acceleration of the ions in the electric field. The ions move in growing spiral orbits and there is strong power absorption by the ions from the electric field. There are no waves at this very low density condition.

For moderately high densities (see Fig. 3) there are ion cyclotron waves. The different lines indicate the different radial mode numbers and, for the high-density condition, all of the modes occur at magnetic fields considerably stronger than the cyclotron field, B_0 . Once the ion cyclotron wave has been set up, the ions do not absorb energy from the electric fields. Therefore there is no damping of the wave.

The second picture in Fig. 3 shows the spectrum for ion cyclotron waves for somewhat lower densities. The various modes now occur at magnetic field values which are somewhat closer to the cyclotron field, B_0 . There is still no damping of the waves. Now in the third picture, the density is quite low and the modes now occur at magnetic fields very close to the cyclotron field, B_0 . This is an intermediate case between the ion cyclotron wave resonance and the single particle resonance. For this condition we have waves, but the ions begin to move in growing spiral orbits and pick up energy from the electric field. In this case we have ion cyclotron waves which are slightly damped.

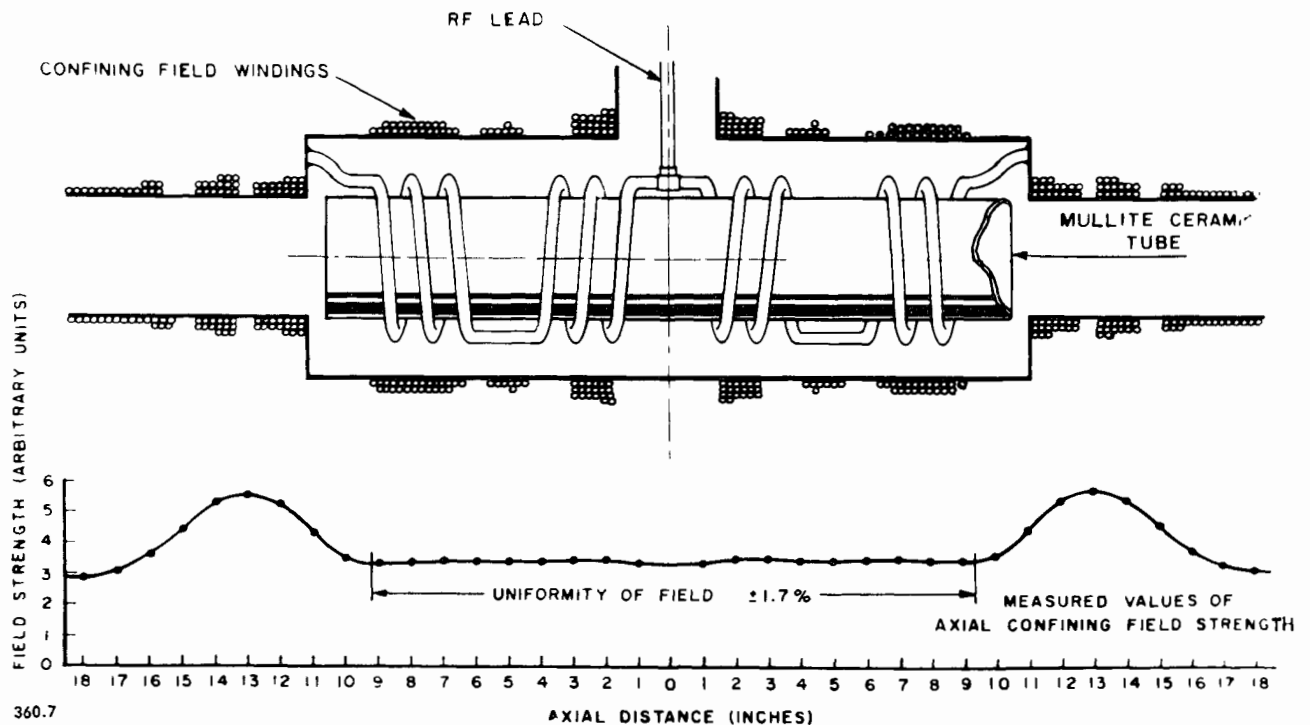


Figure 7. Resonance box, induction coil, field windings and confining field distribution

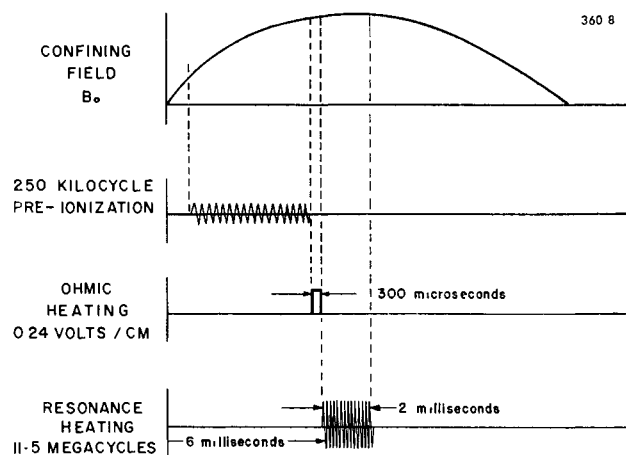


Figure 8. Time sequence for operation of the stellarator

The transition from the undamped waves to the slightly damped waves is particularly interesting because it provides a method for thermalizing the ion cyclotron waves; that is, for transforming the wave energy quickly into heat. It turns out that it is equally valid to talk about wavelengths instead of about densities, and it is the short wavelengths which are damped. To illustrate a plasma heating scheme, we make an analogy with ocean waves running up on a beach. A cross section of an ocean beach is shown in Fig. 4. The waves are moving in toward the shore and, because the water is getting shallower and shallower, the wavelength of the ocean waves becomes shorter and shorter. Finally, in the shallow water, the waves are unable to propagate and they "break"; the wave energy is transformed into heat. Similarly, one might use an induction coil to generate rather long wavelength ion cyclotron waves and allow these waves to propagate along the magnetic lines of force through a region where the magnetic field tapers off and becomes somewhat weaker. The wavelength becomes shorter and shorter and, finally, when the wavelength is sufficiently short the waves damp out and the wave energy goes into heating the ions. It would be in this magnetic "beach" region where the real heating of the gas would take place.

EXPERIMENTAL

The ion cyclotron experiments were performed on the B-65 stellarator. A drawing of this machine is shown in Fig. 5. It is in the shape of a torus or race-track in which toroidal confining magnetic fields up to 20,000 gauss can be produced. It is equipped with a divertor and also with helical windings to give a rotational transform to the confining magnetic field. The large iron-core transformer is used to induce an ohmic heating current which flows along the magnetic lines of force. The resonance box on the far leg of the machine contains the induction coil for the resonance heating. A photograph of the induction coil is presented in Fig. 6. The coil is made up of four sections in which current flows in opposite directions

in adjacent sections of the coil. The coil is thus two wavelengths long and this establishes the wavelength of the ion cyclotron wave which we are trying to generate.

The resonance box is shown in more detail in Fig. 7. The induction coil surrounds a ceramic tube which is 53 cm long. The high voltage rf connection is brought into the center of the coil and the two ends of the coil are grounded. The windings for the confining magnetic field, which is uniform to within $\pm 1.7\%$ over the length of the coil can be seen in Fig. 7. Also, at the bottom of Fig. 7 is a plot of the magnetic field intensity. Beyond each end of the coil there are magnetic mirror fields with a 3-2 mirror ratio. We have not yet installed the magnetic "beach" geometry on a stellarator.

The time sequence of the operations is depicted in Fig. 8. First the confining field rises and reaches a peak in about six milliseconds. While it is rising, an rf electric field directed along the magnetic lines of force breaks down the gas. Ohmic heating is turned on for a time just long enough to reach full ionization. The ohmic heating is then turned off and the resonance heating is turned on for two milliseconds (a diagram of the resonant heating circuit is shown in Fig. 9). The inductance of the coil is resonated with capacitors, and measurements are made of the input voltage and the input current. The phase angle between the input voltage and input current was obtained from Lissajous figures.

The plasma loading versus the strength of the confining magnetic field given in Fig. 10. The plasma loading graph is actually a plot of the input impedance to the parallel resonant circuit, but we have calibrated it in terms of the ratio, W , which is the rf power which goes into the plasma divided by the rf power which would be lost in the resistance of the induction coil if the induction coil had a Q of 300. The upper graph shows the plasma loading with an input power of about 300 watts. Although the working gas is deuterium, hydrogen comes into the discharge as an impurity and there is some loading in the vicinity of the hydrogen cyclotron field. The vertical line in Fig. 8 indicates the cyclotron field value for the hydrogen single particle resonance, obtained from an independent calibration of the magnetic field. At higher values of the confining magnetic field there appears strong plasma loading in the vicinity of the deuterium cyclotron resonance. Again, a vertical line indicates the value for the deuterium single particle resonance. Of particular interest is the asymmetry of the loading

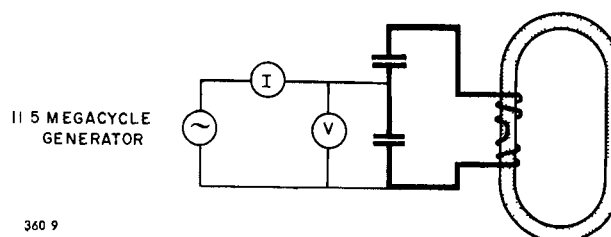


Figure 9. Diagram of resonant heating circuit

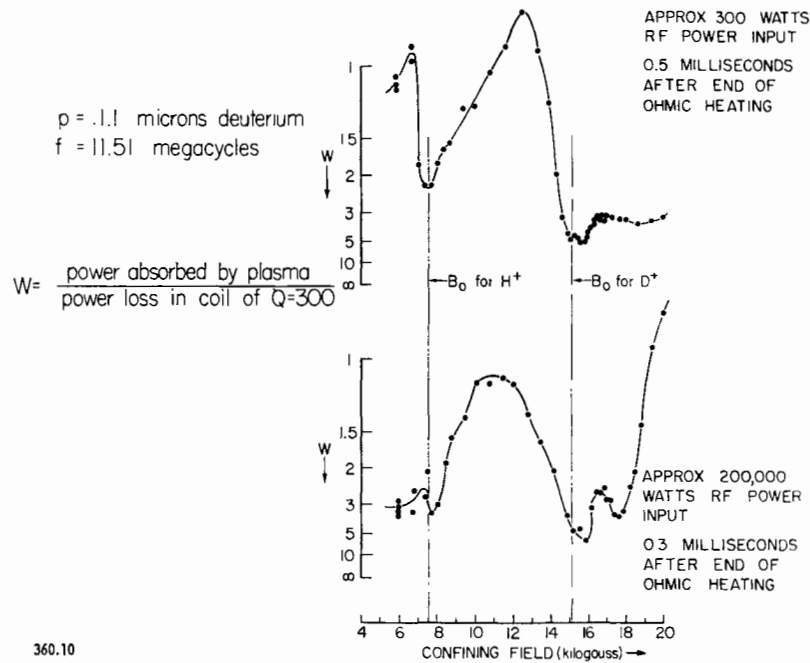


Figure 10. Plasma loading vs. confining field

curve, which shows greater loading at magnetic fields larger than the cyclotron field. At these larger values of the magnetic field one expects ion cyclotron waves to be generated, and the asymmetry in the loading indicates that power is going into this wave generation. The ratio W reaches a value of 5.5 at the peak of the resonance, which means that 5.5 times as much power is going into the gas as would go into an induction coil of Q equal to 300, or that the transfer efficiency from the induction coil to the gas is 85%.

The second loading curve is for an input power of roughly 200,000 watts. The high-power loading curve is remarkably similar to the lower power data, but now there is a clearer separation of the single particle resonance and of the wave generation resonance.

Some other effects which appear at the higher power level in the vicinity of the double peak are illustrated in Fig. 11. The top curve is the plasma loading in the vicinity of the double peak and is the same data as presented in Fig. 10, but is now plotted on an expanded

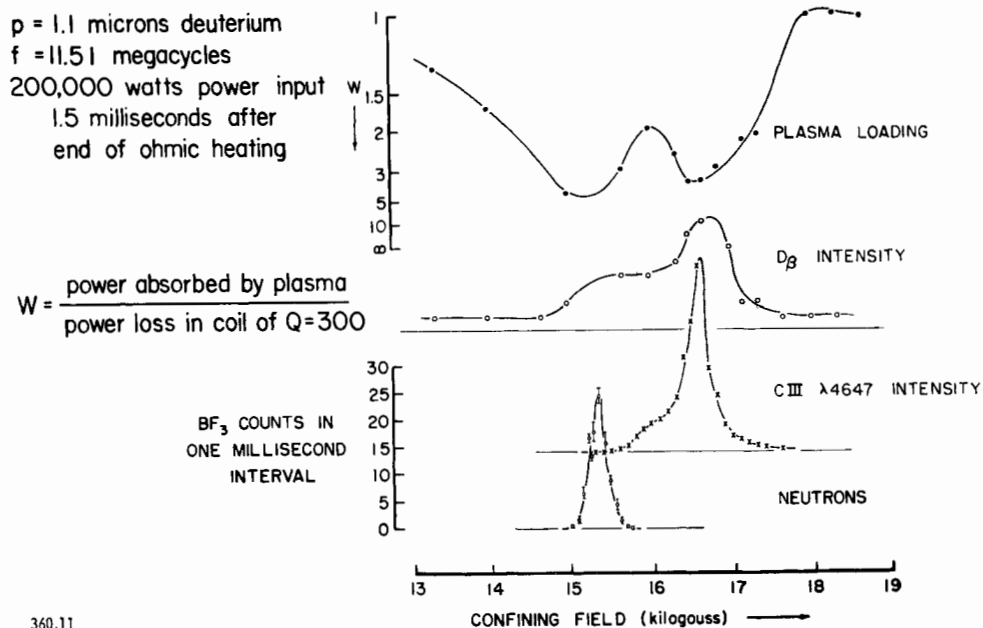


Figure 11. Plasma loading, spectral emission and neutron yield vs. confining field

scale. The second curve is the intensity of the Balmer beta, $D\beta$, spectral line of deuterium at 1.5 m sec after the end of ohmic heating phase. The neutral atoms which produce this light come back into the discharge from the wall of the stellarator tube. The third curve in Fig. 11 is the intensity of the CIII line $\lambda 4647\text{\AA}$. Both of these light intensity curves reach their peaks at the wave generation peak and indicate that the wave generation process is maintaining the electron density and the electron temperature at moderately high values. It seems probable that it is the electron neutralizing current (which flows along the lines of force as part of the ion cyclotron wave mechanism) which is keeping the electron temperature up through the processes of ohmic heating. The Doppler broadening of the CIII line indicates an ion temperature of about 5×10^6 °K and it seems likely that the ions are in approximate thermal equilibrium with the electrons.

The bottom curve in Fig. 11 is the neutron counting rate measured with boron trifluoride counters. The observed neutron flux corresponds to 10^4 to 10^5 neu-

trons coming from the machine per pulse and the counting rate is very sharply peaked around the single particle deuterium resonance. It seems probable that these neutrons come from deuterons which were in the low density region outside the region of the main discharge column and which were accelerated to high velocities by the cyclotron resonance process.

Summarizing the experimental work, both the production of neutrons and the generation of ion cyclotron waves indicate that we are obtaining at least some penetration of the transverse electric fields into the plasma. Altogether, these rather preliminary data would seem to confirm the two extreme examples of behavior which are predicted by theory; namely, the low-density example where the single particle picture is valid and small numbers of particles may be accelerated to high velocities, and the high-density example where ion cyclotron waves are generated. The interesting intermediate case, where one generates ion cyclotron waves and then causes them to thermalize in a "magnetic beach" geometry, has not yet been investigated.

Ion Cyclotron Oscillations and New Injection Methods

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The studies which we are about to describe were carried out in the Physico-Technical Institute of the Academy of Sciences of the Ukrainian SSR at Kharkov. We entered the extraordinarily interesting sphere of thermonuclear physics relatively late; in fact, only this year. We did not wish to embark forthwith on the development of large installations like those mentioned by Academician Artsimovich in his paper, or like the stellarator in the USA or Zeta at Harwell. This was not for lack of enthusiasm or inadequacy of financial resources, but because we were convinced that the necessary temperature of about a hundred million degrees could not be reached by resistance (ohmic) heating. True, our conviction was somewhat shaken when we heard in February of this year of the splendid achievements of Thonemann and his fellow workers at Harwell. But, all the same, we are still convinced that it will only be possible to build working thermonuclear reactors after detailed study of the various methods of supplying additional heat to plasma at a not very high temperature—one of the order of $5 \times 10^5 - 10^6$ K. We therefore concentrated our efforts on the possibility of supplying additional heat to plasma by means other than ohmic heating.

ION CYCLOTRON OSCILLATIONS

Methods where the external energy is supplied directly to the ionic component of the plasma during the period between Coulomb collisions of particles are of special interest. Apparently the most promising method of non-thermal heating is to utilize ion cyclotron oscillations, or other kinds of ionic oscillations, in plasma confined by an external magnetic field. The simplest types of ionic oscillations are those at the ion cyclotron frequency defined by $\omega_{ci} = eH_0/Mc$ where $H_0 > 0$ is the magnetic field strength and M is the ion mass. It must be mentioned that this type of resonance can be obtained only when $\omega_{ci} \gg \omega_p$, where $\omega_p = (4\pi n_e e^2/m)^{1/2}$ is the ordinary electron-plasma frequency. When the den-

sity of the plasma is high, the polarization fields in the plasma must be considered;^{1,2} the resonant frequency then depends not only on H_0 , but also on n , the density of the plasma.

An increased plasma density produces a shift in the resonant frequencies and substantially influences the penetration of external electromagnetic fields into the plasma. The shift of the resonant frequencies is due to the fact that the polarization fields in the plasma accelerate or decelerate particles gyrating in the magnetic field. Thus the sign of the shift in the resonant frequency depends on the direction of the polarization fields relative to the direction of rotation of the plasma particles. The magnitude of the shift depends on the strength of the polarization fields and is determined by the density of the plasma and the length λ_p of the wave excited in the plasma. If the wavelength λ_p is small, the induced polarization fields are also small, as they are proportional to the number of plasma particles situated in a distance $\sim \lambda_p/2$. The magnitude of the shift of the resonant frequency in this case is also small.

When the wavelengths are very long, polarization fields of high intensity can be expected. These will produce large shifts in the resonant frequencies.

Theoretical calculations show that in the first case ($\lambda_p \rightarrow 0$) the resonant frequency is very near to the ion cyclotron frequency ω_{ci} even at very high plasma densities:³

$$\omega = \omega_{ci} \left(1 - \frac{\omega_p^2 \beta_{ph}^2}{\omega_{ce} \omega_{ci}} - \frac{c^2 \lambda_n^2 \beta_{ph}^2}{2 \omega_{ce} \omega_{ci} R_0^2} \right).$$

In the second case ($\lambda_p \rightarrow \infty$) the resonant frequencies at high plasma density are substantially different and approach the value

$$\omega = (\omega_{ci} \omega_{ce})^{1/2}.$$

The values of the resonant frequencies and the depth of penetration of the electromagnetic fields into the plasma, as functions of the plasma parameters, are given in Ref. 3.

It must be mentioned that the short wavelengths in the plasma that are necessary in order to observe

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the resonance near the cyclotron frequency ω_{ci} can be produced either by the proper choice of geometry and parameters of the plasma ("plasma waveguides"⁴), or by the use of an external modulating system.

We have carried out preliminary experiments to prove that these resonances actually occur and to evaluate, at any rate qualitatively, the kinetic ion temperatures which can be achieved by these means.

The "Pyrex" discharge tube of 7-cm diameter and 4.5-cm axial length was mounted between the poles of an electromagnet, giving an axial field up to 10,000 gauss. The poles of the electromagnet were shaped so that the nonuniformity of the magnetic field did not exceed 1% in the region of the discharge plasma. An oscillating electric field was provided by means of parallel plate electrodes mounted outside the discharge tube and coupled to a 400-watt rf source.

Hydrogen was admitted to the discharge tube through a heated palladium tube. Before the experiments the whole system was thoroughly outgassed for 4–5 hours and then rinsed with a stream of hydrogen.

In all these experiments we used the rf discharge produced by the electric field of a capacitor (*E*-discharge), rather than a discharge produced by an rf coil (*H*-discharge). The simplicity of the configuration in the crossed fields favours the *E*-type discharge. Our previous experiments with electron cyclotron resonance showed, moreover, that the experimental determination of the electron cyclotron resonant frequencies depended greatly on the geometry of the experimental arrangement in the case of an *H*-discharge but gave consistent results in a frequency range from 100 Mc up to 4000 Mc in the case of an *E*-discharge.

The choice of a high magnetic field-strength was determined by rather contradictory demands which had to be fulfilled to achieve the best conditions for the observations of ion cyclotron resonance. On the one hand, the selective absorption of energy at the resonance is proportional to the density of the plasma. On the other hand, in our preliminary experiments we wanted to be sufficiently far from the ordinary plasma frequencies ω_p . These demands could be partially fulfilled by the use of strong magnetic fields. The number of cyclotron periods during which the ions increase their energy is determined by the mean time between collisions, and by the value of the magnetic field. The loss of particles accelerated by cyclotron resonance to the walls of the discharge vessel is diminished by an increased magnetic field, because the dimensions of our discharge tube were comparable with the radius of rotation R of the accelerated ions.

The numerical evaluation of the parameters of our experimental set-up showed that the plasma polarization fields can be disregarded up to densities of $10^8 - 5 \times 10^9 \text{ cm}^{-3}$. The energy of the ions was limited by the size of the discharge tube and the strength of the magnetic field, and was several kev,

The change in the conductivity of the plasma at resonance was observed by measuring the plate and grid currents of the rf oscillator or by measuring the strength of the electric field near the tank circuit of the oscillator. Experiments showed these quantities depended in the same way upon the load of the oscillator; due to this, most of our experiments were made with a small antenna measuring the decrease in the strength of the electric field, which was proportional to the power absorbed by the plasma.

The current in the antenna circuit was measured together with the increasing current in the windings of the electromagnet. These measurements were made at a radio frequency of $f = 11 \text{ Mc}$ and a hydrogen pressure of $5 \times 10^{-2} \text{ mm Hg}$. The value of the magnetic field ($\sim 7 \text{ gauss}$) at which resonance occurs corresponds to the true value of the proton cyclotron-resonance frequency. We did not use any auxiliary electric field to start the gaseous discharge and the oscillograph records show that the discharge starts abruptly very near to the resonant frequency. The power absorbed by the discharge is 100–150 watt.

By raising the plate voltage of the rf oscillator, the energy absorbed by the plasma during resonance could be appreciably increased. This leads to an increase of the plasma density, the resonance becomes less sharp, and the maxima of the resonance curves are shifted to a smaller field strength. This shows that at these densities the plasma polarization cannot be neglected, and the values of the shift of the resonance curves are in qualitative accord with the theory of Körper² and Fainberg and Kurilko.¹ In these preliminary experiments we had no experimental means to measure the density of the plasma at resonance. However, assuming that the plasma density depends only slightly on the frequency of the rf field and is mostly determined by the pressure and the absorbed rf energy, we have studied the dependence of ω_{ci}/ω_r (where ω_{ci} is the ion cyclotron frequency and ω_r the frequency at which the absorbed energy attains its maximum) on the frequency of the applied field. These measurements were made at two different power levels of the rf oscillator.

The experimental values are near the theoretical values for a density of $\sim 3 \times 10^8 \text{ cm}^{-3}$.

For the qualitative evaluation of the influence of the collisions of protons with the chamber walls, we investigated the resonances at different diameters of the discharge chamber. By lowering a glass screen into the discharge chamber, we could change the size of the discharge space sufficiently continuously from 1.75 cm to 3.5 cm. The amplitudes display a strong dependence on the size of the discharge space, indicating that the cyclotron radius of the ions is comparable to the size of the discharge space. Very approximate values of the energy gained by the ions due to the cyclotron resonance can be found, supposing that the maximum values of $B_0 R$ is determined by the size of the discharge space. Since $R \sim 2 \text{ cm}$ and $B = 7500 \text{ gauss}$ the energy of ions will be equal to $\sim 2700 \text{ ev}$ or the ion temperature $T_i \sim 3.10^7 \text{ }^\circ\text{K}$. More direct measurements of the

kinetic ion temperatures by Doppler broadening are in progress.

The above values for the energy of the ions are definitely maximum and only a certain part of all the ions can obtain such high energies, due to the high cross sections for charge exchange with neutral molecules.

We think that these preliminary results show great potential promise, but it seems doubtful if they can be used to heat a denser plasma. When the density is greater than 10^{12} – 10^{13} ions/cm³ the resonant frequency approaches the value

$$\omega_r = (\omega_{ce}\omega_{ci})^{\frac{1}{2}}.$$

At such high frequencies the penetration of this dense plasma by electromagnetic fields is greatly diminished. To avoid this difficulty two methods can be envisaged:

1. The ion cyclotron oscillations inside the plasma can be excited, not by an rf field, but by a travelling electromagnetic wave^{1, 5} or by a specially selected configuration of standing waves in the discharge space, as reported by Stix.⁸
2. The ion cyclotron oscillations can be excited by an electron beam modulated at the ion resonant frequency.

COHERENT EXCITATION OF PLASMA WAVES

The interaction of the charged particles with the stationary plasma can be greatly augmented by the inverse of the coherent acceleration effect.⁶ In the passage of a strongly-bunched beam of charged particles through the plasma at rest, there can be large specific losses, at whose expense the electromagnetic field energy increases.⁷ Since the separate bunches of electrons moving through the plasma can be made considerably smaller than the wavelength of the oscillation excited in the plasma, the interaction of bunches with the plasma increases considerably due to coherence.⁶ There is also an additional increase in cooperative interaction if the distances between the bunches are equal to the excited wavelength. This idea was experimentally put to the test in our Institute in the early spring of this year.

The experiments were performed with a pulsed beam of electrons. The energy was ~ 80 keV and the pulse current $I = 1$ amp. The electrons were bunched by an rf electric field of approximately the ordinary electron plasma frequency ($f = 3000$ Mc). The geometrical length of the bunches was about 1 cm, and the number of electrons in each bunch was about 10^9 . This bunched electron beam passed through a cold hydrogen plasma with densities 10^{11} – 10^{12} cm⁻³. The results showed that the specific energy losses dE/dz of the bunched electrons in the plasma approached the value 150 eV/cm, and that they could be increased manifold by decreasing the primary energy of the electrons. The relatively high initial energy of the electrons, 60–80 keV, was chosen only because at

these energies the ordinary ionization and Coulomb losses are exceedingly small, being 10^8 – 10^9 times smaller than the experimental coherent losses. This method of plasma excitation can be applied at greater plasma densities if other types of plasma waves are used (ion plasma, ion cyclotron, acoustical, magneto-acoustical or hydromagnetic), with a longer wavelength to preserve the conditions necessary for coherence.

The high values we find for the specific energy losses dE/dz connected with coherence can be used for the development of a new method for the injection of charged particles into linear or toroidal magnetic traps.

To conclude, we would like to make the following comment: it is not difficult to predict the existence of ion cyclotron oscillations in plasma; the very name reminds one that such oscillations are being observed in hundreds of laboratories all over the world, in machines called "cyclotrons" or "omegatrons". There is an entirely different state of affairs in a plasma of sufficient density, where the ions are effectively shielded by the electron gas from external electric fields. It is due precisely to this circumstance that no one previously observed ion cyclotron oscillations reliably in a sufficiently dense plasma. I had expected that we would be the first to report success in this domain. As you know, our expectations, possibly unduly bold, have not been realized, thanks to Dr. Stix's beautiful paper.⁸

However, the novelty of the problem does not, I hope, deprive additional confirmation of the existence of ion cyclotron resonance of value.

I have already mentioned that we consider the transfer of energy to the ion component of the plasma by means of a modulated beam of electrons to be one of the most promising methods of the supplementary heating of plasma. The thought automatically springs to mind: should we not use the tens and hundreds of thousands of amperes of electron current that flow through apparatus of the Zeta or stellarator type for this purpose? It is possible, of course, to attempt to bunch the stream of electrons flowing into the plasma, albeit very weakly, with the assistance of an external electromagnetic system. But this does not seem to me to be necessary. The task we have set ourselves is the following: How can the energy of the directed electron beam be most quickly and most fully transformed into energy of the ion component? In plasma, as we know, it is possible for a very large number of oscillations of different types to exist; and some of these are related to the electron component of the plasma. If, by judicious selection of externally-controlled parameters, the frequency of one of the types of oscillation of the electron component can be made equal to that of one of the types of oscillation of the ion component, then, in principle, the transfer of energy from the electrons to the ions is entirely possible. Work is being carried out along these lines.

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Electron Cyclotron Resonance, Plasma Wave Guides and Plasmoids

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ELECTRON CYCLOTRON RESONANCE

Investigations dealing with plasma resonance properties are of interest in ascertaining the possibilities of heating the plasma by means of a high-frequency discharge, and in developing methods of investigating the plasma parameters and the physical processes taking place in it.

The present work was carried out to investigate the properties of an electrodeless high-frequency discharge, and plasma parameters under electron cyclotron and polarization resonance conditions.

Electron resonance in plasma has been discussed in other papers.^{1, 2} However, as far as we know these investigations were not concerned with the plasma parameters, i.e., its density and temperature.

The determination of the temperature and density of a plasma produced by means of a high-frequency discharge in hydrogen within an external magnetic field was performed by a double probe method.^{3, 4} The control measurements of density were carried out by means of the method of high-frequency double probes,⁵ the probes being located outside the discharge tube. The high-frequency signal fed to one of the probes is propagated through the plasma and received by the second probe connected with the resonance amplifier. Assuming that the probe field is approximately a dipole field, it is possible to determine the plasma density by the maximum in the transmitted signal which takes place if the measuring signal frequency equals the Langmuir plasma frequency.

The plasma densities measured by the two-probe method and by a high-frequency probe coincide in order of magnitude. The maximum density values obtained with a 200–300 w generator at the frequency of 30–33 megacycles lie in the range of 5×10^{11} – 10^{12} cm⁻³. The volume of the ionized space did not exceed 200 cm³. The resonance was observed by measuring the power output of the generator igniting the discharge as a function of the magne-

tic field at various pressures. At the same time measurements of plasma luminescence intensity within the visible light region were carried out with the aid of a FEU-19† photomultiplier. The resonance curves indicating the power expenditure and light flux coincide on the corresponding scale within measurement errors (pressures being no lower than 10^{-3} mm Hg).

Figure 1 shows the dependence of power output upon the magnetic field value at different pressures. As was to be expected, the width of the resonance curve decreases when the pressure reduces. Further reduction of the pressure, however, causes an increase in the resonance curve width (see Fig. 2). Apparently it can be explained in the following way.

Under cyclotron resonance conditions the electron is accelerated by a high-frequency field within the interval between two collisions. A pressure reduction and a corresponding increase in the interval between two collisions result in an increase in the mean electron energy and the electron orbit radius. Thus, when an electron orbit diameter in the discharge approaches the vessel dimensions, the electron acceleration time is reduced due to collisions with the walls which lead to a broadening of the resonance curve. To check this hypothesis, measurements at another frequency (120 megacycles) have been carried out. The measurements revealed that the resonance curve half-width continues to depend qualitatively upon the pressure, but the pressure at which the broadening of the resonance curve occurs shifts towards higher vacuum.

If the above mechanism actually takes place, it becomes possible to estimate the electron energy at pressures which are close to the value at which the abnormal growth of the resonance curve half-width begins. If the vessel diameter is 4 cm and the magnetic field value in resonance is 17 gauss, the mean electron energy should not exceed 102 eV, which corresponds to the temperature of 1.2×10^6 °K. This value agrees in order of magnitude with the maximum temperature value (4×10^5 °K) obtained

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† Transliteration.

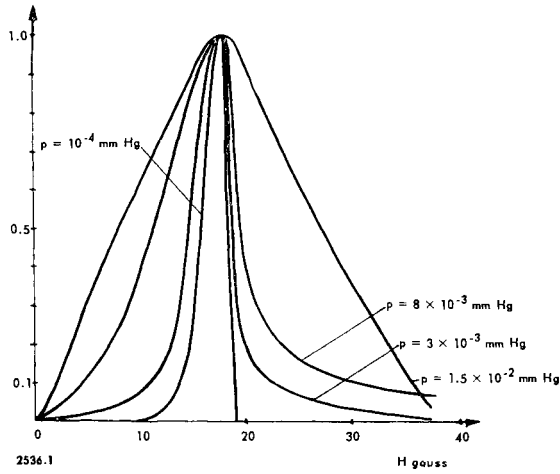


Figure 1. Electron cyclotron resonance curves. Power output W in arbitrary units as a function of magnetic field H , for various pressures p

by the double probe method for the pressure of 3×10^{-3} mm Hg (Fig. 3).

The resonance value of the magnetic field at the generator frequency of 120 megacycles equalled 43 gauss. The vessel diameter being 10 cm, the corresponding temperature was $1.5-4.5 \times 10^7$ °K.

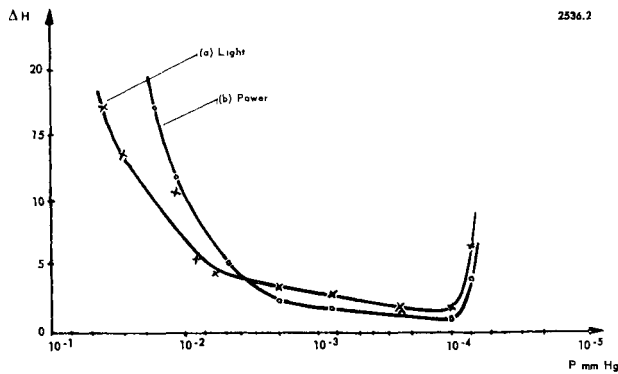


Figure 2. Electron cyclotron resonance width as a function of pressure p : (a) as determined by light measurements; (b) as determined by power measurements

It should be noted that at the frequency of 33 megacycles the magnetic field value corresponding to the electronic cyclotron resonance equals 12 gauss instead of 17 gauss obtained experimentally. Such a difference in magnetic field intensities with an inductive coupling between the generator and the discharge, and with an external magnetic field perpendicular to the axis of the coil causing the discharge, was also observed in other experiments dealing with cyclotron resonance in plasma. At the frequency of 120 megacycles resonance was observed for the magnetic field value corresponding to cyclotron resonance. However, during this experiment capacitive coupling between the generator and the discharge was used. The possibility of changing the resonance conditions by changing the mode of excitation ensues from the relations (1), (2) given below.

When the resonance took place at the frequency of 120 megacycles, the plasma luminescence inside

the discharge volume almost completely disappeared simultaneously with the beginning of the increase in the resonance half-width. This was accompanied by an increase in generator power expenditure and the appearance of blue luminescence of the retort walls, which is usual during the bombardment of glass with sufficiently fast electrons.

Plasma Resonance

Along with the cyclotron resonance determined by the condition $\omega_c = eH/mc$ there takes place a polarization resonance, in which the resonance frequency is determined not only by the magnetic field intensity, but by the plasma density as well. In this case the resonance frequency and the electron velocity are determined by the following relations.

Low Densities:

$$\omega_p^2 \ll \omega_c^2; \quad \omega = \omega_p \left(1 + \frac{\omega_p^2}{\omega_c^2} \right).$$

(a) Inductive coupling between the generator and the load ($E_\phi(a)$ is given at the plasma boundary):

$$v_r = \frac{\omega_c^2}{\omega_p^2} \frac{r}{a} \frac{eE_\phi(a)}{m\omega_c}, \quad v_\phi = iv_r. \quad (1)$$

(b) Capacitive coupling between the generator and the load ($E_r(a)$ is given at the plasma boundary):

$$v_r = \frac{\omega_c}{2\nu} \frac{r}{a} \frac{eE_r(a)}{m\omega_c}, \quad v_\phi = iv_r. \quad (1)$$

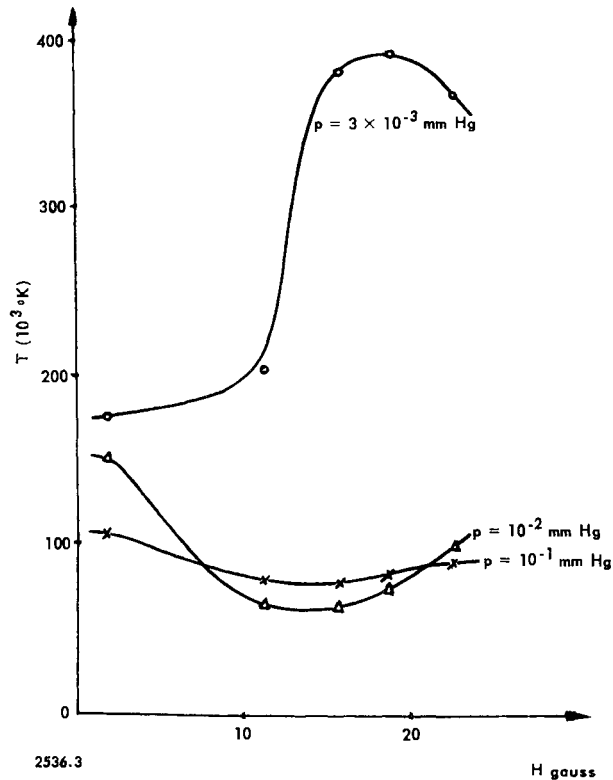


Figure 3. Electron temperature T as a function of the applied magnetic field H , for various pressures p . Generator frequency = 33 Mc

High Densities:

$$\omega_p^2 \gg \omega_c^2; \quad \omega = \omega_p + \frac{1}{2}\omega_c$$

(a) Inductive coupling between the generator and the load:

$$v_r = \frac{\omega_c}{\omega_p} \frac{r}{a} \frac{eE_\phi(a)}{m\omega_c}, \quad v_\phi = iv_r. \quad (2)$$

(b) Capacitive coupling between the generator and the load:

$$v_r = \frac{\omega_c^2}{2\omega_p \nu} \frac{r}{a} \frac{eE_r(a)}{m\omega_c}, \quad v_\phi = iv_r. \quad (2)$$

Here ω = generator frequency, ω_p = Langmuir electron frequency, ω_c = Larmor electron frequency, ν = collision frequency, v_r and v_ϕ = components of the electron velocity in the plasma, and a = plasma column radius. A photomultiplier was used to indicate the polarization resonance and to determine the maximum plasma luminescence. The increase in the plasma density was achieved by increasing the high-frequency generator capacity. Figure 4 shows the dependence of the photomultiplier indications upon the magnetic field intensity at different anode currents of the high-frequency generator. The plasma density increases with an increase in the above current, and the resonance curve is shifted into the region of large magnetic fields.

Investigation of noises generated by the plasma produced by a high-frequency discharge is of great interest. Measurements of noise levels as a function of the magnetic field were carried out. For this

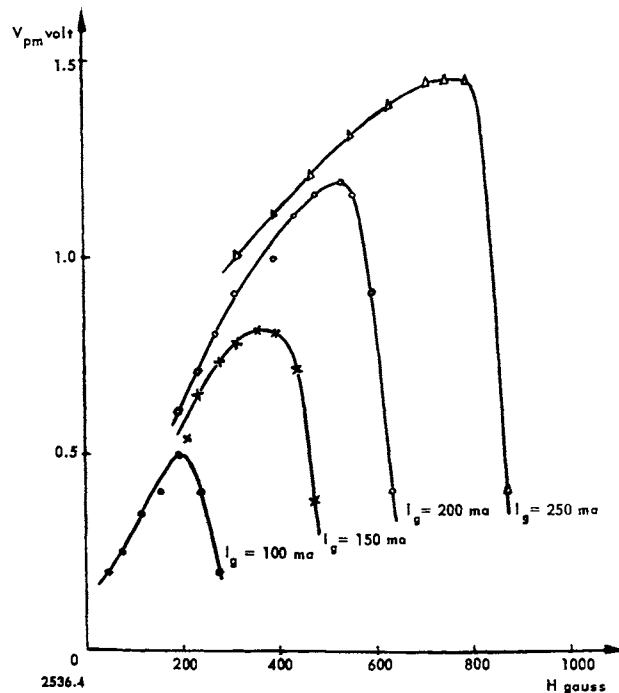


Figure 4. Light intensity as a function of magnetic field H for various currents I_g of the hf generator. (Photomultiplier voltage V_{pm} as ordinate)

purpose an insulated probe was placed in the plasma, and the induced signal was fed to an oscillograph. Oscillations at the frequency of 1 megacycle and relaxation oscillations were discovered among the noises. An abrupt noise decrease in the region of the cyclotron resonance (see Fig. 5) is characteristic of the noise level dependence on the magnetic field intensity.

PLASMA WAVE GUIDES

To investigate the possibilities of high-frequency plasma heating, it is necessary to study the penetration of the high-frequency field into confined plasma (plasma wave guides).

To construct linear accelerators, it is necessary to have an effective moderating system where the phase velocity of electromagnetic wave propagation would be lower than the velocity of light, and a simultaneous phase and radial stability of accelerated particles could be provided. Anisotropic plasma wave guides meet these requirements, in particular plasma wave guides in a magnetic field.⁸⁻¹²

This work includes an investigation of plasma wave guides. The experiments also determined dispersion properties, i.e., the dependence of the phase velocity upon the frequency at the given plasma density. They also determined the topography of high-frequency fields.

The plasma wave guide was formed with the help of a high-frequency discharge in a quartz tube 30 mm in diameter and 1800 mm long, where the vacuum was sustained at about 10^{-3} mm Hg. The high frequency gas discharge was ignited at one end of the tube by a 500 watt high-frequency generator at the frequency of 116 megacycles. The discharge plasma filled the whole tube, its density falling off with the distance from the generator.

A high-frequency signal with a frequency changing within 50–500 megacycles, was fed to the plasma from a separate low-power generator placed opposite the ignition point.

Moving a loop connected with a resonance circuit along the wave guide, we could observe how H_ϕ , the ϕ component of the magnetic field of the standing wave, propagated along the plasma wave guide. The standing wave field distribution is shown in Fig. 6. The distance along the wave guide in cm is plotted on the abscissa axis, the H_ϕ values in arbitrary units being plotted on the ordinate axis. A plasma wave guide in a constant longitudinal magnetic field H_0 is characterized by curve I; curve II was obtained for the case $H_0 = 0$. The distance between the mean minimums equals a half wavelength in the plasma wave guide. The ratio of this wavelength to the wavelength λ_0 in the free space determines the value of $\beta = v_{ph}/c$. It should be noted that the plasma density differed somewhat from one experiment to another because the ignition conditions changed. The results of measuring the dependence of the phase velocity βc upon frequency f with a magnetic field (β_H) and without it (β_0) are summarized in Table 1.

It follows from Table 1 that with frequency changes from 50 to 500 megacycles, the phase velocity changed from 0.16c to 0.33c.

The influence of the constant magnetic field upon the phase velocity is not great in the given region of plasma wave guide parameter variation, and this fact coincides with theoretical predictions.

Table 1. Dependence of Phase Velocity upon Frequency in a Plasma Wave Guide

$f(\text{Mc})$. . .	50	140	150	200	300	400	500
β_0	0.14	0.167	0.2	0.196	0.18	0.307	0.33
β_H	0.16	0.169	0.2	0.283	0.22	0.32	0
$n \times 10^{-9}$.	—	9.1	4.7	4.6	9.0	9.7	9

It should be pointed out that the density data given in Table 1 refer to a plasma wave guide with no external magnetic field. The superposition of the magnetic field resulted in a change of the plasma density by 10%–15%.

The determination of the plasma density was performed by two methods. One method determined the displacement $\Delta\omega$ of the proper frequency of a resonant cavity along the axis of which the plasma wave guide was placed. The frequency displacement $\Delta\omega$ is determined by the plasma density $n_0 = k\Delta\omega$ where k is a coefficient depending upon the geometry of the resonant cavity and the plasma wave guide.¹³ The second method was that of a high-frequency probe.⁵ The results of both methods are in accord, showing that the plasma density in the wave guide changed from 4×10^9 to $9.8 \times 10^9 \text{ cm}^{-3}$.

Measurements along the wave guide radius of the electric field component E_z were carried out with the help of a wire probe. These measurements indicate that within the phase velocities of about 0.15 c–0.2 c the E_z drop from the wave guide boundary to the centre is insignificant. Theoretical calculations show that within these phase velocities the field E_z should fall off by 10–20%. Having determined the $E_z(r)$ ratio outside the wave guide we can find the phase wave velocity. For a typical case we have

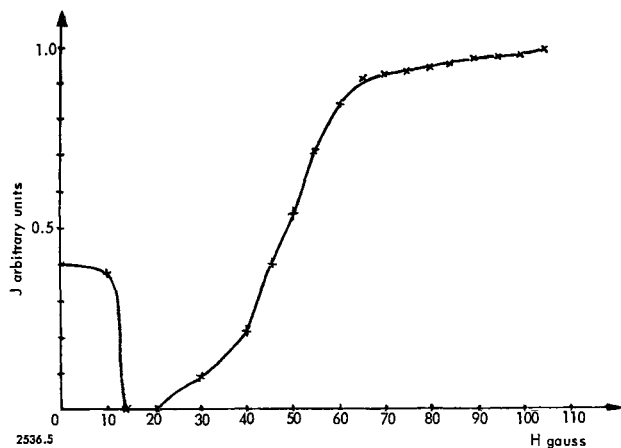


Figure 5. Noise level J as a function of magnetic field H

thus obtained $\beta = 0.2$ which agrees well with the measurement data produced by the standing wave method.

PLASMOIDS

One of the possible methods of investigating the relaxation effects inside a plasma which is subject to the external magnetic fields, is studying the behaviour of plasmoids confined in space during their movement in non-uniform magnetic fields. We attach particular importance to cases where adiabatic conditions obtain only for the electron component of the plasma in the interaction with external magnetic fields.

To obtain a sufficiently fast moving plasma, a modified plasma source of the Bostick type¹⁴ was used. The modification consisted in adding a cylinder nozzle 10 mm in diameter and 20 mm long to the Bostick source, thus reducing the divergence of the plasmoid to 20–30°. Hereafter the term "plasmoid" will mean a neutral plasma beam, modulated in density and stringently limited in space and time. We found in the plasmoid no self-organizing properties associated with its own currents or magnetic moments. All measurements were conducted at a distance of about 100 cm, and if there were any currents and magnetic fields close to the source (at ~ 10 cm), as stated by Bostick,^{14, 15} they were extinguished under the conditions of our experiment and could not influence the measurements. The plasma density modulation was due to the presence of ions of different masses (chiefly of oxygen and copper) which within 10 μsec of movement could consolidate into closely located, separate plasmoids. The ion density estimation of the leading plasmoid gave $1\text{--}5 \times 10^9$ particles/ cm^3 . The mean plasmoid speed was $\sim 8 \times 10^6 \text{ cm/sec}$.

Figure 7 shows the scheme of the set-up. The plasma, leaving the source (1), after covering a certain distance gets into a non-uniform magnetic field created by a short solenoid (2) which has a magnetic field gradient of about 7 gauss/cm.

The basic measurements were carried out at a magnetic density of ~ 100 gauss.

Different magnetic (4) and electric (3) probes were used to measure the plasma parameters. The magnetic probes were coils which could both encompass the glass tube and be inserted inside, NS remaining constant (NS is the product of the number of turns in the coil by the area). The Faraday cylinder (3) shown in Fig. 7 served as electric probe. It contained a measuring electrode 8 mm in diameter placed in an isolated cylinder, which, in its turn, was separated by a copper cylinder from the bulk of the plasma. The magnetic and electric probes were connected with an oscillograph having a low resistance input (100 cm). Signals arising in the magnetic probes were caused by magnetic field changes in the coil at the moment the plasma was crossing it. The signals were of diamagnetic type, i.e. the magnetic field was reduced at the moment the plasma was crossing it. The magnetic probes located on the chamber axis were helpful in finding the diamagnetic signal which was at a maxi-

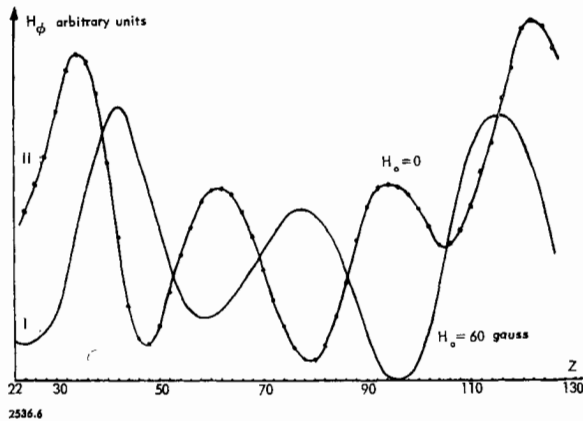


Figure 6. Standing-wave field distribution in a plasma wave guide

mum when the plasma entered the magnetic field. The signal disappeared gradually as the plasma was sinking in the magnetic field, but when it left the latter, the diamagnetic signal of the opposite sign appeared again 3-4 times farther from the lens centre. The outgoing diamagnetic signal was 3-4 times lower than the corresponding entrance signal. An investigation of the dependence of the diamagnetic signal upon the magnetic field value showed that the signal reached its maximum value when the magnetic field intensity was 60 gauss.

An examination of the plasmoid movement through a non-uniform magnetic field carried out with the aid of electric probes leads to the conclusion that diamagnetic signals can be observed at the moment the electron component of the polarized plasma crosses the magnetic probe.

As the magnetic field of a short solenoid is characterized by non-uniformity along z and r , the presence of an H_z gradient must lead to the polarization of the plasma along the z axis and consequently to the appearance of uncompensated charges in certain parts of the plasmoid.

The character of the signals obtained from electron probes is given in Fig. 8a, which shows that plasma crossing a heterogeneous magnetic field becomes polarized in the longitudinal direction and has the appearance of a sign-alternating chain. Charge topography in the plasmoid during its movement through

a heterogeneous magnetic field was obtained by means of electric probes. Fig. 8b, c, d shows the dependence of the probe current upon time for different positions of the probe in the magnetic field. The position of the short solenoid is marked by a dashed line and the probe location with respect to the chamber axis is shown to the right of each diagram.

The diagrams indicate that electrons (the first negative peak) entering the heterogeneous magnetic peak) leave the plasmoid periphery and concentrate around the plasmoid axis. This process is accompanied by a positive charge formation over the periphery. Thus a plasma polarization takes place in the radial direction which results from the presence of an H gradient along r . The first negative peak disappears at some distance in front of the magnetic field, and the plasmoid front proves to be positively charged. This shows that besides pressing electrons towards the axis, the magnetic field slows them down, thus enabling the ions to outstrip the electrons. Analysing the behaviour of the first and the second negative peaks we can see that the appearance and disappearance of electrical polarization is asymmetric with respect to the value of median lens density. This is verified by diamagnetic probe signals, as mentioned above. The behaviour of the ion peaks is also asymmetric with respect to the magnetic field entrance and exit.

Knowing the charge distribution (shown in Fig. 8b, c, d), the ion velocity, and the particle density, we can estimate the polarization electric fields arising during the plasma movement through a heterogeneous magnetic field; such an estimate shows that an electric field of 500 v/cm can appear between the first positive and negative peaks. Such local fields, naturally, must cause a relative movement of plasma particles and change the plasmoid configuration. The presence of electric fields is also proved by an observed drastic 1.5-2-fold increase of the ion velocity which occurs at a certain distance from the median plane of the magnetic field. This ion acceleration approximately coincides with the moment of the first negative peak disappearance when the ions come to the front. To summarize, the plasmoid movement through a heterogeneous magnetic field leads to a longitudinal and transverse plasma polarization and to a change in the configuration of the plasma formation.

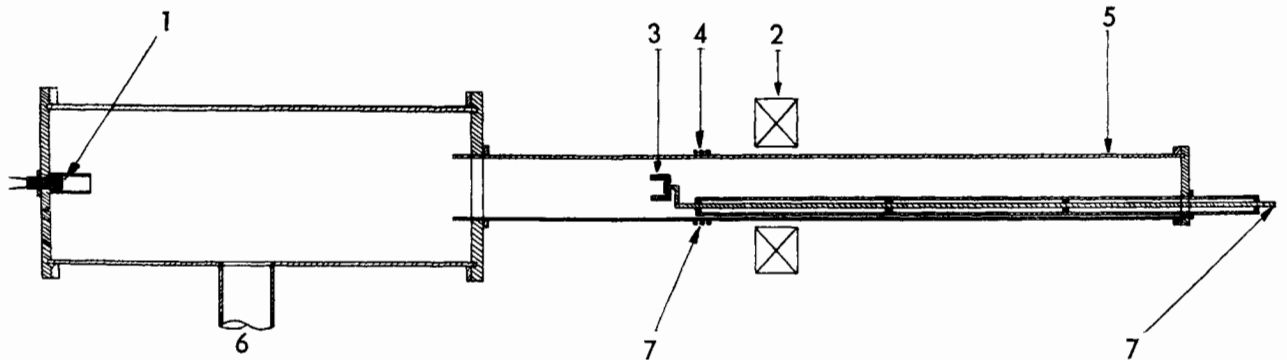


Figure 7. Apparatus for plasmoid studies. 1, plasma source; 2, solenoid; 3, electrostatic probe; 4, magnetic probe; 5, glass tube; 6, pump; 7, oscilloscope

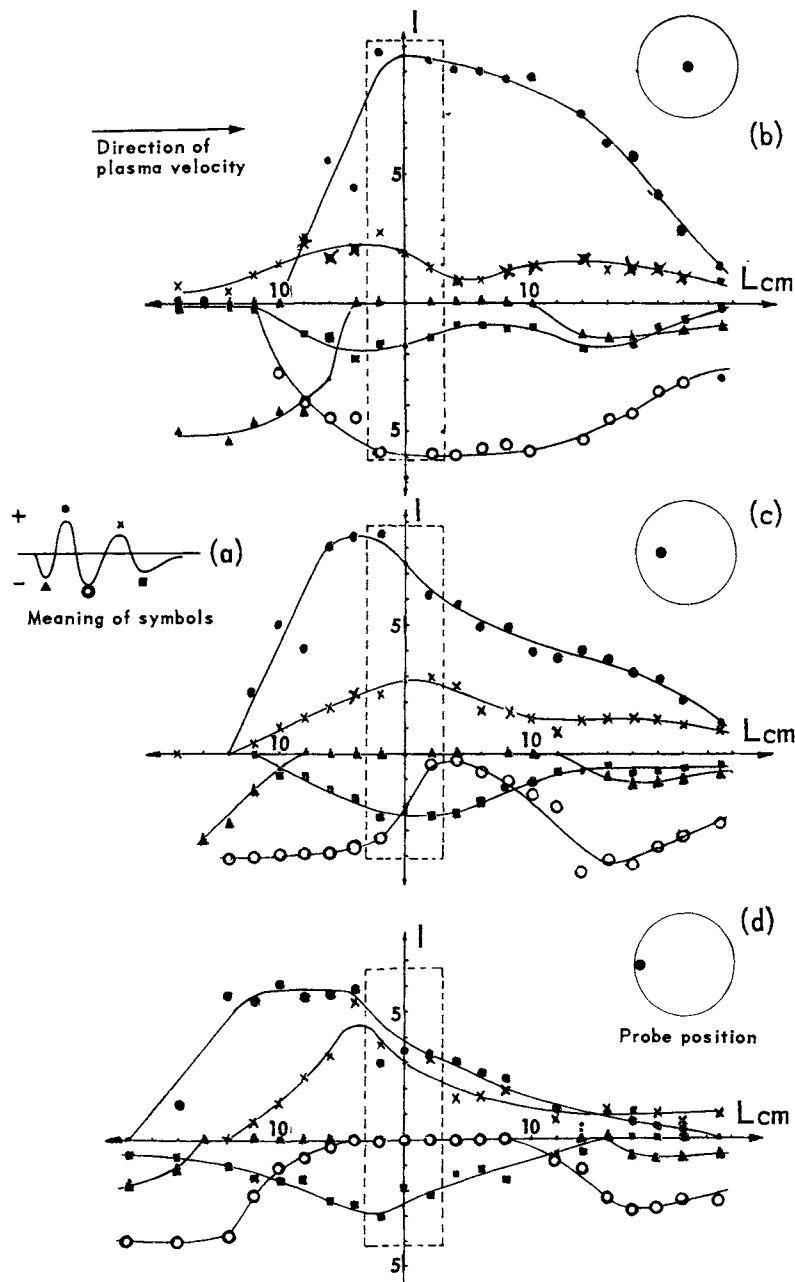


Figure 8. Plasmoid signals

The appearance of polarization electric fields and the movement of plasma particles inside the plasmoid caused by them probably brings about an additional energy exchange between the ion and electron plasma components.

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The Oak Ridge Thermonuclear Experiment

By C. F. Barnett, P. R. Bell, J. S. Luce, E. D. Shipley and A. Simon *

The production of a self-maintaining thermonuclear reaction in a mass of deuterium, or deuterium and tritium, gas is the goal toward which many different experimental programs are directed. One of the most difficult problems to be solved is that of ignition, or the initial heating of the fuel gas to a temperature high enough to make the reaction self-maintaining. The Oak Ridge Group is attempting to solve this ignition problem by trapping an accelerated beam in a magnetic field container by a change of the charge-to-mass ratio of the particles inside the field. By this means we hope to accumulate a high concentration of energetic ions and hold them until they have had time to randomize into a truly thermalized plasma at a high ultimate temperature.

The success of the trapping method depends on the properties of the mechanism used to induce the change of charge-to-mass ratio. We have found that certain specialized arc discharges can cause the dissociation of energetic deuterium molecular ions into atomic ions and neutral atoms with high efficiency. The atomic ions are trapped in the field by virtue of having half the radius of curvature of the original molecular ions. A successful device must produce efficient trapping, but at the same time it must not cause a large loss of the trapped ions through charge exchange or exert a strong cooling effect upon the resulting plasma. The special arc discharges developed by our group¹ have proved to be satisfactory in this respect as will be shown in the following sections.

The ignition method is being tested by the injection of deuterium molecular ions at an energy of 600 kev into a direct current mirror machine (DCX) where they pass through one of these special discharges, an energetic carbon arc. The 300-kev atomic ions produced by dissociation are trapped into a closed orbit near the central plane of the machine, while the resultant neutral atoms and the remaining undissociated

beam leave the machine at once. The trapped ions are in the form of a highly organized beam. The ions of this beam cause ionization of the residual gas in the machine and also become thermalized by interaction with electrons and other ions.

DCX THEORY

The conditions under which an injected beam will give rise to a thermonuclear plasma are considered below. The requirements for overcoming the neutral gas background are considered in the next section. The steady state plasma temperatures are then studied in the section thereafter. Attention is paid in this and the succeeding sections to the effects of cold electron influx, impurities, heat transfer to the walls and electrons in the arc.

Burnout

The initial neutral gas pressure in DCX will lie somewhere in the range between 10^{-6} and 10^{-8} mm of Hg, depending upon the vacuum system used. Under these conditions the trapped beam will remain organized. Each deuteron will lose very little energy before being lost by charge exchange with a neutral gas atom. The essential reason for this is that the charge-exchange cross section (σ_{ex}) for 300-kev deuterons in deuterium is about 2×10^{-18} cm², while the effective energy-loss cross section is less than 10^{-19} cm². The effective Coulomb cross section (σ_c) for 90° deflection by multiple collisions is about 10^{-23} cm². As a result, there will be little energy degradation of the input beam, and a plasma will not be formed unless the neutral gas density can be further reduced.

The injected ions, however, tend to destroy the neutral gas atoms in two ways. The first, and principal, mechanism is by ionization. Since the ionization cross section (σ_i) at 300 kev is about 4×10^{-17} cm², there will be about 20 neutral atoms ionized by each hot ion before it suffers a charge exchange. The slow ions which are produced scatter rapidly and leave the system by following the magnetic lines to the end walls or the arc cathode. In addition to this, charge exchange itself also destroys a neutral atom, since the fast neutral atom which is formed will then fly to the walls and bury itself with little sputtering.

If the slow ions, which will be neutralized upon striking a solid surface behind the mirrors, can be

* Oak Ridge National Laboratory operated by Union Carbide Corporation for the US Atomic Energy Commission.

The work reported in this paper is the outcome of the joint effort of the members of the Thermonuclear Experimental Division with effective assistance from members of other divisions of this Laboratory. Active participation in early theoretical work by A. E. Ruark and H. S. Snyder is greatly acknowledged and the continuing interest of these men together with other consultants is likewise recognized with great appreciation.

prevented from re-entering the system, the trapped beam will effect a strong pumping action on the system. However, in addition to this, there is a direct means of virtually eliminating neutral gas from the plasma interior even in the presence of a finite external pressure. If the rate of ionization of neutral gas inside the plasma can be made greater than the rate at which neutral atoms enter, a process will take place which we call "burnout". The neutral gas density will decrease, permitting an increase in the ion density and leading to an even greater rate of ionization. As a result, once this critical point is passed, the system will tend to clean itself up quickly. As a practical fact, the influx of gas represented by the beam necessary to achieve burnout is so great that a significant fraction of the beam's pumping action must be realized in order to keep the system's pressure from rising disastrously.

An approximate expression for the critical injected current, I_{crit} , is obtained directly from the consideration above.

$$I_{crit} \cong I_0 \sigma_{ex} / (\sigma_{ex} + \sigma_i). \quad (1)$$

The ratio of cross sections represents the fraction of ions removed per neutral atom destroyed, and I_0 is the total equilibrium current of neutrals into the volume. If N_0 is the neutral density external to the plasma region, v_0 the kinetic velocity of these neutrals, and S the surface area of the plasma, then

$$I_0 = \frac{1}{2} S N_0 v_0. \quad (2)$$

More accurately, one can represent the time behavior of both the ion density, n_+ , in the plasma region and the neutral density n_0 , by the coupled equations:

$$dn_+/dt = (I_+/V) - n_+^2 \sigma_e v P - n_+ n_0 \sigma_{ex} v \quad (3)$$

$$dn_0/dt = (N_0 v_0 S / 4V) - (n_0 v_0 S / 4V) - n_+ n_0 (\sigma_{ex} + \sigma_i) v. \quad (4)$$

Here I_+ is the injected current, v is the ion velocity, and P is the probability of a randomly directed particle having its velocity in the loss cone of the mirror. The first term on the right of Eq. (3) represents the steady injection of ions into the plasma volume, V . The second term accounts for mirror loss by Coulomb scattering (which plays no role until burnout has actually occurred), and the last term represents the loss of ions through charge exchange. Similarly, in Eq. (4), there is an influx of neutral atoms into the plasma region, a loss by streaming out, and destructions by both ionization and charge exchange. Note that N_0 is the fixed neutral gas density external to the plasma.

These equations have been solved by the use of an analog computer. The results are shown in Fig. 1 for the case of a plasma sphere whose radius is about equal to the Larmor diameter of a 300-keV deuteron in a field of 10 kilogauss. The neutral gas density is normalized to be unity when equal to the external manifold density, N_0 , and the ion density is normalized to unity when it has reached a maximum value consistent with mirror scattering loss only. Note that these results are plotted for three values of the pres-

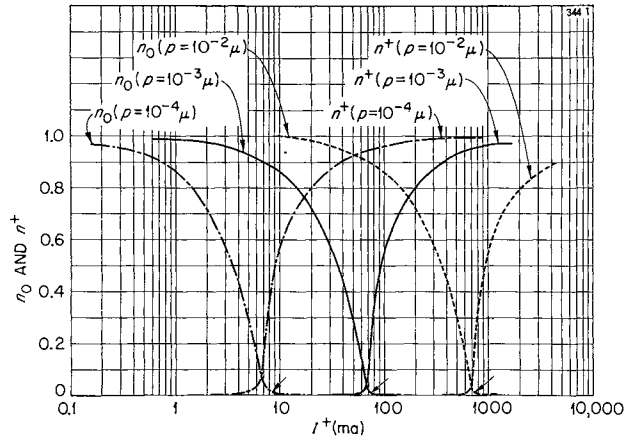


Figure 1. Steady state ion and neutral atom densities as a function of input current for various pressures

sure. The curves are shifted linearly with pressure, as would be expected from the simple criterion of Eq. (1). The critical currents predicted by Eq. (1) are shown in this figure by use of arrows. This equation is seen to give a good measure of the point at which the neutral gas density has dropped appreciably and the ion density has started to increase. For currents well above the critical value the normalized neutral-atom and ion densities approach the asymptotic values given below:

$$n_0 \rightarrow \frac{v_0 S}{4 (\sigma_i + \sigma_{ex})} \left(\frac{\sigma_e P}{v I_+ V} \right)^{\frac{1}{2}} \quad (5)$$

$$n_+ \rightarrow 1 - \frac{N_0 v_0 S}{8 I_+} \frac{\sigma_{ex}}{\sigma_i + \sigma_{ex}}. \quad (6)$$

From Fig. 1 it may be seen that the critical current at a pressure of 10^{-6} mm is about 80 ma and at a pressure of 10^{-8} mm is about 800 μ a. It should be remembered that these results are for the injected ion current and that the total current of molecular ions into the machine must be larger than this value by the reciprocal of the fractional breakup actually occurring in the arc.

Finally, it should be pointed out that a considerable assist to the burnout process would be given by pre-ionizing a large fraction of the background gas. Various schemes for accomplishing this are under consideration.

Energy Transfer

Once burnout has been achieved, charge exchange will no longer be the dominant mechanism. The hot ions will begin to slow down by transferring energy to the colder electrons and ions which may be present. This process will continue until the background electrons and ions are so hot that energy transfer rates are reduced. When this occurs, appreciable Coulomb scattering, as well as nuclear reactions, can start to take place, and the thermonuclear condition has been reached. It is of great interest to inquire as to the actual values of the energies or temperatures at which the plasma clouds will form. The proper

way to calculate this is to use the Boltzmann equation (or the Fokker-Planck equations) and solve for the steady state solutions. Such computations are under way by use of high speed digital computers.

There is a simple model, however, which gives answers far more easily and which may be moderately reliable. In this model, the injected ions are assumed to degrade smoothly and uniformly, i.e., without dispersion, until they go into a Maxwellian cloud of ions at some average energy E_+ . The ions finally are lost from the system by Coulomb scattering via the mirrors. Similarly, cold electrons which may be entering the system are heated in a uniform fashion from some input energy E_e until they go into a plasma of electrons at an average energy E_- . The model is illustrated in Fig. 2. Solid lines in the drawing represent particle paths in energy space. The rate of electron loss from the system is adjusted so that total space-charge neutrality is conserved. The dotted lines in the diagram represent the general directions of energy flow. For example, the degrading ion beam heats both the ion plasma and the electron plasma below it, while the ion plasma transfers energy both to the electron plasma and the electrons which are heating up. The electron plasma loses energy by bremsstrahlung as well as by heating of the input electrons. Direct energy transfer from the input ions to the input electrons has been neglected in this model.

The rate of energy transfer by Coulomb collisions from a "test" particle to a Maxwellian distribution of field particles has been given by Chandrasekhar.² The requirement of net energy balance in the ion and electron plasma respectively leads to two equations. Ion conservation and space-charge neutrality yield two additional equations. These four coupled, nonlinear, integral equations in four unknowns, $\bar{E}_+$, $\bar{E}_-$, n_+ , n_- , may be reduced directly to two coupled equations in two unknowns, $\bar{E}_+$ and $\bar{E}_-$. These two equations depend on several parameters, of course, such as the input ion energy, E_0 , the input electron energy, E_e , the mirror loss probability, P , and the ratio, λ , of electron input current to ion input current. They are, however, independent of the actual input ion current I_+ and the plasma volume.

These equations may be solved readily by hand computation. Some of the results are shown in Table 1. Details of the equations and additional results will be given in future publications. The results in Table 1 are for the case $E_0 = 300$ kev, $E_e = 1$ volt, and $P = 0.3$. These results are conser-

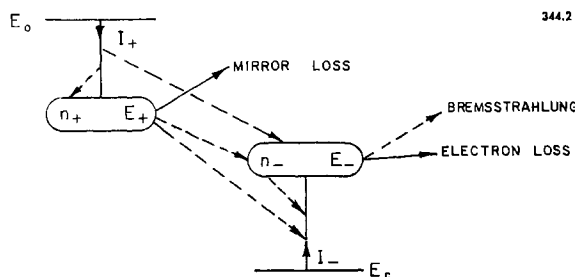


Figure 2. Idealized model of the DCX plasma

Table 1. Equilibrium Mean Plasma Energies for Various Ratios (λ) of Electron Input to Fast-Ion Input

λ	E_+ (kev)	E_- (kev)
0	291	221
1	219	78
2	186	56
5	134	33
10	95	21
15	74	15
20	60	12

vative in some respects. For example, it is assumed that the entering electrons must heat all the way up to the average electron plasma temperature before leaving the system. In addition, an influx of cold electrons which is 20 times larger than the ion input ($\lambda = 20$) seems rather extreme. On the other hand, heat conduction to the walls, impurity radiation, and possible cool electrons in the arc have been neglected. These will be discussed in the next section.

In the most favorable case ($\lambda = 0$) the final energies are very high. With an input of 1 ma of ions into a spherical plasma volume whose radius is equal to the Larmor diameter of a 300-kev deuteron in a field of 10 kilogauss, one obtains an ion density of $n_+ \cong 10^{13}$ cm⁻³. The total neutron yield from the working volume would then be about 10^{13} sec⁻¹. It is important to note that the occurrence of such a large neutron yield would be an unambiguous signal of the arrival at a thermonuclear state. For example, if one were to fire 1 ma of deuterons into a block of solid deuterium ice, the resultant neutron yield would be only about 10^8 sec⁻¹. An increase in the total yield from 10^8 to 10^{13} could only be accounted for by the heating of electrons and the formation of a hot thermonuclear plasma.

Additional Effects

Impurities may be troublesome either by virtue of a high nuclear charge, and hence increased bremsstrahlung, or as a result of being partially ionized and producing resonance excitation radiation. These will now be considered.

It is an interesting property of steady-state mirror machines that the effect of impurities of high mean ion charge, $\bar{Z}$, is to decrease the working density in the volume rather than to depress the temperature. The steady state ion equation has the form:

$$I_+ = \{(Z^2)_{av} n_+\}^2 (\sigma_c v)_{av} P V \quad (7)$$

where σ_c is now the effective Coulomb cross section for scattering through 90° by multiple collisions, for $\bar{Z} = 1$, and $(Z^2)_{av}$ is the mean-square ion charge. On the other hand, the power radiated in bremsstrahlung has the form:

$$p \sim (Z^2)_{av} n_+ n_- f(E_-) \quad (8)$$

where $f(E_-)$ is a function of the electron energy only. At constant ion input and energy, increasing the impurity level decreases the ion density in such a way that the product $(Z^2)_{av} n_+$ stays constant. Since this same product occurs in the bremsstrahlung term,

it is clear that this term is increased only by virtue of an increase in n_- , which is proportional to $\bar{Z}$. This is often a very small effect. For example, with 10% carbon as an impurity, $\bar{Z} = 1.5$ and $(Z^2)_{av} = 4.5$. The resulting changes in the values in Table 1 are minor. For $\lambda = 0$, E_+ changes from 290 to 285 kev. Of course, the density is decreased by a factor of 4.5.

Probably the more serious problem is the possible existence of partially ionized impurities which have excitation cross sections of atomic dimensions ($\cong 10^{-16}$ cm²). In many cases, especially where the impurity consists of a 3-electron system (e.g., carbon IV), the ionization energy may be considerably larger than the excitation level. If radiation by these partially stripped impurities is sufficiently large, the electron energy will not rise high enough to ionize and destroy the impurities. The system will be stopped and no plasma will form. This possibility has been studied by the use of a model very similar to that employed in the temperature determinations described in the previous section. The initial calculations assumed carbon IV as an impurity with an excitation level at 8 volts and an ionization level at 64.5 volts. It was found that with a 300-kev initial injection energy, even an impurity density of 5% failed to prevent the electrons from rising to temperatures of the order of kev. In actuality, this would have completely burned out the impurities, i.e., stripped them and removed them as a source of trouble. It would appear then that in the method used in DCX, which involves heating from above, the problem of impurities should not be a serious one.

One intrinsic difficulty of the ORNL ignition scheme is the fact that the trapped ion is just one ion Larmor diameter from the injection snout. Calculations of the heat transfer rate from a plasma to the walls over a distance of one Larmor diameter assuming a linear temperature variation over this distance, give rather large heat transfer rates, many times larger than the bremsstrahlung loss. However, calculations of this sort must be highly suspect. For one thing, the actual plasma density near the walls will be sharply decreased from that in the interior of the plasma because of the high Coulomb scattering rate of these cold particles and their subsequent loss through the ends. Furthermore, the following very rough calculation of the maximum possible energy transfer rate, out of the plasma, indicates that the final particle energy should still be quite high. Since the collision rate per unit volume is approximately $\frac{1}{2}n_+^2\sigma_c v$, the total collisions per second within a Larmor radius, r_0 , of the wall is $\frac{1}{2}n_+^2\sigma_c v r_0 S$, where S is the wall surface area. If the entire energy of both colliding particles, $2E_+$, is transferred out of the plasma per collision, then

$$I_+(E_0 - E_+) = n_+^2 \sigma_c v r_0 S E_+ \quad (9)$$

in the steady state. The left hand side represents the power input in the plasma. By use of the ion conservation relation,

$$I_+ = n_+^2 \sigma_c v P V, \quad (10)$$

this expression may be solved for E_+ . The result is:

$$E_+ = E_0 / (1 + r_0 S / V P). \quad (11)$$

Now $P \cong 0.3$ and $r_0 S \leq V$. Hence

$$E_+ \geq \frac{1}{4} E_0 = 75 \text{ kev}. \quad (12)$$

Of course, this calculation is extremely crude. Nevertheless, it does indicate that a large energy depression of the system will not be caused by collision heat transfer mechanisms.

Another factor is the arc itself. There could be a considerable density of cool electrons in the arc regions through which most of the particles in the plasma pass quite frequently. If the plasma particles continued to pass through the arc there could be appreciable energy degradation occurring. Fortunately, the ions should miss the arc after a short while because of their small-angle Coulomb deflections in the plasma itself. For example, a 300-kev ion moves on a circle of 5 inch radius in a field of about 10 kilogauss. Since the arc radius is about $\frac{1}{4}$ inch, a net scattering of $1/20$ radian (i.e., $\langle \theta^2 \rangle \cong 0.0025$) is necessary in order that a large number of ions miss the arc. It may be shown that if the density of electrons in the arc is not excessive (10^{14} or less), sufficient scattering will occur before the energy has seriously degraded.

Even if the ions do not miss the arc, reasonable plasma temperatures may be obtained. Calculations of the resultant plasma energy in the presence of a given electron density n_- at a fixed energy E_- have been made. The results are quite sensitive to the actual energy distribution of the electrons and their density. The reason for this is that the most rapid heat transfer is from an ion to those electrons whose *velocity* is smaller than the ion *velocity*. A Maxwellian distribution always has a tail of cold electrons which will soak up the energy from the faster ions. On the other hand, there is experimental evidence that the electrons in the arc are far from Maxwellian. For example, the high ratio of hard ultra-violet radiation to optical radiation in the arc (30 to 1) indicates that almost all the electrons are in the 10 to 100 volt energy range. There is also the high electric field gradient (~ 1 v/in.), combined with the long mean free paths, to be reckoned with.

Our preliminary results have been obtained by assuming a Maxwellian distribution, however. The effective electron density should be chosen to be equal to the density in the arc reduced by the fraction of the time a typical particle actually spends in the arc region. This fraction will be of the order of 10^{-2} to 10^{-3} . Since the electron density in the arc is probably about 10^{14} , the effective density should be about 10^{11} to 10^{12} . The results for $E_- = 100$ ev and a variety of values of (effective) n_- are shown in Table 2. It should be remembered that these results will be grossly pessimistic if the very cold electrons (1 ev or less) are missing. As a final note, it should be remembered that the arc is thought of as an ignition scheme rather than as a permanent feature of a thermonuclear device. For example, if the arc

Table 2. Effect of Arc on Plasma Operation

Input ion current per unit volume of plasma, I_+/V , required to sustain various plasma-ion temperatures, E_+ , when the plasma is in contact with a Maxwell-distributed electron bath (the arc) with fixed mean energy 100 ev, effective density n_- . For DCX, V is about 10^4 cm^3

$E_+ \text{ (kev)}$	$n_- \text{ (cm}^{-3}\text{)}$	$I_+/V \text{ (ma/cm}^3\text{)}$
1	10^{11}	3.7×10^{-5}
1	10^{12}	3.7×10^{-3}
10	10^{11}	1.6×10^{-3}
10	10^{12}	1.6×10^{-1}
100	10^{11}	1.3×10^{-1}
100	10^{12}	13

should hold down the full development of the plasma temperature, one might then shut off the arc after the initial plasma has formed and use the residual plasma for further breakup of the molecular ions and heating. Indeed, if the temperature of the residual plasma is above the ignition point, it would be possible to shut off high energy injection entirely and turn to injection of cold neutral atoms to increase the plasma further.

Further details of long range plans for thermonuclear devices based on the ORNL ignition scheme will be presented at a later date. However, one feature is clear. The present set-up will not allow the creation of a self-sustaining plasma because the reaction products will be sufficiently energetic so that they can reach the walls. One way of removing this difficulty is to compress the plasma after it has been formed. At the same time, high energy injection would be discontinued, and the injection of energetic or cold neutral atoms would then begin. Needless to say, the present version of DCX makes no attempt to deal with economics but is, instead, concerned solely with the problem of creating a thermonuclear plasma.

DCX EXPERIMENT

The direct-current mirror machine (DCX) used in the present experiments is shown in Fig. 3. The vacuum chamber and coils are in the lower half of the picture with the 600-kev accelerator mounted above. The transformer-rectifier set that supplies the accelerator is just beyond the right side of the picture. The ion source used for molecular ion beams up to one milliamperere is of the Moak-Good type.³ The beam from the ion source is focused into the accelerator by an Einzel lens which has provision for rapidly turning the beam on and off by means of deflecting plates in the central element. After acceleration, the beam is magnetically deflected through an angle of about 10° , both to aid in directing the beam through the arc and to remove the unwanted ionic species from the beam. The molecular ion beam, entering DCX in the central plane, strikes the arc at the point of closest approach of the beam to the magnetic axis. The point is chosen so that the dissociated atomic ion beam has a circular orbit concentric with the magnetic axis.

A dual vacuum system is used to obtain very low pressures. Fig. 4 is a diagrammatic drawing of the

equipment. The inner chamber, or liner, can be heated to 400°C to remove occluded gases. Pressures as low as 3×10^{-8} mm of Hg have been obtained in this liner. The breakup arc enters the inner liner through graphite baffles, which must be insulated from the liner to prevent heavy currents flowing from the arc column to the baffle plates.

The energetic carbon arc¹ that is used to dissociate the molecular ions in DCX requires some description. When a gas arc is struck between graphite electrodes in a magnetic field, a spontaneous conversion to a carbon arc follows, if the arc power supply is large enough. When the gas feed is discontinued, the pressure in the container falls rapidly, and the arc column becomes less brilliant and very sharply defined and steady. The arc will permit chamber pressures below 10^{-7} mm Hg with pure electrodes. Since the electrodes are placed off-axis in the magnetic field, the arc column is strongly curved in following the magnetic field lines. Fig. 5 is a photograph of the arc taken from a port at the anode end of the equipment. The baffles are absent, so that the whole arc length can be seen.

In usual operation, the arc current is from 200 to 300 amp, and the arc voltage is approximately 150 v. The high efficiency of the arc for the dissociation of D_2^+ ions which was measured in other experiments⁴ has been confirmed by measurements in DCX. The incoming beam and the dissociated beam were measured with a Faraday cup, and dissociation efficiencies

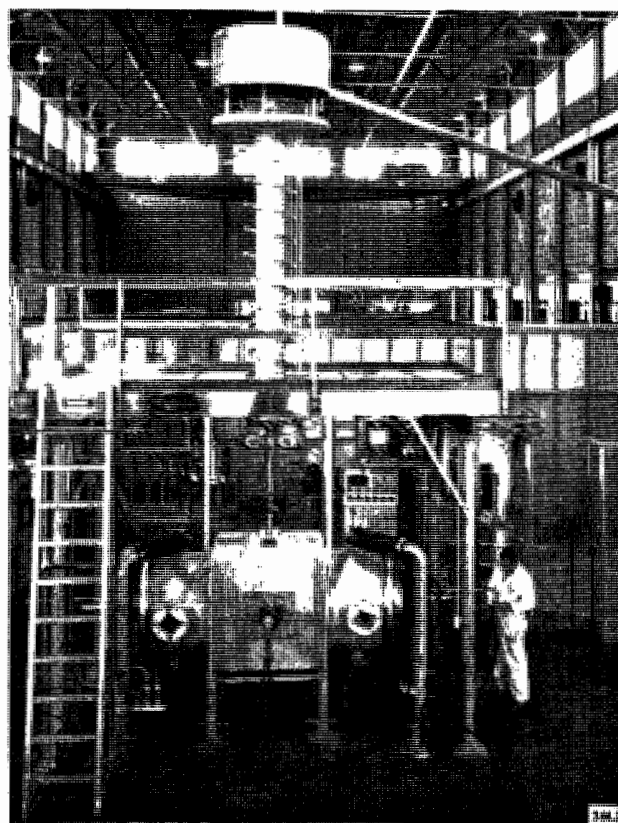


Figure 3. Photograph of the Oak Ridge Thermonuclear Experiment (DCX)

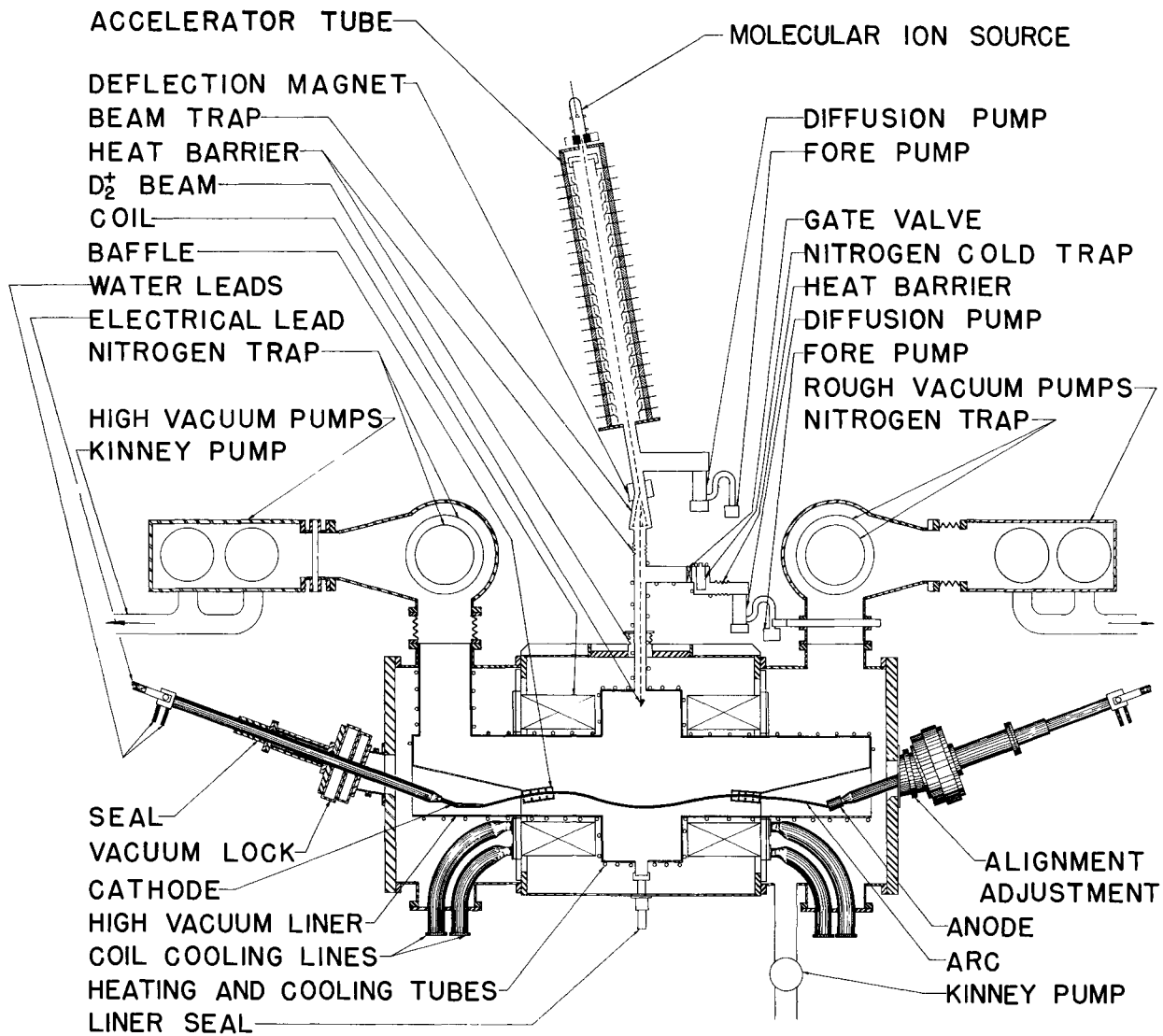


Figure 4. DCX schematic drawing

as high as 40% have been obtained for a D_2^+ energy of 600 keV.

The arc further assists in the experiments by exhibiting a strong pumping action. Neutral atoms entering the arc column are ionized and then are constrained by the magnetic field to follow the arc column through the baffle into the cathode region of the arc. An auxiliary experiment has demonstrated that an arc of this length gas a pumping speed of 7500 liters per second for deuterium gas in the pressure region around 10^{-6} mm Hg. A further discussion of the arc is found in the paper by Luce.¹

The magnetic mirror coils produce a field, on the axis, of 10,000 gauss at the central plane and 20,000 gauss at the coil centers. The coil exciting power of 2.7 megawatts is removed by direct water cooling of the coils. With a D_2^+ energy of 600 keV, the magnetic field is adjusted to give a circular orbit of 10.6-in. diameter for the 300-keV atomic ions.

The energetic neutral atoms produced in the ring by charge exchange are measured by neutral atom detectors and provide a measure of the intensity of the ion ring, its profile, and its buildup and decay time when the molecular ion beam is switched on and off. The neutral atom detector customarily used consists of a Faraday cup behind a 25-micro-inch nickel foil. This foil is thick enough to make the beam leaving the foil essentially an equilibrium beam with respect to charge, irrespective of the charge state of the incident particles. The intensity of the trapped ring can also be measured by the current it induces in a toroidal winding which is linked by the ring. Neutron detectors, bremsstrahlung spectrometers, and optical spectrographs are being used to study the trapped ring and the resulting plasma.

The operation of DCX down to pressures of 10^{-6} mm Hg has been satisfactory. The off-axis operation of the arc is easier than operation in uniform fields



Figure 5. Energetic carbon arc in DCX. The anode is at the left side

since the cathode can be arranged so that the part of the ion stream that misses the cathode tip does not strike the cathode mounting or insulators.

When DCX is operated without arc baffles, the trapped ring is visible because of a large number of incandescent particles or sparks which mark its location. These are small carbon particles which have been torn off the arc electrodes and which fly about invisibly until they are heated to incandescence in the trapped ring. So many carbon particles would have a disastrous effect on the trapped ring by absorbing most of the ions. The baffles remove all but a very few of them. Under typical conditions about one particle per second is seen.

The trapped ring can be seen faintly at higher pressures, but at a pressure of 5×10^{-7} mm Hg a very faint glow fills the central region out to a diameter of 10 or 11 inches with a similar axial length.

Recent runs using hydrogen have been made at a reduced magnetic field to maintain the same ring

diameter. The decay times measured with the neutral atom detector have been as long as 10 milliseconds at a pressure of 5×10^{-7} mm Hg. They have been found to vary inversely with the observed pressure, supporting the belief that charge exchange is the dominant cause of particle loss in the range covered so far. The 10-millisecond decay time corresponds to a "wrap-up" of about 90,000 turns. The injected current of H_2^+ was $200 \mu a$ and the measured dissociated current was $40 \mu a$. The current in the trapped ring was thus 3.5 to 4 amp.

The ring could not be seen directly, even with this circulating current, but the occasional sparks showed that the greater part of the current was concentrated in a toroidal region about three inches minor diameter and ten inches major diameter. At the indicated pressure, if the residual gas is assumed to be principally nitrogen, it is possible to determine by how much the current-to-pressure ratio must be increased in order to reach the calculated burnout conditions. For a surface area of the ring of approximately 2000 cm^2 , a pressure of 5×10^{-7} mm Hg, and an input H^+ current $40 \mu a$, it appears that 14 times more input current or less neutral atom influx is required.

By flushing the system with hydrogen during bakeout so that most of the residual gas is hydrogen, it is hoped to reduce the factor lacking for burnout to about 4. For a deuteron beam, burnout is expected to be more difficult to achieve. Experimental steps are being taken toward increasing the input current and reducing background pressure.

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Intense Gaseous Discharges

By J. S. Luce *

Much of the gas discharge research at the Oak Ridge National Laboratory had its beginnings in the ion-source development work carried on under the war-time Electromagnetic Isotope Separation Project which was begun at the University of California Radiation Laboratory.¹ Since the summer of 1954 many of the studies have been motivated by the needs of the thermonuclear program. These studies have led to the evolution of several new types of arcs which exhibit certain interesting characteristics. These discharges, all of which require an axial magnetic field for stable operation, appear to possess properties which are qualitatively different from those of previously investigated arcs. This paper describes the development of these arcs, their use in the ORNL thermonuclear program, and possible applications of more general interest. A more thorough study of such discharges may eventually lead to a better understanding of plasma physics.

In February 1955, the observation was made at ORNL that energetic deuterons could be trapped in a magnetic field by the injection and dissociation of deuterium molecular ions.² The idea is illustrated in Fig. 1.

The essential feature of this method is the injection of a D_2^+ ion across a magnetic field and the subsequent dissociation of this ion into a D^+ ion and a D^0 atom or into two D^+ ions and an electron. In either case, each resulting deuteron has half the momentum of the original molecule and hence half the Larmor radius in the magnetic field. If dissociation occurs over a certain range of the molecular ion orbit, the resultant D^+ is trapped and does not return to strike the injector. It has been shown by calculation^{3, 4} that if a sufficient number of deuterons with energies of the order of hundreds of kilo electron volts could be trapped, a thermonuclear plasma would be formed that would maintain a high temperature and could then be fed by much less energetic fuel.

Much of the effort in the last three years has been directed toward the development and exploitation of devices designed to produce intense beams of molecular ions and devices to dissociate these ions with high efficiency. This program has culminated in the

ORNL experimental device, DCX,⁴ in which an attempt is being made to achieve a reacting plasma by the means just outlined.

The ion-source development program has led to the construction of sources capable of producing 2.5 amperes of D_2^+ ions and five amperes of D^+ ions. This program is described briefly in the Appendix.

The first part of the paper is devoted to the discharge now used for dissociating molecular ions, which has been given the name *energetic carbon arc*. A brief description of the arc is followed by an account of its evolution and development. The experiments which established its high efficiency for dissociation are discussed in some detail; some experimental results on arc properties are described; and details of techniques of operation complete the discussion.

The last part of the paper describes experiments on the newly-discovered energetic deuterium arcs and some speculations on their possible uses.

ENERGETIC CARBON ARC

The discovery which, more than any other, made the high-energy molecular-ion injection and trapping scheme appear feasible was that of a high-current dc arc which was found to have a remarkably high efficiency for dissociating molecular ions. This arc is run between graphite electrodes in a magnetic field of several kilogauss and is supported by carbon from the electrodes with no supplementary gas input.

The arc is ordinarily struck as a dc gas arc with a radio-frequency voltage applied for initial breakdown. After the discharge has become established, the gas flow is completely stopped. Initially, there is high local pressure (1 to 5 microns Hg) resulting from outgassing, and a secondary plasma is observed around the arc. When the pressure has dropped below about 0.1 micron Hg, the secondary plasma disappears, the light emitted by the anode increases, and the discharge is observed to separate into bright blue filaments which move slowly across the magnetic field. These filaments terminate at bright cathode spots located on the cathode rim.

Most of the arc current appears to be carried in a hollow cylinder whose diameter is roughly that of the cathode, customarily about half an inch. Figure 2 shows a set of wafers cut from thin graphite plates as the arc burned through them. The photograph was

* Oak Ridge National Laboratory, operated by Union Carbide Corporation for the U.S. Atomic Energy Commission.

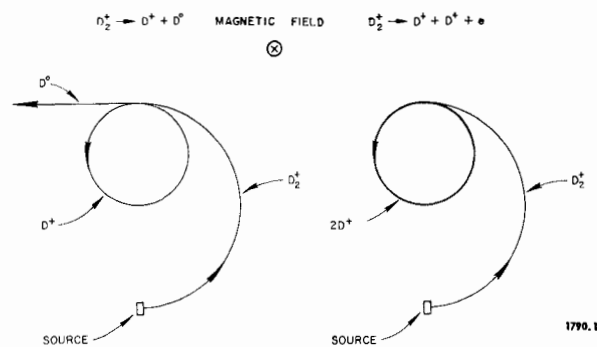


Figure 1. The trapping of energetic deuterons in a magnetic field by dissociation of deuterium molecular ions

made shortly after the arc was extinguished; the carbon pieces were still quite hot. In arcs which are several feet long, the current channel appears to converge before reaching the anode.

Associated with the production and disappearance of filaments, there appear to be electrical or thermal gradients at the cathode which cause small pieces of hot carbon to be ejected at high speed (10 to 20 meters per second). Many of these particles rebound from solid objects without disintegrating. A photograph of an arc and some of these fragments is shown in Fig. 3.

It has recently been discovered that this arc is but one of a broader class of arcs, all of which are characterized by electron energies of tens to hundreds of electron volts and, it is believed, by comparable ion energies. These have been given the generic name *energetic arcs*; some preliminary results on their behavior are given in a later section.

Developments Leading to its Discovery

The initial investigations in the program were made with arcs of the type developed for the isotope separator (mass spectrometer) ion sources.¹ It was with such an arc that the ORNL experiments bearing on the diffusion of ions across a magnetic field⁵ were done. A typical arrangement of such an arc is shown in Fig. 4.

The electrode which defined the arc column, although useful in the original application, proved to be a serious obstacle to achieving high arc currents because of excessive electron drain between the filament and the defining electrode. Accordingly, modifications were attempted which dispensed with the defining electrode. Figure 5 shows an early arrangement of this kind.

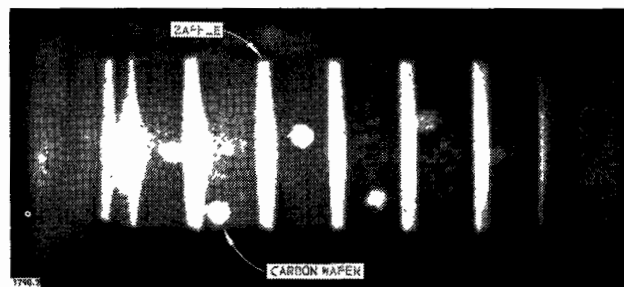


Figure 2. Half-inch carbon wafers, cut from graphite plates by the arc

A sufficient flow of gas is supplied through a hole in the cathode to provide ionization and space-charge neutralization at the cathode face with minimal increase in pressure in the rest of the system. The discharge is defined by the boundary of the cathode itself. In striking, a radio-frequency discharge is customarily used to provide initial breakdown of the gas. In operation, a sheath forms at the cathode face, providing an electric field to accelerate the ionizing electrons. Arcs of this type have been operated with currents up to 5000 amp, and many of their properties have been studied.

Their markedly different behavior under different pressure conditions is worth noting. It was found that at 50 microns pressure and no magnetic field the arc would blow up to a diameter of several feet (Fig. 6). At a pressure of 0.5 microns the arc would spiral in several patterns depending on the current and voltage applied (Fig. 7).

Experiments designed to measure the efficiency of gas arcs of this kind for dissociating molecular ions

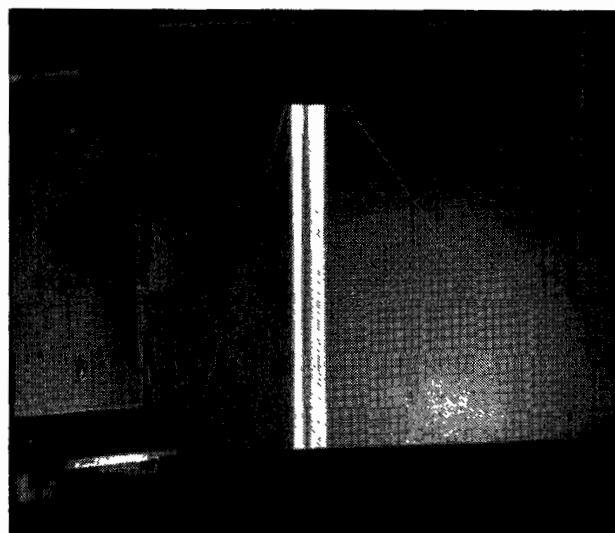


Figure 3. Twelve-inch energetic carbon arc

were unsuccessful because of high gas pressure. In the course of attempts to operate these arcs at successively lower pressures, it was found that under proper conditions a discharge could be made to operate between carbon electrodes without a sustaining gas input. It was later found possible to operate this type of discharge in lengths of 6 feet or more. The low gas pressure associated with this carbon arc made it immediately evident that this was a strong candidate for use as a dissociating agent for molecular ions, and experiments to test its efficiency were undertaken.

Molecular-ion Dissociation Experiments

Of most immediate interest to the thermonuclear program has been the energetic carbon arc's efficiency for dissociating deuterium molecular ions D_2^+ which pass through it. The initial experiments were conducted with 20 kev D_2^+ ions in a uniform magnetic field of about three kilogauss. The ions were produced in a modified source of the type originally used for isotope separation. The exit aperture of the source

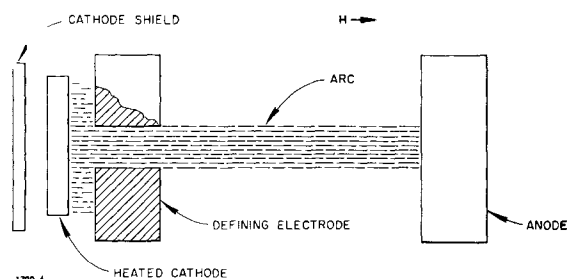


Figure 4. Arc with defining electrode

was a slit oriented parallel to the magnetic field, which produced a ribbon-shaped ion beam. The carbon arc was placed at the opposite end of the orbit diameter, a location of first-order focus for the beam, and current collectors were located on the anticipated paths of the molecular beam and the dissociated atomic beam. The configuration is shown in Fig. 8.

The current collectors were water-cooled and were provided with raised portions around their edges to prevent the loss of secondary electrons. The target for the atomic beam was well shielded from the arc by a water-cooled copper box. An average figure for the molecular ion current during these experiments was 60 ma. The pressure was always under 3×10^{-5} mm of Hg, and the arc current was 300 amp.

The intensity of the atomic beam was determined both electrically and calorimetrically. The latter determination was accomplished by measuring the temperature rise of a metered flow of cooling water. The two sets of data agreed to within a few per cent, about as closely as would be expected with the known intrinsic measurement errors.

The major result of the experiment was the observation that 40% of the molecular ion beam was dissociated. Allowing for the beam's imperfect focus, it has been calculated that 70% of the molecular ions which passed through the arc were dissociated. A subsequent measurement made with 700 amp of arc current was made inexact by beam losses due to high pressure but indicated that approximately 90% dissociation was achieved.

No metered loss of ions was detected in the electrical measurements. The sum of the currents in the $R/2$ and the residual D_2^+ beam (plates A and B, Fig. 8) always equalled the current reading of the D_2^+ beam before dissociation.

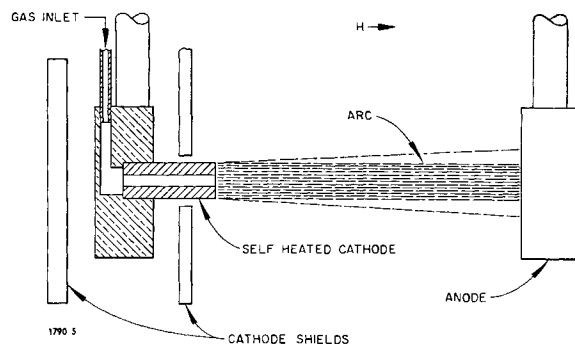


Figure 5. Cathode-controlled dc gas arc

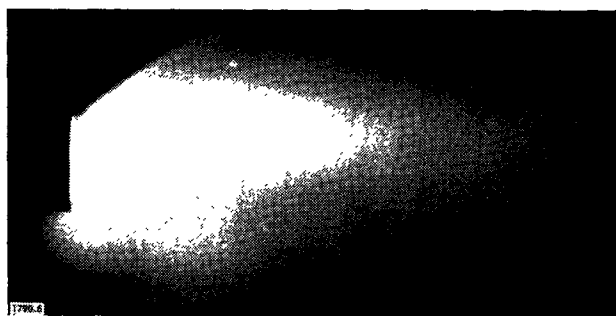


Figure 6. Behavior of 500 amp argon arc at 50 microns pressure with no magnetic field

That dissociation occurs only within the arc was shown by the good focus of the dissociated beam. The existence of this focus is indicated by the following observations: (1) probe-scanning of the focal line of the D_2^+ beam (at the arc position) and the focal line of the dissociated D^+ (at the position of collector B in Fig. 8) showed no appreciable difference in width of focus; (2) stainless steel strips placed at the focal line of the dissociated beam were readily melted through even when a beam current as small as 10 ma was used. Such a strip is shown in Fig. 9.

Only brief studies were made on the circulating deuteron beam. These were sufficient to demonstrate that the deuterons did remain in the system for a substantial number of turns and to suggest that they did not lose appreciable energy before leaving the system. An indication of the shape of the circulating beam was obtained by permitting it to burn through thin aluminum foils. A comparison of the pattern made by a single-pass beam with that made by a circulating beam was obtained by making use of a movable intercepting plate, as shown in Fig. 10.

The respective foils are shown in Fig. 11; the one on the right shows that all of the circulating particles have the same radius of curvature, indicating negligible energy loss in the arc.

The extremely encouraging results of the dissociation experiments led to the decision to incorporate the energetic carbon arc in a device designed to create a steady-state high-temperature plasma by the injection and trapping of energetic deuterons. This device, DCX, and the basis for its design are described in another paper.⁴ One crucial point in

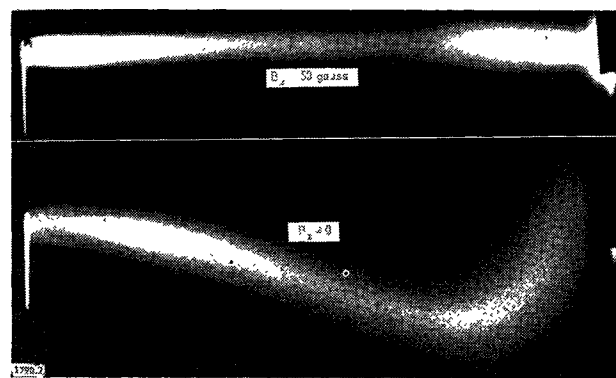


Figure 7. Behavior of 500 amp twelve-inch argon arc at 0.5 microns pressure and two magnetic field values

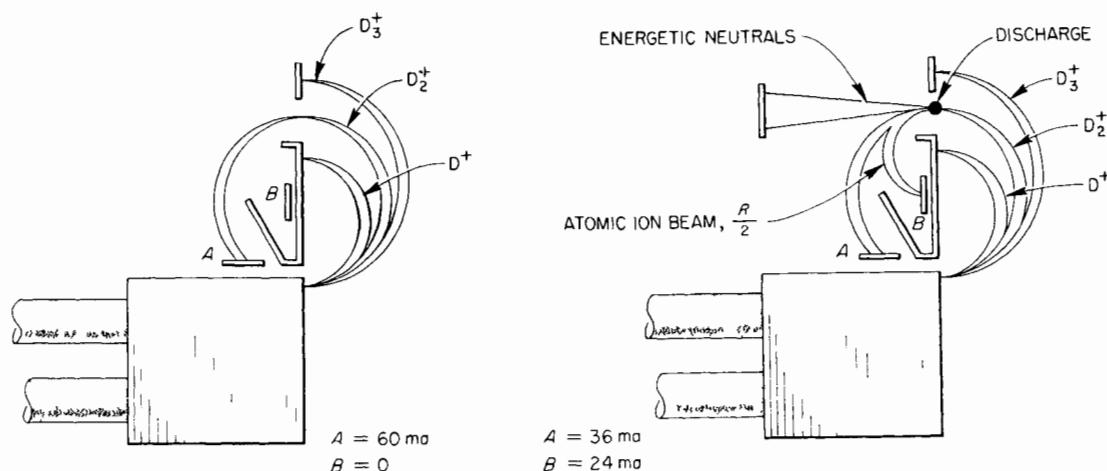


Figure 8. Experiment to measure dissociation efficiency of carbon arc

determining the feasibility of the 600 kev D_2^+ injection energy chosen for DCX was that the dissociation efficiency of the arc remain substantial at that energy. Experiments⁶ verified that this was so.

Properties of the Energetic Carbon Arc

The remarkably high dissociation efficiency of the carbon arc and the prospect of forming a thermonuclear plasma adjacent to it have occasioned a considerable interest in its detailed properties. The measured dissociation efficiency for 600 kev D_2^+ ions⁶ implies that the product of particle density and cross section for the event is of the order of 0.1 cm^{-1} . Thus, either both quantities are extremely high in terms of ordinarily expected values ($n \sim 10^{13} \text{ cm}^{-3}$, $\sigma \sim 10^{-16} \text{ cm}^2$), or one quantity is anomalously high. Studies so far have failed to provide unambiguous answers to these and other key questions. Therefore, only a list of experimental results will be given here; no attempt will be made to use these results to analyze the arc.

1. The energetic carbon arc has been operated at lengths of from 6 inches to 6 feet. The arc voltage, with an arc current of 250 amp, was found to be 55 v for a 12-inch length and to increase roughly linearly to 120 v for the 6-foot length. It was found to be a slowly-varying function of the arc current, the voltage increasing for increasing current. For any configuration, there was a critical current below which the arc would not continue to run. For a 12-inch arc, this current was as low as 50 amp for a few seconds.

2. Measurements were made of the arc's efficiency for stripping additional electrons from 20 kev N^+ and He^+ . The experimental configuration was the same as that shown in Fig. 8. The change of charge on the stripped ions reduced their radius of curvature by a factor of 2. The results were: for nitrogen, 12% stripping efficiency; for helium, almost zero.

3. Spectroscopic observations⁷ showed lines requiring the presence of 43 ev electrons for their excitation. The dominant spectra were those of C^+ , C^{++} , and

C^{+++} . The relative intensities of these lines varied with arc length, the spectra corresponding to higher stages of ionization increasing in intensity as the arc length became greater. The resonance line of neutral carbon at $2478 \text{ \AA}$ could not be detected in the center of a 6-foot arc. Doppler shifts observed in the long arc indicated ion rotational energies as high as 50 ev. Velocities parallel to the axis were found to correspond to energies of only a few ev except in the vicinity of the cathode.

4. An experiment was performed to measure the amount of energy radiated from a one-foot section at the center of a 6-foot arc. Figure 12 shows the arrangement of the apparatus for this measurement. A water-cooled cylinder was placed around the arc and shielded from the electrode radiation by baffle plates which fit closely around the arc. The power absorbed by the cylinder was determined from the temperature rise of its cooling water, whose flow was measured. With an arc current of 240 amp and an applied potential of 120 v, it was found that the energy transferred to the cylinder varied between 2.5 and 3 kw, depending somewhat upon pressure. If this number constitutes a good average figure for the power radiated along the arc's length, about half of the total arc power input is thus accounted for. Experiments⁶ suggest that most of this radiation comes off in the form of hard ultraviolet light ($\lambda \sim 1200 \text{ \AA}$). This is a plausible finding in view of the fact that the resonance radiation from C^+ , C^{++} , and C^{+++} lies in the hard ultraviolet region. Radiation balance calculations yield a probable ion density range of 10^{13} to $10^{14} \text{ ions/cm}^3$ depending on the electron energy assumed.

5. Probe measurements and arc-current studies showed that the operation of the arc was accompanied by considerable rf oscillation with frequencies at least up to tens of megacycles.

6. Preliminary results of microwave experiments⁸ indicated the arc to be strongly absorbing for four-millimeter and twelve-millimeter waves. The sug-

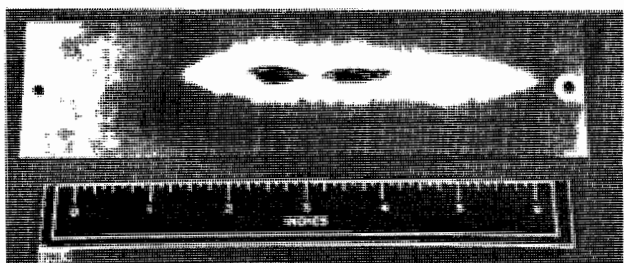


Figure 9. A 0.005-inch stainless steel strip burned through by a beam of deuterons produced by dissociation of D_2^+ in a carbon arc

gested density is above 7×10^{13} electrons/cm³, if the theory is adequate.

7. Studies of the vacuum properties of the arc^{9, 10} indicated not only that it would be useful as an ion pump, but also that the deposited carbon adsorbs gases and thereby provides additional pumping action. This action was found to be strong enough to make it possible to valve off the diffusion pumps of apparatus in which the arc was operating. The base pressure under such circumstances was of the order of 10^{-5} mm Hg. When the arc was run through a chamber from which the electrodes were excluded, a base pressure of 2×10^{-7} mm Hg was achieved in the (unbaked) chamber.

Techniques for Carbon Arc Operation

In the two years since its discovery, the energetic carbon arc has been operated with several markedly different magnetic field configurations. Initially, it was run in the uniform field of a mass separator unit with its length limited to about 12 inches. There, the assembly was arranged roughly as shown in Fig. 5 with the anode at ground potential. The major auxiliary components were the floating baffle plates which prevented damaging strike-overs from the cathode assembly.

The feasibility of operating a 6-foot carbon arc was first demonstrated in the long solenoid shown in Fig. 13. Typical electrode assemblies are shown

in Fig. 14. These are inserted in opposite ends of the solenoid. In the experiments it was found convenient to replace the floating cathode shields with pieces of quartz tubing slipped over the cathode holder and with a floating baffle plate to prevent the backstreaming ions that miss the cathode proper from striking the cathode holder.

In the magnetic mirror apparatus, DCX, the carbon arc has been operated in both on-axis and off-axis positions (Fig. 15). The electrodes are customarily placed as much as 10 inches outside the magnetic field maxima. With the arc off-axis it was found that the backstreaming ions no longer struck the supporting assembly for the cathode; the need for the quartz cover was thus eliminated. This method of operation also provided the first opportunity to study these ions. It was found that an insulated plate in their path acquired a positive potential as high as 110 v with respect to the cathode. When the plate was operated at anode potential, it drew a current of 20 amperes. The total arc voltage was about 140 v.

In order to reduce the conductance for gas flow between the two parts of the DCX dual vacuum system, it was found necessary to pass the arc through a series of close-fitting baffles at each end of the inner chamber. This method of arc operation presented two problems, one of mechanical alignment and the other of baffle insulation. The alignment problem was solved by burning holes through the baffles with the arc at low current and with some gas feed, then mechanically reaming the holes to a larger diameter. In order to achieve stable arc operation through the baffles it was found necessary to insulate the baffle holders carefully, since inadequate insulation allowed a part of the arc current to be by-passed through the baffle assembly.

ENERGETIC DEUTERIUM ARCS

The discovery of a deuterium arc capable of dissociating 20 kev D_2^+ ions resulted from a proposal by

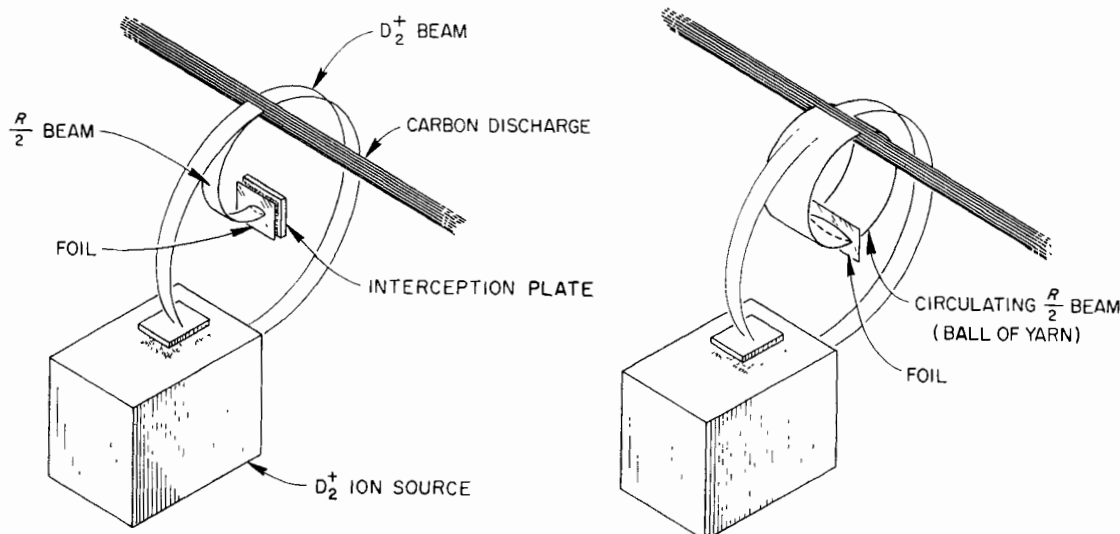


Figure 10. Foil-burning experiment to demonstrate beam build-up

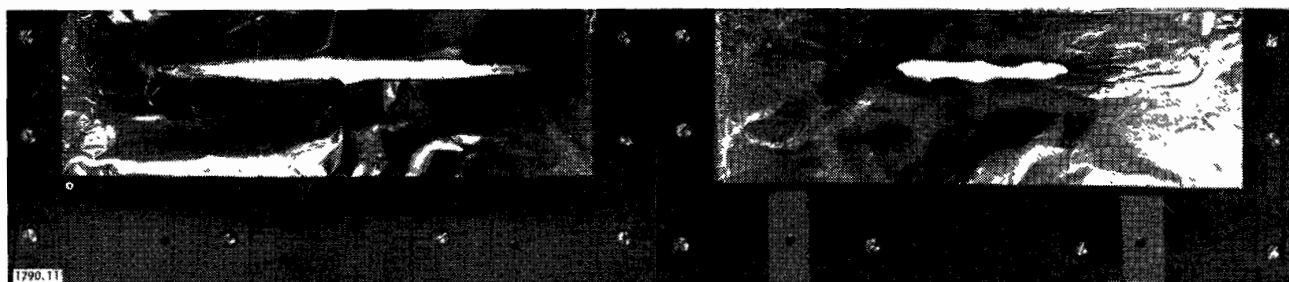


Figure 11. Foils burned through by deuteron beams in a uniform magnetic field. Width of single-pass beam is 2 inches

P. R. Bell. He suggested that a gas arc could be made to operate in a manner similar to the carbon arc if some way were found to limit sharply the extraneous gas emitted into the vacuum chamber. He proposed to achieve this by improving the tubular anodes (which had formerly failed to produce adequate gas arcs) by making them longer and making them fit the arc closely. With this improvement, gas fed in at the anode base would have to pass through a considerable length of arc before it could be emitted into the vacuum chamber and would thus have a high probability of being ionized.

Several attempts were made with a tungsten anode 5 inches long and bored to an inner diameter of five-eighths of an inch. The outer diameter of the tungsten cathode was half an inch. Electrons from the cathode were thus expected to clear the anode wall by one-sixteenth inch. Eventually, a low pressure arc (5×10^{-5} mm Hg) was established with a current of 150 amp and a potential of 200 volts applied across a 4-inch gap. An arc of this type was demonstrated to have a 10% efficiency for dissociating 20 kev deuterium ions.

A subsequent discovery, which will be described, led to an improved version that provided 25% dissociation, although in all these cases tungsten impurities may have played a part. The electrode configuration is shown in Fig. 16. Investigations of this arc are still in their initial stages, and very little is known about the mechanism of its operation. The arc has, however, displayed some unexpected properties. Investigation of these has led to several variations in arc configuration and has suggested possible applications of much broader interest than originally anticipated.

One curious feature of these arcs is the fact that they can be run from the inner surface of a hollow cathode, thus depending on emission across the magnetic field.

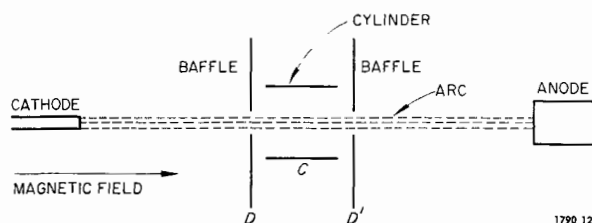


Figure 12. Experiment for determination of power radiated by the arc

Hollow cathode arcs are not uncommon in applications in which little or no magnetic field is present, but in this case fields up to 6000 gauss have been used and no decrease in the emission observed. One hypothesis which has been advanced to explain this fact is the following. Electrons are accelerated radially from the inside wall of the cathode across a very thin sheath. Some of those that suffer collisions before returning to the wall are trapped and thereupon are drawn out of the cathode by an axial potential gradient in the arc. The idea is demonstrated in Fig. 17.

The hypothesis is supported by the observation that at low pressure the arc current appears to be carried in a thin hollow cylinder close to the inside cathode wall. Under certain conditions, the current flow to the anode was found to take place in a well-defined ring, giving rise to intense local heating. Figure 18 shows the result of insufficient cooling of a tungsten anode. Improved cooling was achieved for both the cathode and anode by inserting the electrodes into thick copper jackets, as shown in Fig. 19. The hole in the copper was made undersized

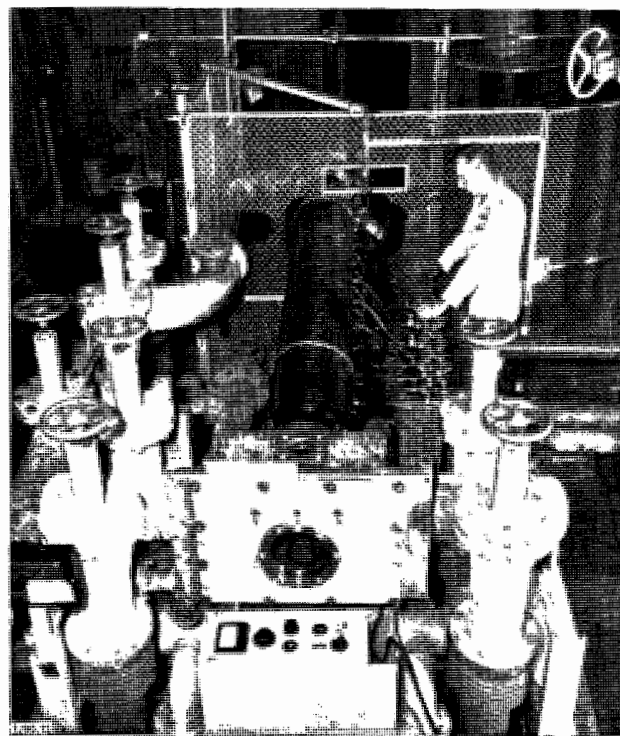


Figure 13. The ORNL long solenoid apparatus

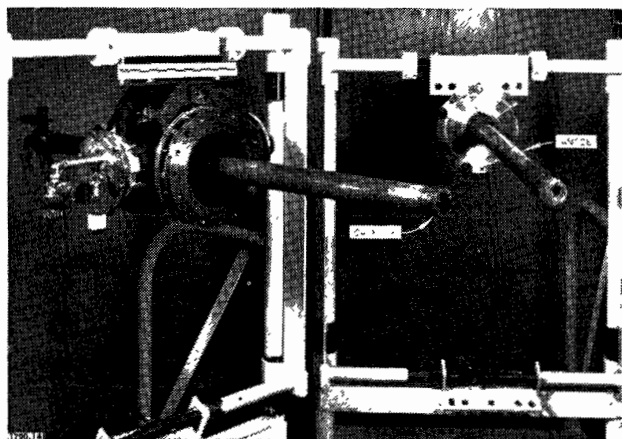


Figure 14. High-current vacuum carbon arc electrode assemblies

to the extent that it was necessary to heat the copper to several hundred degrees in order to insert the tungsten. During operation the copper was water-cooled; the differential expansion, which occurred when the tungsten was heated by the arc, produced a very tight fit and resulted in good heat transfer.

As might be expected from the cross-field emission, the deuterium arc was found to require a much higher applied voltage than the carbon arc. The arc voltage was found to vary inversely with the gas input. The exact part played by the hollow anode is hard to understand, since the highest efficiency for molecular ion dissociation was achieved when all of the supporting gas flow was into the cathode. This might have been due to contamination (tungsten) from the anode.



Figure 15. Off-axis arc in DCX. About three feet of arc are visible

This last observation led to experiments with arcs running between hollow cathodes and solid anodes. For these experiments, anode heating was minimized by the use of a rotating anode assembly, which effectively spread the electron bombardment over a larger surface area. Both carbon and copper anodes have been used, the former arranged to permit anode gas feed. These experiments are still in progress and will be described in a future report.

The need to reduce anode heating with its consequent emission of impurities led to energetic arc experiments with a reflex (Phillips Ion Gauge, or PIG) geometry. In one arrangement, hollow cathodes were used at each end of a long tubular anode. An insulated plate substituted for one of the cathodes was found to charge negatively with respect to the

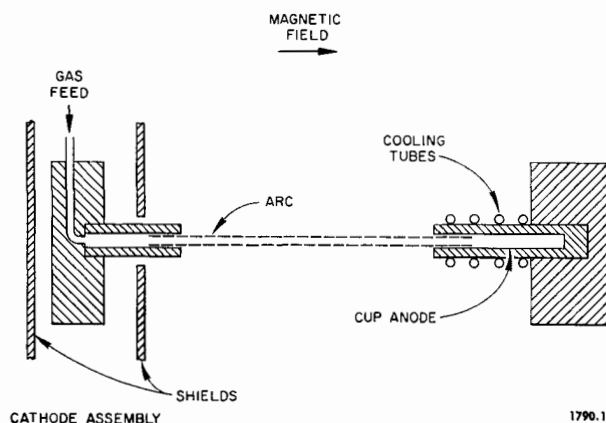


Figure 16. An energetic deuterium arc with a cup anode

remaining cathode to a voltage as great as the anode-to-cathode voltage. This phenomenon is believed to be closely related to the "Mode II" arc behavior studied at ORNL.¹¹ The arrangement is shown in Fig. 20.

This arc has even higher resistance than that shown in Fig. 16. It has been operated at potentials up to one kilovolt. A power supply which will provide 2.5 kv is now under construction.

Formerly, arcs with large cross-sectional areas were almost impossible to obtain because of current limitations and uneven emission. Even in small arcs, emission tends to run from cathode spots. The cross-field emission described here provides a large surface area from which electrons may enter the system. The total current required to run the arc is considerably reduced since the electron stream is confined to a thin cylindrical sheet.

When the pressure is lowered below some critical value, adequate space-charge neutralization is provided inside the cathode but not in the main volume. In one test, a quarter-inch diameter hollow stream of electrons was "blown up" to more than 3 inches in diameter in a distance of 5 inches. The mechanism for this spreading is not entirely understood, inasmuch as the arc was operating in a 3000 gauss magnetic field, but it does seem clear that it

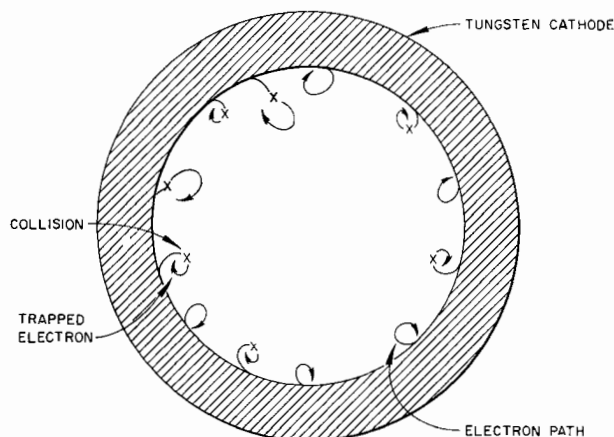


Figure 17. Possible mechanism for arc operation from inside a hollow cathode

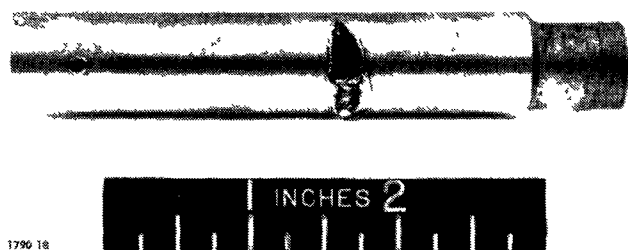


Figure 18. Hollow tungsten anode burned through by an energetic deuterium arc

was associated with imperfect space-charge neutralization. The oscillatory electric fields produced when streaming electrons are present in a plasma often give rise to organized ion motion of considerable energy. This effect has been observed by several investigators.¹¹ The high-voltage, high-resistance characteristics of the energetic PIG arc should enhance this process.

One of the first projects undertaken by the ORNL Thermonuclear group was the development of a system whereby a high potential (50 kv) could be sustained between a solid gas arc and a surrounding hollow gas arc running coaxially, or between an arc and a coaxial cylinder.¹² It is notable that in the high-voltage energetic PIG arc, there has been a return to a very similar geometry.

It is interesting to extrapolate to the future somewhat and to speculate on possible applications of this new class of arcs. Since fairly high electron and ion energies appear to be involved (of the order of that which would be imparted by the arc potential), an attractive proposition would be to pinch off a section of arc between two magnetic mirrors and compress it so as to multiply the energy and perhaps thereby achieve ignition of a thermonuclear plasma. The design of an experiment to test this concept is now under way. The energetic PIG arc concept is applicable also to toroidal geometry, with or without compression (see Fig. 21). The most exciting possibility presented by the hollow energetic arcs, however, is that of surrounding a reacting plasma with an arc wall. Should this concept be brought to its ideal

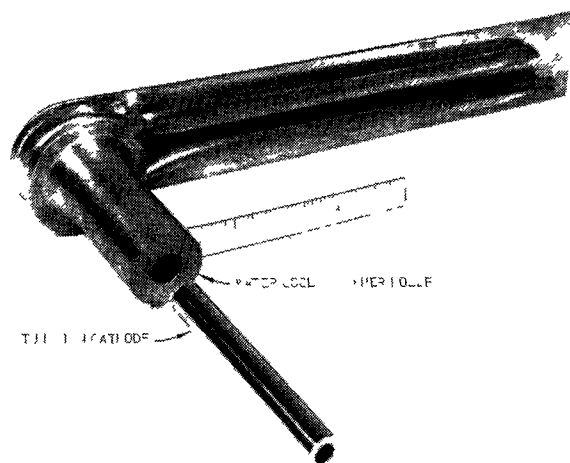


Figure 19. Electrode and holder for energetic gas arc

fruition, the sputtering problem would be eliminated, because a thermonuclear plasma contained inside a hollow arc would be protected from cold neutral atoms which would normally flood in from the walls of the vessel.

The applications will almost certainly not be limited to the thermonuclear field. For instance, P. B. Moon¹³ has considered the breakup of H_2^+ , using a gas target, as a method of injection into proton synchrotrons. The energetic arcs, capable of high dissociation efficiency, sharply localized dissociating action, and low-pressure operation would be admirably suited to such a use. Heavy-ion accelerators could also make profitable use of such devices for stripping successive electrons from the ions to be accelerated.

CONCLUSION

The ORNL concept of producing a high-temperature plasma by the dissociation-trapping of energetic ions was brought within the bounds of feasibility by the development of high-current molecular ion sources and by the discovery of the energetic carbon arc. An experiment, DCX, is now under way to determine

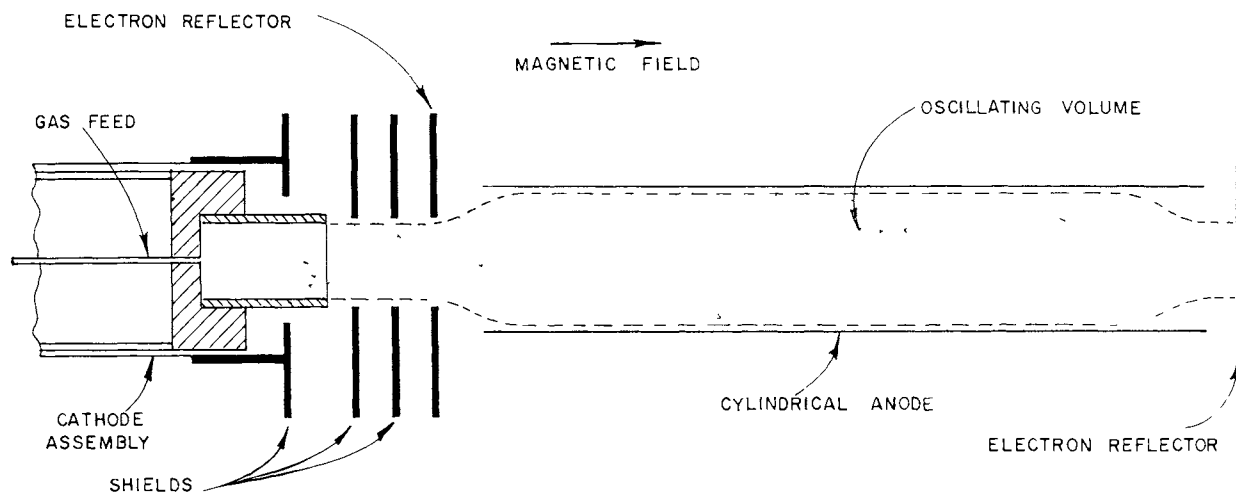


Figure 20. Energetic PIG arc experiment

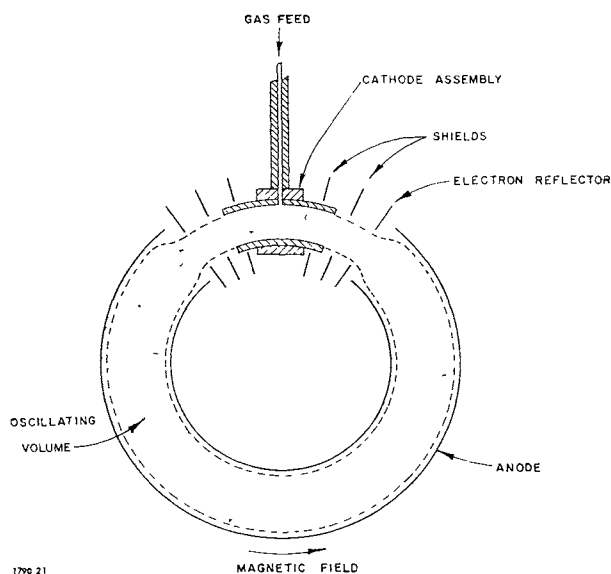


Figure 21. Energetic PIG arc in toroidal geometry

whether this trapping system is, in fact, capable of producing a thermonuclear plasma. None of the experimental results to date has indicated the presence of insurmountable obstacles to the achievement of this goal. However, if such an obstacle should be found to be presented by the energetic carbon arc, there is now available for substitution the energetic deuterium arc. In addition to the obvious advantages of an arc which does not contain heavy, partially-stripped ions, the deuterium arc has been shown to be capable of a flexibility of operation which should permit a considerable measure of control over its effects on the plasma. In addition, the deuterium arc appears to be susceptible to other, and perhaps

even more interesting, applications than that of a dissociating agent. A few of these have been mentioned above.

The study of energetic arcs is clearly in its infancy. Some of these arcs may even now constitute plasmas which have extremely high temperatures.

APPENDIX

Development of Ion Sources yielding Ampere Currents of Atomic or Molecular Deuterium Ions

The approach to the establishment of an energetic deuterium plasma by molecular dissociation and its promise in the thermonuclear program pre-suppose the existence of intense currents of D_2^+ ions for injection purposes. In 1955, certain modifications of calutron hot-cathode sources were explored, and operating conditions in hydrogen were sought, under which the molecular ion fraction of the output current could be maximized. It was soon found that, in such an ion source, the output can be controlled at will so as to give an ion yield that is either predominantly atomic or predominantly molecular, and that the most important determining factor is simply the pressure. The atomic ions are favored at high pressures (red color of discharge) and the molecular ions are favored at low pressures (blue color). This behavior can be qualitatively explained on the basis of a difference in the mean energies of the electrons in the ion source plasma. The minimum electron energies for the various processes are as follows:

1. $H_2 + e \rightarrow H + H + e \dots 11.4 \text{ ev}$
2. $H + e \rightarrow H^+ + e + e \dots 13.5 \text{ ev}$
3. $H_2 + e \rightarrow H_2^+ + e + e \dots 15.6 \text{ ev}$

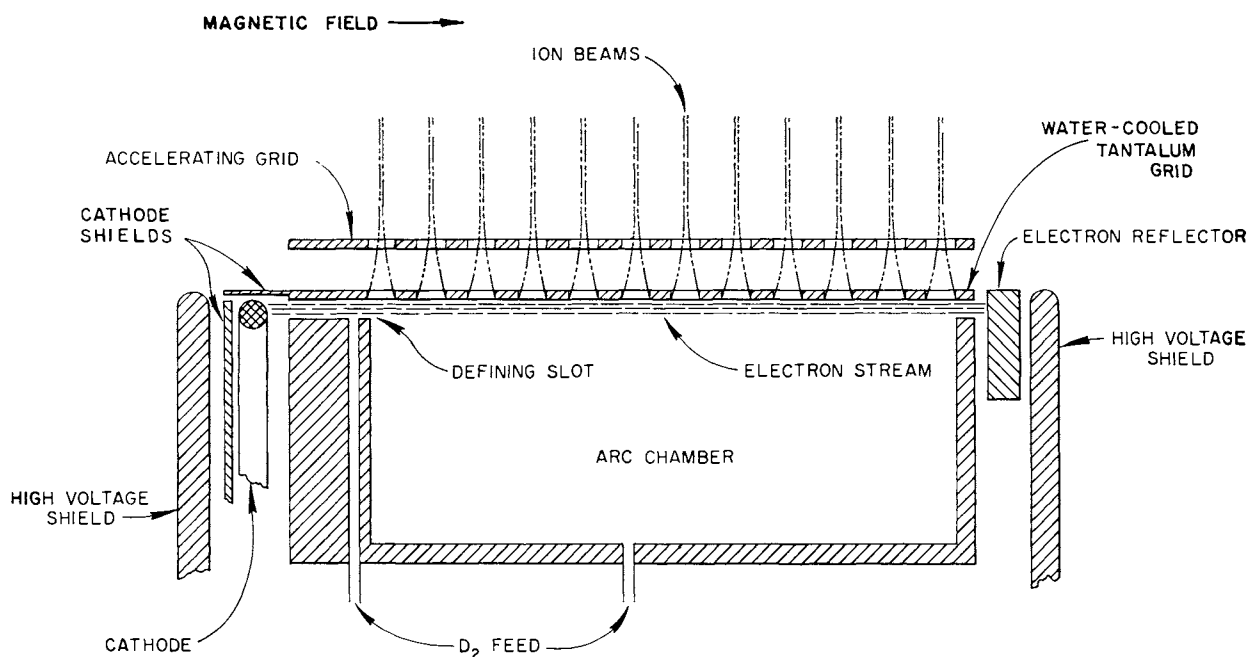


Figure 22. Grid ion source

from which it would appear that if multiple collisions degrade the energy of the electrons under conditions of high pressure, a combination of the first two processes might dominate over the third process. A corollary of such an explanation would be that process 3 is primarily responsible for the production of molecular ions rather than a succession of processes 1 and 2 followed by recombination. The physics of the situation has been discussed in greater detail by Massey and Burhop¹⁴

The use of metal walls in an ion source to enhance the output of the molecular species is well known, in the adaptation of the calutron source for this purpose, water-cooled tantalum has been found to be suitable, and indeed it is especially important in the exit slit of the device. Figure 22 shows a "grid" ion source in its adaptation for work with deuterium. The over-all length of the device is about 8 inches. In the output, H^+ dominates at high pressure and current, H_3^+ at intermediate pressure and current, and H_2^+ at low pressure and current. The pressure setting suffices to allow one to choose between the following operating ranges

	Voltage ^a	Current ^a
Favorable to H^+	80 v	15 amp
Favorable to H_3^+	100	10
Favorable to H_2^+	150	5

^a Measured between cathode and arc chamber

An output of 2.5 amp of ions with an H_2^+ content of 85% has been achieved from a grid source of this kind having an arc area of three square inches. Under appropriately altered conditions, five amperes of ions have been extracted from one square inch of arc area, of which 95% were atomic.

ACKNOWLEDGEMENTS

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Effect of an Applied Pressure Gradient on a Magnetically Collimated Arc

By R. V. Neidigh* and C. H. Weaver†

This report describes experimental observations made in connection with a magnetically collimated arc having an applied pressure gradient along its length and presents possible explanations of the phenomena observed. It is believed to be pertinent to thermonuclear research because it involves the transport of a plasma across a magnetic field and the acceleration of ions without use of solid electrodes and furnishes evidence concerning the behavior inside magnetically collimated arc discharges as the pressure is decreased. The observations are repeatable to an unusual degree and are believed to be sufficiently interesting to be reported at this time, even though a thorough understanding of the entire mechanism involved has not been reached.

The apparatus used for the study was so designed that a pressure gradient could be established along the center-line of the arc, between the limits of 10^{-3} and 10^{-6} mm Hg. A schematic diagram of this apparatus is shown in Fig. 1. The two end plates are made of one-half inch copper, are approximately six inches square, and are at ground (zero) potential. The circular holes in the end plates through which the arc passes are approximately one-fourth inch in diameter, and a tube feeds into the left hole from an external gas supply. The hot cathode and associated shield shown at the left plate are connected to a voltage source, so that they can be operated at voltages from 0–500 v with respect to ground, and at currents from 0 amp to 5 amp. The shield shown behind the right plate is floating electrically, being connected to ground through a high resistance voltmeter. The probe used to study the region adjacent to the arc is made up of a one-sixteenth inch diameter copper wire covered, except at its extreme end, by a quartz tube and an external copper shield that is grounded. Thus the probe makes available an electrical signal from one "point" in the region adjacent to the arc. The pressure in the entire closed region is controlled by the gas feed rate and pumping rate. The pressure is measured by an ion gauge

(not shown) placed inside the hole (defining slot) of the left end plate. Gases used were H_2 , He, N_2 , Ne, Ar, Kr and Xe; the results are qualitatively the same for all these gases.

Two distinct types of gas discharges are obtainable in this apparatus, and are here designated Mode I and Mode II. The Mode I type of arc exists when a negative voltage is applied to the hot cathode and pressures in both the defining slot and the probed region are of the order of 10^{-3} mm Hg. In such case, the arc is as shown in Fig. 2 (upper), with edges clearly defined, with uniform brilliance along its entire length and with associated colors corresponding to the type of gas being fed in. The right-hand insulated shield assumes a potential approximately equal to that applied to the hot cathode and shows signs of intense particle bombardment (heat and wear). With this type of discharge only small dc potentials and small random oscillations are observed on the point probe adjacent to the arc.

Consider now that the pressure external to the defining slot is decreased by increased pumping, while the pressure in the defining slot is maintained constant (say at 10^{-3} mm Hg) by the gas feed. Then, when the range of 10^{-4} – 10^{-5} mm Hg is reached, an abrupt visual change takes place in the arc as indicated in Fig. 2 (lower). From this figure it is seen that after the transition takes place a mildly glowing plasma exists in the region surrounding the arc and the intensity of the central arc column diminishes markedly toward the right end plate. This type of arc is called a Mode II arc and is further characterized by a marked cooling off of the right-hand shield, which assumes a negative potential approximately twice that of the hot cathode potential.

The following experimental observations were made in the course of a detailed study of the Mode II arc:

1. Figure 3 shows a typical periodic potential variation observed on a floating point probe inserted in the glow surrounding the central arc column. Typical peak to peak potentials are of the order of 100 v. It was also found that when floating point probes were placed around the arc at equal angular intervals equidistant from the arc, signals were

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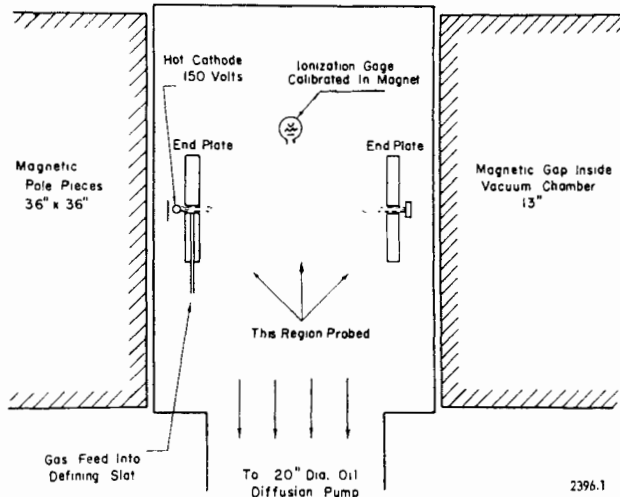


Figure 1. A schematic diagram of the apparatus which was used to produce the Mode II arc. The terminal plate at the right-hand end of the arc column is insulated, and typically assumes a potential of about minus 300 volts with respect to ground. Both end plates are grounded

observed that were identical except for a phase displacement identical to angular displacement. The phase lag was in the direction of positive ion rotation; that is, for a magnetic field into the paper, the signal on some probe B, 60° counterclockwise from some probe A, would lag that of probe A by 60 electrical degrees (one-sixth of a period). The frequencies observed varied from 20,000 to 100,000 cycles per second, depending on the system parameters (gas feed, pressure, accelerating voltage, etc.).

This correspondence of time and space phase permitted the diagram of Fig. 4 to be constructed. The interpretation of this diagram is as follows: if any point away from the arc ribbon is held fixed as the pattern is rotated, the values of the contours crossing the point at any time give the instantaneous

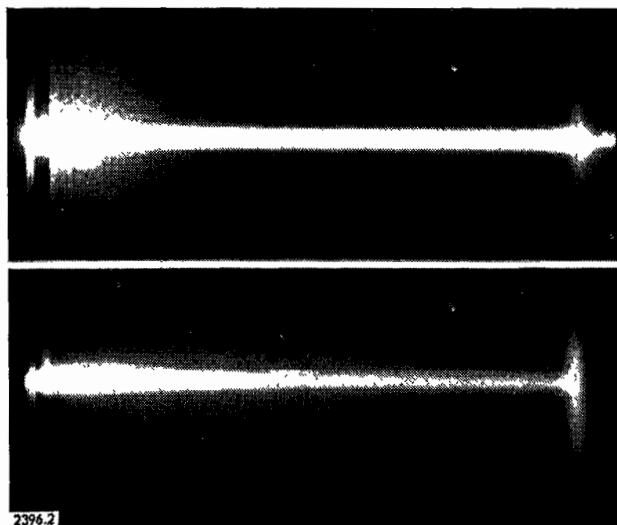


Figure 2. The upper photo is the normal arc without a pressure gradient along its length. The lower photo is the corresponding arc with the pressure gradient. All other parameters are the same

potential at that time; and in fact, the exact reverse of this interpretation was used to obtain the plot. Thus this plot shows the complete time varying signal picked up by a floating point probe anywhere near the arc. It should be emphasized that the potentials shown are total potentials (as is also the case in Fig. 3), with positive up and zero at the bottom of the trace.

2. A marked degree of sputtering was observed on the right-hand end plate, with none on the left except when a mirror coil was placed at the right-hand plate. The circular pattern had its center at the arc center and its edges coincided with the edges of the external glow. Figure 5 shows such a pattern, with a baffle placed just above the plate in order to secure some indication of the paths taken by the bombarding particles.

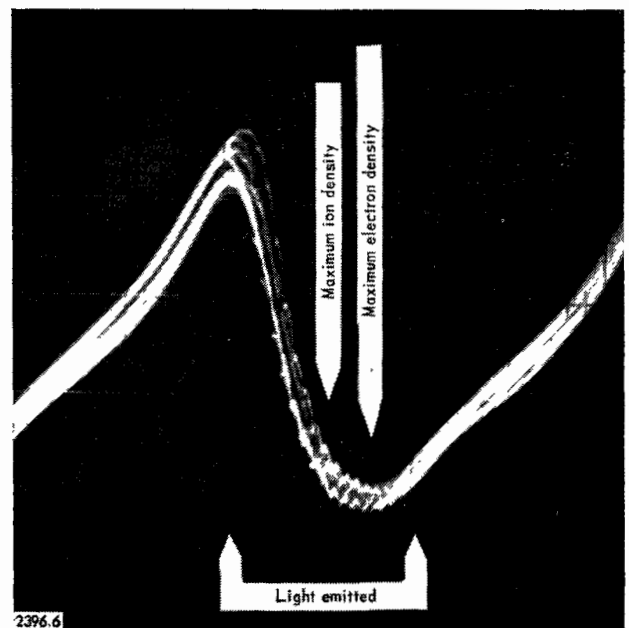


Figure 3. The floating-probe potential as a function of time. This is a photo of several oscilloscope traces superimposed by triggering the oscilloscope at a given probe potential. Time is from left to right. The period of the fundamental component of the signal is 40 micro-seconds and represents one revolution of the plasma potential. The peak-to-peak potential is about 70 v in this case. Current measurements to biased probes located ions and electrons in positions shown. Light emitted was observed with a photo-multiplier

3. A Faraday collector with a curved entrance channel was used as shown in Fig. 6 (left). The current measured was in such a direction as to neutralize positive particles striking the collector. The assembly has considerable length into the paper in order to accept particles having a velocity component along the field. Values of current received are shown in Fig. 6 (right).

4. Spectroscopic examination of the Mode II discharge by J. R. McNally of the Oak Ridge National Laboratory showed the slant or Doppler effect illustrated in Fig. 7. In this photograph, the central

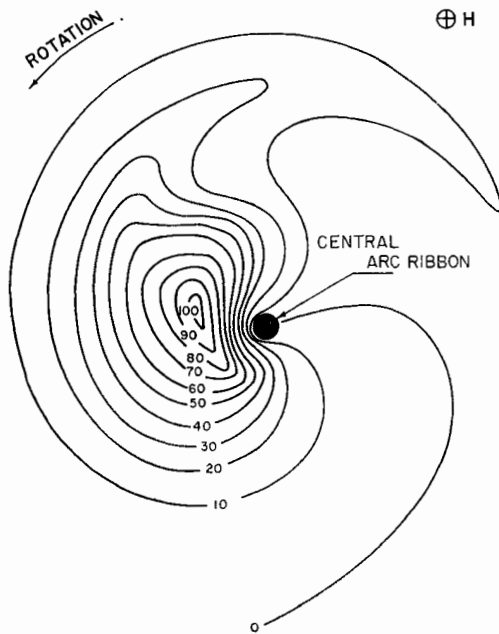


Figure 4. Floating-probe equipotential lines in the plasma surrounding the arc in Mode II. The numbers indicate positive potentials with respect to the end walls. These are typical of the potentials assumed by a floating probe at any instant. The field rotates as a unit about the arc in the direction indicated

arc column is across the paper, with edges at the top and bottom of the intense central glow. Thus it can be seen that the slant inside the central arc column is opposite that outside the arc column.

5. Signals were observed on a floating probe biased in steps over a range of positive and negative values. These time varying signals were compared with an unbiased signal observed on a probe at effectively the same point, in order to establish a time interdependence. Large *increases* of low energy electrons were indicated at the valleys of the unbiased signal; small increases of high energy ions were observed just to the left of the valleys. These results are indicated by vertical arrows on Fig. 3.

6. Light emitted from the glow surrounding the central arc columns was observed by means of a photo-multiplier cell and compared with the signal from an unbiased probe located in the region under examination. Results indicated that the flow was made up of light pulses that always occurred in the valleys of the unbiased probe signal. These results are also shown in Fig. 3.

The experimental results listed above indicate certain apparent characteristics of the Mode II discharge. First, it would appear that positive ions are shot out of the region of the arc centerline, with velocity components across the field and toward the right end plate. This is indicated by the sputtering patterns, the Faraday cup collector signals, and Doppler slant effects observed spectroscopically. Second, it would appear that the energies associated with such ions can amount to as much as twice the potential placed on the hot cathode. This is indicated by sputtering patterns and collector currents obtained for a given gas, hot cathode potential and magnetic field. Third, it would appear that a rotating dis-

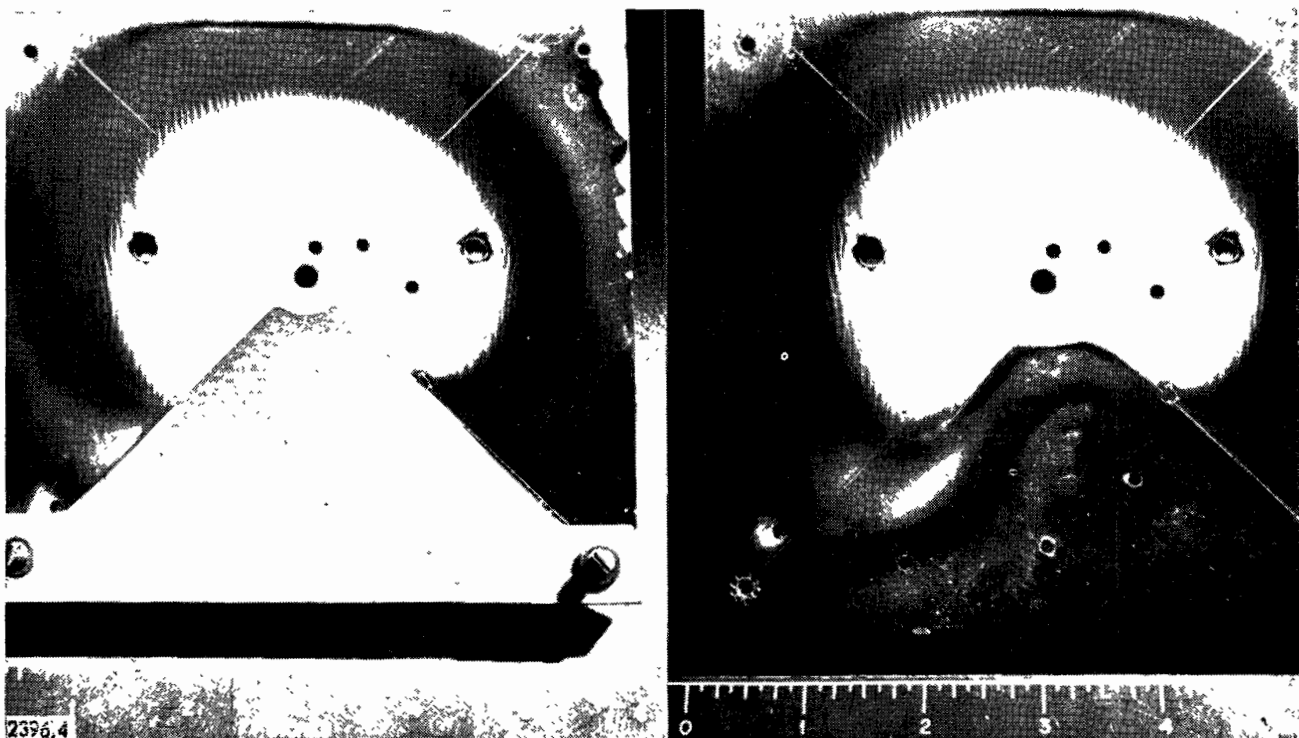


Figure 5. Cold cathode end plate with the obstacle in place (left) and with the obstacle removed (right). Note the azimuthal displacement of the shadow with respect to the obstacle. The black dots are holes in the end plate and are not german to this experiment

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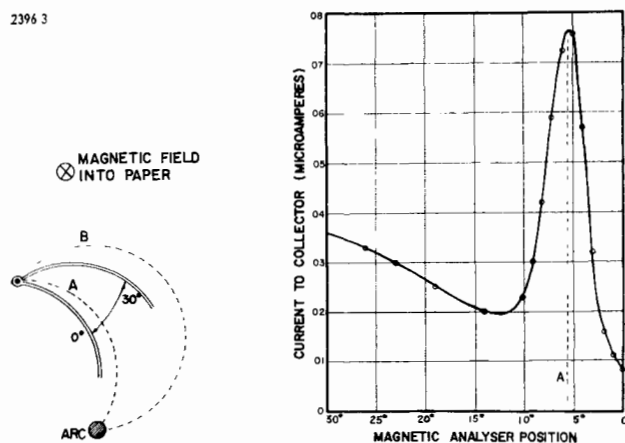


Figure 6. The schematic diagram on the left shows the Faraday collector with the curved entrance channel. It could be rotated 30° about the collector as shown. The peak current to the collector occurs when the channel is in the position of the dotted line at A as is shown on the graph at the right

charge is set up to one side of the arc centerline, having a rough wedge-shaped cross-section and extending between the end plates. The rotation is always in the direction a positive ion would take and the speed of rotation is in the range of 20,000 to 100,000 revolutions per second. This is indicated by the frequency, phase and shape of the electrical signal to the floating probe, by the light signal to the photomultiplier and by the large increase in low energy electrons observed in the valleys of the floating probe signal.

It is important to note here again that the frequency of oscillation (speed of rotation) was in no way related to the Larmour frequency of any ion of the gas under study.

From this, it would appear that the Mode II type of discharge can be thought of as made up of two phenomena, linked together by the accelerated ions. The first phenomenon occurs outside the central arc column and is the subject of the above discussion. The second occurs inside the arc column and furnishes the accelerated ions; it is the subject of studies now in progress. (One difficulty here is the tremendous damage sustained by any probe inside the central arc column.)

It is possible to formulate a tentative model of the rotating plasma effect external to the arc column. Thus, at the onset of Mode II, positive ions are sprayed into the region surrounding the column. Some of these ions undergo charge-exchange with the neutrals in the region, forming slow ions, and thus a positive potential is built up. This is indicated by the smoothly rising portion of the floating probe signal shown in Fig. 3.

The ions that do not undergo charge-exchange strike the right-hand end plate, causing sputtering and giving rise to free electrons. Thus, the increasing potential

and available electrons eventually cause a gas discharge along the magnetic field at some point away from the central column. This is indicated by the steep potential drop shown in Fig. 3.

The discharge now establishes itself in some mode compatible with the given boundary conditions. It cannot remain still, since it has destroyed the potential that established it. However, examination of the potential map of Fig. 4 shows that a radial E field directed outward is associated with the high potential. When electrons are accelerated into the high potential region from the end plate, they are shifted in the direction a positive ion would rotate (counterclockwise) by the effect of this field crossed with the magnetic field. Thus, propagation of the discharge is favored in the direction of this shift due to the ionizing effect of the shifted electrons at the edge of the discharge.

Another asymmetry which favors rotation can be seen from Figs. 4 and 3. Positive ions accelerated from the arc column into the zero potential region of Fig. 4 are favored over those accelerated into other regions, as is shown from the biased probe studies (see Fig. 3). This means that the greatest number of ions are striking the right-hand end plate on the

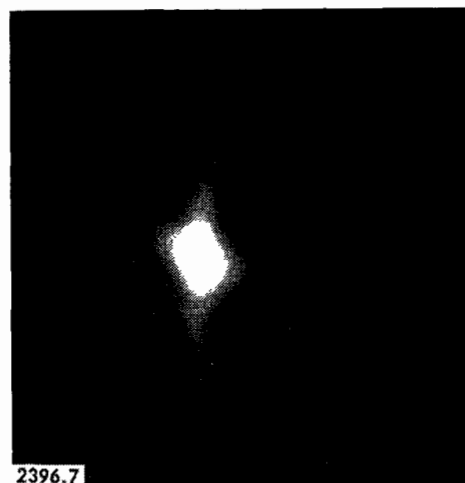


Figure 7. $H\alpha$, showing intensity distribution across the arc column. Note asymmetries in the region near the arc column due to ordered Doppler motion, presumably of H^+ observed as radiative-recombination spectra

counterclockwise edge of the discharge and again counterclockwise rotation is favored.

The overall picture of the rotating discharge is then as follows: The discharge always rotates into a high potential region and leaves behind it a region of zero potential. However, the continuous spray of ions rebuilds this potential at a steady rate, thus maintaining the high potential region toward which the discharge moves. The speed of rotation is governed by the time necessary for the potential to rebuild at any one point and therefore *should* be a function of system parameters other than the magnetic field.

Theoretical and Experimental Discussion of Ixion, a Possible Thermonuclear Device

By K. Boyer, J. E. Hammel, C. L. Longmire, D. Nagle,
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The plasma drift in crossed (radial) electric and (axial) magnetic fields is the essential mechanism in a thermonuclear device called Ixion. It has been demonstrated by Baker and collaborators¹ that large amounts of energy can be stored in such drift rotation of high density plasmas. Post and collaborators² have investigated the use of "magnetic mirror" configurations in which the axial magnetic intensity increases with distance away from the median plane. The essential principle of Ixion consists of applying electric fields to magnetic mirrors in such a way as to provide additional mechanisms for both heating and containment of the plasma:

(a) Appreciable heating may be obtained, i.e., useful cyclotron motion may be superimposed on the drift motion, in those cases where the ion motion is generated abruptly, for example when molecules suddenly become ionized in the presence of the crossed electric and magnetic fields. This results in the ions taking up cycloidal motions composed of roughly equal portions of drift and cyclotron motion.

(b) The azimuthal plasma drift enhances the axial confinement which occurs in a magnetic mirror as a result of the cyclotron motion of the particles about the $\mathbf{B}$ lines. The existence of this centrifugal mirror enhancement, produced by the $\mathbf{B} \times \mathbf{E}$ drift, was recognized independently by both the UCRL and Los Alamos Groups.

One must provide extra magnetic pressure in Ixion devices in order to contain the centrifugal body forces, even in the absence of cyclotron (thermal) motion. This limits the amount of plasma which can be contained.

In the following we present theoretical discussions of the diamagnetic effects and of the mirror enhancement brought about by the ion centrifugal force. This is followed by a discussion of the experiments.

THEORETICAL DISCUSSION

Diamagnetic Effects of the $\mathbf{E} \times \mathbf{B}$ Drift

In this section we treat steady-state diamagnetic

and centrifugal effects for an ideal Ixion device of infinite length, ignoring axial dependences and mirror effects. The results are used to predict the behavior of Ixion as a capacitive circuit element.

Figure 1 is a diagram of an experimental model which illustrates the basic geometry of the device. We choose a cylindrical coordinate system with the z axis oriented along the axis of the machine and let $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ represent unit vectors in the r , θ , and z directions. The inner and outer electrodes are located at $r = a$ and $r = b$, respectively. (In the present model, Ixion 3, the inner electrode is replaced by a plasma "center rod".) Ignoring dependences upon the z coordinate, we write the magnetic and electric fields in the form

$$\mathbf{B} = B(r)\mathbf{e}_z, \quad \mathbf{E} = E(r)\mathbf{e}_r. \quad (1)$$

Ions of mass M , charge e , and electrons of mass m , charge $-e$ occupy the region $a \leq r \leq b$, and assumed electric neutrality leads to equal particle densities $N(r)$ for both species. The first-order drifts due to the combined effects of electromagnetic and centrifugal forces are purely in the θ direction and there is a net diamagnetic drift current density due to the centrifugal force. Letting v_D and v_e stand for the ion and electron drift velocities, and using the definitions $v_L = eBr/Mc$, $v_E = cE/B$, we may express the balance of radial forces in the form

$$v_D^2 + v_L v_D + v_L v_E = 0 \\ (m/M)v_e^2 - v_L v_e - v_L v_E = 0. \quad (2)$$

The solution of Eq. (2) for the ion drift velocity is

$$v_D = \frac{1}{2}v_L[-1 + (1 - 4v_E/v_L)^{\frac{1}{2}}], \quad (3)$$

so that $v_D \cong v_E$ for $v_E/v_L \ll 1$.

Combining Eqs. (2) we obtain an azimuthal diamagnetic drift-current density

$$j_\theta = (Ne/c)(v_D - v_e) = - (NM/Br) \\ \times [v_D^2 + (m/M)v_e^2]. \quad (4)$$

If ϕ is the applied electrostatic potential, the assumed electric neutrality leads to an electric field given by

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$$E(r) = \frac{\phi}{\ln(b/a)} \frac{1}{r}. \quad (5)$$

The diamagnetic current density of Eq. (4) decreases the magnetic field in the central region, creating an outward magnetic pressure gradient which balances the centrifugal body forces. The magnetic field distribution is found by solving the $\nabla \times \mathbf{B}$ Maxwell equation together with Eqs. (4) and (5). This procedure is simplified if N is arbitrarily taken to be independent of r and if the solutions of Eq. (2) for v_D and v_e are expanded in powers of v_E/v_L , only the first order terms being retained. The resulting solution is

$$B^4(r) = 8\pi \left(1 + \frac{m}{M}\right) \left(\frac{\phi}{\ln b/a}\right)^2 N M c^2 \left(\frac{1}{a^2} - \frac{1}{r^2}\right), \quad (6)$$

which satisfies the condition that the applied magnetic field be completely cancelled at $r = a$. This condition may be taken as a criterion for magnetic containment and leads to the following relationship between ϕ and the magnetic field $B(b)$ at the outer electrode:

$$\left(1 + \frac{m}{M}\right) \left(\frac{\phi}{\ln b/a}\right)^2 = \frac{B^4(b)}{8\pi N M c^2} \frac{a^2 b^2}{b^2 - a^2}. \quad (7)$$

With $B(b) = 10$ kilogauss and an average applied electric field of 64 esu, the maximum contained particle density is roughly $2 \times 10^{13} \text{ cm}^{-3}$.

The initial applied magnetic field B_0 may be related to the field distribution of Eq. (6) by assuming that the flux inside the machine is the same before and after the onset of plasma drift. Such an assumption yields the relationship

$$B_0 = \frac{2}{b^2 - a^2} \int_a^b dr r B(r). \quad (8)$$

In the present approximation the net change in magnetic energy density and the particle kinetic energy density are, respectively,

$$W_B = \frac{B^2 - B_0^2}{8\pi} \quad \text{and} \quad W_p = \frac{1}{2} N (m + M) v_E^2. \quad (9)$$

The total energy per unit length is that of a capacitor with capacitance per unit length given by

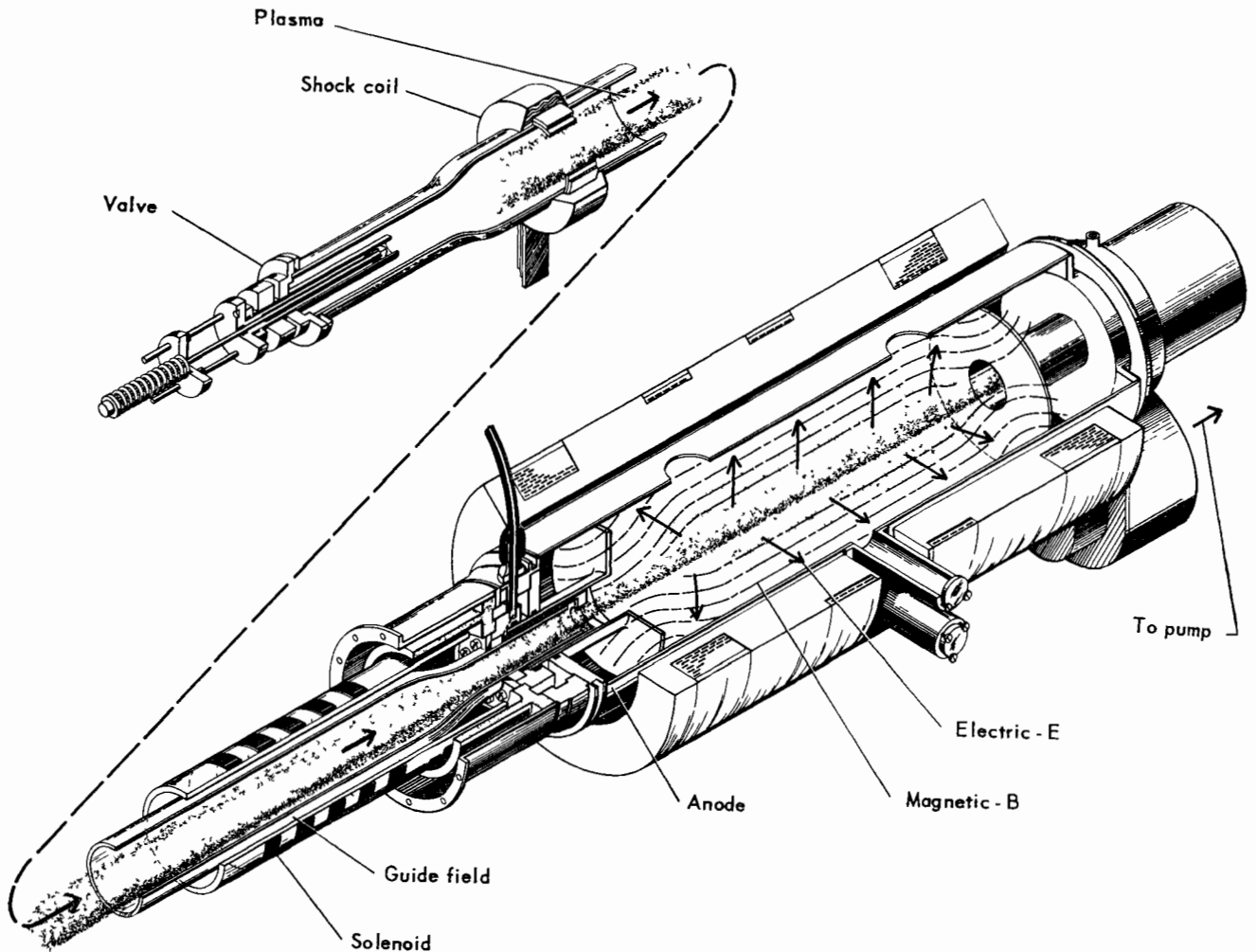


Figure 1. Diagram of Model III of Ixion, a deuterium plasma machine utilizing the $\mathbf{E} \times \mathbf{B}$ drift of a radial electric field and an axial magnetic field

$$C = \frac{1}{\phi} \frac{\partial}{\partial \phi} \int_a^b dr 2\pi r (W_B + W_p). \quad (10)$$

By use of Eqs. (3), (5), (6) and (9) this result may be expressed in the form

$$C = 2\pi \left(1 + \frac{m}{M}\right) (\ln \alpha)^{-1} \frac{NM c^2}{B_0^2} F(\alpha), \quad (11)$$

where

$$\alpha = b/a$$

$$F(\alpha) = 2\gamma^2 (\ln \alpha)^{-1} \int_1^\alpha dx \left[\frac{(1 - \gamma^4)x^2 - 1}{\gamma^2 x + (x^2 - 1)^{\frac{1}{2}}} + \frac{1}{2(x^2 - 1)^{\frac{1}{2}}} \right] \\ \gamma = \frac{2}{\alpha^2 - 1} \int_1^\alpha dx [x^2(x^2 - 1)]^{\frac{1}{2}}. \quad (12)$$

The effective dielectric constant is given by

$$\varepsilon = 4\pi \left(1 + \frac{m}{M}\right) \frac{NM c^2}{B_0^2} F(\alpha) = \varepsilon_p F(\alpha). \quad (13)$$

Thus the usual dielectric constant ε_p of the polarized plasma is modified by the factor $F(\alpha)$ which depends upon the geometry of the system. A numerical evaluation of Eq. (12) for $\alpha = 4$ and $\alpha = 8$ gives the values 1.49 and 1.46 for F , respectively. It is seen that the centrifugal effect leads to an increase of the dielectric constant.

Particle Orbits and Mirror Confinement in Ixion

In this section we present a more detailed treatment of the particle orbits, with particular attention to the effect of the magnetic mirrors at the ends of the machine. It will be shown that the rotation about the axis of symmetry increases the confinement provided by these mirrors.

The magnetic field, which has no θ component and which has axial symmetry, can be derived from a vector potential of form

$$\mathbf{A} = A(r, z) \mathbf{e}_\theta; \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (14)$$

It is convenient to define a flux function by the relation

$$\psi(r, z) = (e/c) r A(r, z), \quad (15)$$

which implies that $(2\pi c/e)\psi$ is the flux enclosed by a circle of radius r . The potential energy of a particle of charge e is given by

$$V(r, z) = e\phi(r, z), \quad (16)$$

where ϕ is the electrostatic potential. $\mathbf{B}$ lines are assumed to become approximately equipotentials due to the free longitudinal motion of electrons. It follows that $\mathbf{V} = \nabla(\psi)$. The Hamiltonian of a particle under the influence of these fields is

$$H = \frac{p_z^2}{2M} + \frac{p_r^2}{2M} + \frac{(p_\theta - \psi)^2}{2Mr^2} + V, \quad (17)$$

where p_z , p_r , and p_θ are the canonical momenta appropriate to the cylindrical coordinate system. Since H is independent of θ , $p_\theta = mr^2\dot{\theta} + \psi$ is a constant of motion. Therefore, Eq. (17) may be regarded as the Hamiltonian of a particle moving in two Cartesian dimensions in an effective potential $\mathcal{V}$ given by the sum of the last two terms in Eq. (17). We shall assume that the particle is positively charged, so that $\mathcal{V}$ is positive.

We first consider the case $V = 0$. Ixion particles start their cycloidal orbits from rest, and hence positive particles have positive p_θ . For a given value of z , $p_\theta - \psi$ will then vanish at some radius. The effective potential $\mathcal{V}$ also vanishes and thus has a minimum at this radius. The locus in the $r - z$ plane of these minima forms the centerline of a potential trough, with the centerline following a $\mathbf{B}$ line. The two-dimensional particle moves in this trough, oscillating between the two flux surfaces for which $\mathcal{V}$ equals the total energy.

If there is an applied electric field, the equipotential surfaces of $\mathcal{V}$ will be altered. For a given value of z , the minimum of $\mathcal{V}$ occurs where $\partial\mathcal{V}/\partial r = 0$. From the form of $\mathcal{V}$ it follows that this condition holds when the relation

$$p_\theta - \psi = \frac{1}{2} Mr v_L [-1 + (1 - 4v_E/v_L)^{\frac{1}{2}}] \quad (18)$$

is satisfied, where $v_L = eB_z r/Mc$, $v_E = cE_r/B_z$ as before. If E_r were zero, this would give the minimum at $\psi = p_\theta$ as above. Comparing with Eq. (3) we see that motion along the bottom of the potential trough implies that the expression

$$p_\theta - \psi = -Mr v_D \quad (19)$$

is satisfied. Since $-Mr v_D = L_D$, the angular momentum of the drift motion, Eq. (19) states that

$$L_D + \psi = p_\theta = \text{constant}. \quad (20)$$

Thus the quantity $L_D + \psi$ is an adiabatic invariant for the general drift case, a well-known result which can be proved in a variety of ways and which can be used directly to calculate the centrifugal confinement effect of the mirror fields. If E_r is negative (outer electrode positive), the sign in Eq. (19) can be shown to lead to an orbit which moves outward across flux surfaces as it enters the mirror region, and conversely.

To calculate the mirror confinement effect in the presence of the $\mathbf{E} \times \mathbf{B}$ rotation it is convenient to perform a transformation to eliminate the radial electric field. This is done by introducing an equivalent flux function ψ' which includes the radial dependence of V . For given p_θ and z , ψ' is defined by the relation

$$\frac{(p_\theta - \psi)^2}{2Mr^2} + V(r, z) = \frac{(p_\theta - \psi')^2}{2Mr^2} + K(z), \quad (21)$$

where $K(z)$ is the minimum value of the left hand side for the given value of z . This allows $p_\theta - \psi'$ to vanish at the minimum, and allows ψ' to be a smooth monotonic function of r , as is required of a flux function. Equation (17) becomes

$$H = \frac{p_z^2}{2M} + \frac{p_r^2}{2M} + \frac{(\phi_\theta - \psi')^2}{2Mr^2} + K(z), \quad (22)$$

corresponding to the Hamiltonian of a particle in a mirror magnetic flux distribution ψ' and an axial electric field with potential $K(z)$. The mirror reflection problem is reduced to that of finding the effect of $K(z)$. We first determine $K(z)$ in the approximation $v_E/v_L \ll 1$. This is done by expanding $(\phi_\theta - \psi)^2/2Mr^2$ and V about the point $r = r_0$. A brief calculation then shows that $K(z)$ is the negative of the term $(\phi_\theta - \psi)^2/2Mr^2$ evaluated at the minimum of $\mathcal{V}$, and hence Eq. (19) enables us to write

$$K(z) = -\frac{1}{2}Mv_E^2, \quad (23)$$

where v_E is the $\mathbf{E} \times \mathbf{B}$ drift velocity evaluated on the same ψ' flux surface for each z . On a given flux surface, $B \propto 1/r^2$ and $E \propto 1/r$ so that $K(z) \propto -1/B(z)$. Thus $K(z)$ is more negative in the center of the machine than in the mirror, leading to a suppression of particle leakage. To evaluate this effect quantitatively we use the law of energy conservation for motion along a flux tube in the form

$$w_\perp(c) + w_\parallel(c) + K(c) = w_\perp(m) + w_\parallel(m) + K(m), \quad (24)$$

where $w_\perp$ and $w_\parallel$ are the perpendicular and parallel kinetic energies, and (c) refers to the center and (m) to the mirror. From adiabatic theory we have

$$w_\perp(m) = w_\perp(c)B_m/B_c. \quad (25)$$

In order to pass through the mirror, a particle must have $w_\parallel(m) \geq 0$. By use of Eqs. (23), (24), (25), and the relation $K(z) \propto -1/B(z)$ this condition becomes

$$w_\parallel(c) \geq w_\perp(c)[(B_m/B_c) - 1] + \frac{1}{2}Mv_E^2(c)(1 - B_c/B_m). \quad (26)$$

The last term gives the effect of $K(z)$, making it more difficult for a particle to leak. Physically, the improved containment arises from the centrifugal force of the $\mathbf{E} \times \mathbf{B}$ rotation which keeps particles from approaching the axis, as they must in order to pass through the mirror. In an Ixion machine, $\frac{1}{2}Mv_E^2$ may be about equal to the total Larmor energy $w_\perp$.

EXPERIMENTAL RESULTS

We report on the successive operation of three models of Ixion, covering a period of about eight months up to April 17, 1958. The models differed primarily in the ranges of electric and magnetic fields which were used and in the series inductances of the electrical leads which in each case connected Ixion to energized capacitor banks. Models I and II had metallic center electrodes, while the latest version, Ixion III, has a "plasma center electrode".

Figure 1 shows Ixion III. The vacuum envelope is a stainless steel cylinder 24 cm in diameter and 86 cm long which is normally evacuated between discharges to 2×10^{-6} mm Hg. The magnetic field is provided by pulse coils, shown surrounding the

vacuum envelope. The magnetic intensity is typically 9 kilogauss at the median plane and increases with distance from the median plane to a maximum of 20 kilogauss ("mirror throat"). Five coaxial cables at each end of the device (one of which is shown) connect it through 0.1 microhenry to a 75 microfarad capacitor bank. Deuterium gas is admitted for approximately 100 microseconds through the valve, which is opened electromagnetically with precise timing. Following the opening of the valve, a portion of the gas is ionized and driven along the axis into Ixion proper by the magnetic shock coil and the guide field. Details of the operation of this and similar injectors are described by J. Marshall,³ who developed them. In operation a luminous cylindrical region of plasma about 3 cm in diameter (indicated by the heavy shading in Fig. 1) is visible after the injector is fired. This plasma center electrode is in contact with a pair of tungsten ring electrodes located at the mirror throats. Each ring electrode connects

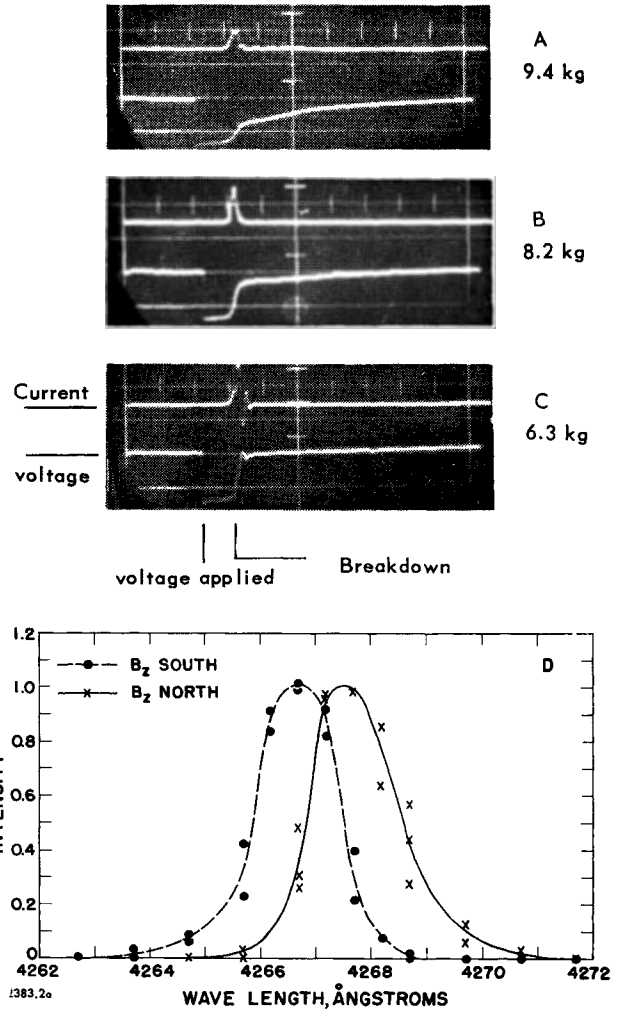


Figure 2. Oscillograms A, B and C represent the behavior of Ixion III for three different values of applied axial magnetic field and fixed applied voltage and injected deuterium pressure. Graph D shows line profiles of C^{II} 4267 Å light intensity vs. monochromator wavelength setting for opposite directions of applied axial magnetic field

(along the path shown dashed in Fig. 1) to the center conductors of five of the coaxial cables. A pair of hat-shaped insulators prevents arcing at the ends of the container.

Figures 2A, 2B and 2C are typical oscillograms taken at a sweep speed of $200 \mu\text{sec}/\text{cm}$, showing the time behavior of the voltage and current at the terminals of Ixion. About one micron of D_2 was injected, followed 400 microseconds later by excitation of the shock coil and simultaneous initiation of the oscilloscope sweep. For about 200 microseconds after application of the voltage (about 10 kv) negligible current was drawn, even though plasma from the injector had arrived promptly. The initial slope of the voltage trace was caused by the external circuit. After the 200 μsec delay, which could be shortened or lengthened as the relative delay between plasma injection and voltage application was lengthened or shortened, a sudden discharge occurred. This discharge was accompanied in the cases of Fig. 2A and 2B by (a) the occurrence of a sudden impulse of current (about 20,000 amperes) which lasted about 25 μsec and transferred a major portion of the capacitor-bank charge to Ixion, (b) a drop of the voltage of the system to about one half its initial value as charge was shared between Ixion and the capacitor bank, and (c) intense emission of light due to atomic deuterium as well as to impurities. Following the impulsive discharge, the voltage decayed to zero in about 500 μsec .

In the case of Fig. 2C the discharge was oscillatory, having a period characteristic of the cable inductance and bank capacitance. This latter type of oscillogram was obtained if the magnetic field was lowered, the voltage raised, or the pressure raised excessively. Here Ixion presented a resistance of a small fraction of an ohm. Such oscillograms indicate that under oscillatory conditions the plasma is not contained. It was found, however, that the voltage for "break-down", or loss of containment, varied according to a much lower power of B than the second, as would be indicated by Eq. (7).

In connection with the behavior illustrated in Figs. 2A and 2B, the current spike and sudden drop in voltage are interpreted as being caused by the development of a rotating plasma containing internal energy of cyclotron motion. Ixion at this time presents an impedance to the external circuit which is partially capacitive and partially resistive. The absence of the oscillations which might be expected owing to circuit inductance at the time of the voltage drop may be caused by the damping effects of plasma heating. The subsequent decay of the voltage is attributed to the loss of plasma to the walls. Tuck has shown that because of azimuthal momentum transfer between ions and electrons, an outward radial drift of the plasma is to be expected.

In order to see how well the capacitance of Ixion could be predicted by the considerations of Section II, the following experiments were performed. Data of the type illustrated in Fig. 2A were taken for various values of applied magnetic field B_0 . The ratio Q/V

of the total charge transferred to Ixion divided by its voltage immediately after discharge was found to vary approximately as the inverse square of B_0 as predicted by Eq. (11), although the exponent could be made to vary between 1.6 and 3.3, depending upon discharge conditions. For $B_0 = 8$ kilogauss and an injected D_2 pressure of 1.5 microns, Q/V was typically 42 microfarads.

In order to determine what portions of the charge Q corresponded to reversible initiation of plasma rotation and to irreversible processes, the following measurement was made. After the process of transfer of the charge Q to Ixion and while its voltage has the resulting value V , Ixion was short-circuited by means of a timed ignitron, and the amount of charge corresponding to the short circuit was measured. It was found that $1/5$ of the original charge Q could be recovered, corresponding to the value 8.4 microfarads for its capacitance under the foregoing conditions of applied magnetic field and injected D_2 pressure. Assigning F in Eq. (13) the value 1.5, this capacitance corresponded to an amount of rotating deuterium plasma of equivalent pressure 5.6 microns. The discrepancy between this figure and the injected pressure of 1.5 microns may be accounted for either by the pressure of impurities or by larger values for the enhancement function F than would be calculated on the particular assumptions used in Section II.

The circuit behavior illustrated in Figs. 2A and 2B has been obtained only with Ixion III. In an earlier model (Ixion II) with a metal center rod, it was found that large diamagnetic signals during the oscillatory discharge (Fig. 2C) could be detected by means of a magnetic probe coil inserted along its axis. The diamagnetic signals corresponded to almost complete removal of the axial magnetic field from the center of the machine. The diamagnetic effect decreased to zero as the probe was moved axially from the central plane to the mirror throat.

The high impedance or "waiting" phase was investigated in detail with Ixion II. It was shown that during this phase an electron sheath surrounded the anode and kept the electric field from penetrating the main body of the gas. The same effect can partly explain the waiting period in Ixion III. However, the situation is complicated by the fact that the bulk of the gas from the valve reaches Ixion III about a millisecond later than the "plasma electrode". Some experiments were performed which indicated that the waiting time could be reduced by using auxiliary electrodes designed to help the electric field to penetrate the gas. Work is continuing in this direction.

Bursts of neutrons appear at the time of the current spike of Fig. 2C. At 40 kilovolts applied potential there are roughly 10^5 neutrons per discharge. Even at 10 kilovolts neutrons are detected under conditions where the deuteron drift energy is less than 200 volts. At 10 kilovolts, the neutrons can be made to disappear by raising the magnetic field to the point where the capacitive type of signature (Fig. 2B) is obtained. The neutrons have not been satisfactorily accounted for.

Several experimental facts demonstrate that the present behavior of Ixion is strongly influenced by impurities. We find that the discharge releases gas from the walls, the amount being about one micron in the case of the non-oscillatory discharges, and several microns for the other discharges. The analysis of spectrograms reveals about 30 spectral lines, among which those of H, CII, FeI, FeII, SiIII, SiII, and OII are prominent. Time-resolved studies show the impurity light appearing very soon after the start of the current spike and lasting about as long as the current. For H and D the light lasts as long as 100 μ sec after the spike of current.

Time-resolved studies of the light were made with a 0.5-meter Ebert-type monochromator looking into the lower window shown in Fig. 1, along a line perpendicular to the axis and 11 cm below it at the median plane. Changing the wavelength setting of the monochromator in one or one-half Ångstrom steps on successive discharges, line profiles were obtained. The resolution of the monochromator was about 1.5Å full width at half maximum. Figure 2D shows profiles of the line CII (4267Å). The data were taken for B_z pointing north (solid line) and B_z south (dashed

line) without any other changes. The lines are seen to be asymmetric and somewhat displaced from one another, presumably because of the Doppler effect of the plasma drift. The amount of Doppler shift here indicates a drift velocity of about 4×10^6 cm sec⁻¹, a result consistent with what we calculate from the applied electric and magnetic fields. It appears that the impurities are spinning. Similar studies on deuterium Balmer lines showed extreme broadening and some structure for which at present we can not satisfactorily account.

We believe that these experiments demonstrate the containment of a rotating plasma, although impurities are limiting the energy that can be contained.

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Liquid Conductor Model of Instabilities in a Pinched Discharge

By A. Dattner,* B. Lehnert* and S. Lundquist†

The problem of the stability of a pinched discharge has been studied with great interest during the last decade because of its importance in the field of controlled fusion research. A number of workers (Cousins and Ware,¹ Kurchatov,² Golovin³ and many others) have reported experimental results showing a pronounced tendency of a pinched discharge to be unstable. The problem has been discussed theoretically by Kruskal and Schwarzschild,⁴ Tayler,⁵ Shafranov,⁶ Rosenbluth,⁷ etc. Their discussions show that a cylindrical column of ionized gas carrying electric current along the axis cannot be kept in confinement by the current's own magnetic field alone. Other means have to be used to stabilize the system, such as an external longitudinal magnetic field and conducting walls. The above methods have been applied in ZETA, giving fairly satisfactory results.

However, in order to understand the nature of the problem, one would like to be able to study the structure of the instabilities in great detail. In this respect the experimental physicists doing pinched gas discharge experiments seem to have been handicapped by the great speed with which the instability develops as well as by the light coming from impurities instead of the main body of pinched gas. In the present work we have used a liquid conductor to eliminate these difficulties.

APPARATUS

The apparatus used is schematically shown in Fig. 1. The vessel A having a circular hole in the bottom contains mercury. The liquid runs through the hole, thus creating a cylindrical jet of 4-mm diameter and 14 cm length. The bottom of the vessel is designed so as to prevent the formation of vortices. The bottom container B collects the falling mercury which then is pumped back to A. The circular plates C are soldered to A and B respectively and serve as connections between the current circuit and the mercury stream. The leads are distributed

around the plates in order to give no appreciable magnetic field in the vicinity of the beam. A, B and C are made of copper and the supports are insulators. The current source is a battery of 6-volt accumulators.

The whole apparatus can be put into a magnet giving an axial magnetic field variable up to 8000 gauss. The current used in the experiment was variable up to 400 amps.

PERFORMANCE OF THE EXPERIMENT

The time of passage of a volume element of mercury from A to B is roughly 0.15 sec. During this time, all the processes we wish to study have to be completed. Since the processes were too fast for visual study they were recorded with a high-speed film camera, Eastman Kodak type III, taking up to 3300 frames per second. The stream of mercury was situated between a powerful light source and the camera, and in this way the shadow of the mercury was filmed.

A study of a pinch was made with and without the axial magnetic field. In cases with a magnetic field, the currents and fields were chosen so as to give a longitudinal magnetic field equal to or three times the azimuthal field at the boundary of the mercury stream.

DISCUSSION OF THE RESULTS

Two representative photographs are shown in Figs. 2 and 3. Figure 2 shows a sausage-type instability arising when the external axial magnetic field is absent. This seems to be the most common type under such conditions. In most cases, the column of mercury remains remarkably symmetric in relation to the axis in spite of the deformation, but in a few cases combined sausage- and kink-type instabilities were observed. The characteristic time for the growth of an instability of the sausage type, from the moment when the diameter starts to decrease and until it has decreased by a factor of two, is of the order of 30–50 msec for currents of the order of 200–350 amps.

The mechanism of such an instability can be qualitatively explained by reference to the fact that a

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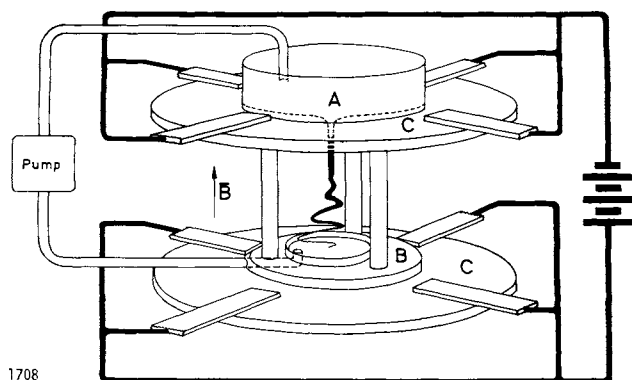


Figure 1. The apparatus

local decrease in the radius of the column causes an increase in the azimuthal magnetic field since $B \sim 1/r$. The consequence of this is an increase of the inward force, $\mathbf{F} \sim \mathbf{I} \times \mathbf{B}$, reducing the diameter still more, etc. The mercury in the compressed region is pressed upwards and downwards, increasing the diameter above and below the pinch. Above and underneath this region the same type of depression is then likely to occur again giving rise to a number of sausages, each of which seems to approach a spherical shape. In this connection, surface tension should not be forgotten.

Figure 3 shows the process in a longitudinal magnetic field. The situation in this case is different. As soon as the stream of mercury is no longer parallel to the axis, a force acts on the component of the current perpendicular to the axis. A semicircular kink in the vertical plane is turned towards a horizontal position because its upper part is rotated in one direction while its lower part is rotated in the opposite direction. Once it has reached a nearly horizontal position all its parts receive an acceleration outwards until it breaks up. The photograph is taken at such a moment. One can say that the stream tends to approach such a shape that the magnetic field generated is parallel to the external magnetic field. The radial velocity of the expanding spiral is of the order of 2 m/sec at currents of the order of 300 amps and magnetic fields of the order of 300 gauss. The total length of the spiral can reach 30 cm without breaking up, which is more than a straight, freely falling jet of mercury of the same diameter can reach without breaking up into droplets. This can be explained by the fact that the latter breaks up under the action of the surface tension which, in the case of the spiral, does not act sufficiently quickly. The sausage-type instability is also too slow a process to occur. In some cases, however, sausages along the spiral were observed.

CONCLUSIONS

The study of the results shows that in the case without an external magnetic field there is a striking similarity between the behavior of the pinch in a stream of mercury and in an ionized gas column.

The stabilizing action of the surface tension is small and the instabilities develop easily.

The case with an external magnetic field is somewhat more complicated. The diffusion velocity of the field lines given by Lundquist⁸ is

$$v = (\mu\sigma L_c)^{-1},$$

where μ = the constant of permeability for mercury $= 4\pi \times 10^{-7}$ v/sec/amp/m, σ = the conductivity of mercury $= 10^6$ ohm⁻¹m⁻¹ and L_c = a characteristic length $= 0.5 \times 10^{-2}$ m, thus $v \approx 200$ m/s while the radial velocity of the expanding spiral is of the order of 2 m/s. Hence the magnetic lines of force are not frozen into the medium; they can "escape" from the medium. In this case the magnetic field has no stabilizing effect. The influence of conducting walls around the mercury column will be studied in forthcoming experiments.

ACKNOWLEDGEMENTS

The investigation would hardly have been possible without the valuable aid of Mr. L. F. T. Koch who handled the high-speed camera and gave welcome assistance in all the photographic work. We wish to thank Mr. E. Dahlberg for many stimulating discussions. The technical assistance of Mr. A. Spångberg is gratefully acknowledged.

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Figure 2. Sausage instability
for $B = 0$ Figure 3. Spiral instability
for $B = 300$ gauss

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Magnetic Compression of Shock Preheated Deuterium

By Alan C. Kolb *

In order to use externally generated magnetic fields to compress a high-density deuterium plasma (10^{16} – 10^{18} ions/cm³) with a ratio of the external magnetic pressure to the gas pressure near unity ($\beta \simeq 1$), preheating is necessary to establish a high conductivity initially. The utilization of magnetically driven axial shock waves for preheating has been under investigation in one phase of the controlled fusion research at the Naval Research Laboratory and is discussed in the first part of this paper. The magnetic compression of the preheated plasma is accomplished by discharging a large condenser bank into a coil system surrounding the shock tube. With this general arrangement, axial magnetic fields of the order 100–500 kilogauss can be generated with currents of 1.5 – 8×10^6 amp. The high magnetic pressure raises the gas to high temperatures and high-energy radiation has been observed with a scintillation counter for time intervals of the order 10 μ sec.

The second part of this paper is concerned with the study of the magnetic compression and stability of high- β plasmas without the complications introduced by the generation of axial shock waves with an auxiliary capacitor discharge. In order to carry out experiments of this type, in which a pulsed electrodeless discharge is used to compress a plasma with $\beta \simeq 1$, the final high-temperature plasma is actually produced in four steps:

(a) Partial ionization, by means of a 27 Mc, 1 kw rf electrodeless discharge.

(b) Preionization, by means of a high-power, high-frequency capacitor discharge through a single-turn coil with magnetic mirrors surrounding the discharge tube.

(c) Shock pre-heating ($T \sim 100$ ev), by axial and radial shock waves set up in the ionized gas when the main bank begins to discharge into the coil with magnetic mirrors.

(d) Adiabatic compression, by means of the continued increase of the magnetic field after the transient shock phenomena have terminated.

With this latter technique it has been possible to establish, in a reproducible way, a fully ionized, spectroscopically pure plasma with a compression ratio greater than 100.

PROPAGATION OF STRONG SHOCK WAVES CONFINED BY MAGNETIC FIELDS

A deuterium plasma can be accelerated along a quartz tube by striking a high current discharge (100–300 k amp in 0.3–0.5 μ sec) between a pair of electrodes at one or both ends of a quartz tube. A magnetic field perpendicular to both the current path and the axis of the tube causes the acceleration and resultant shock wave.¹ To prevent cooling by heat conduction to the tube walls an axial magnetic field along the tube is used to drive the plasma radially away from the walls.² A shock tube with field coils is shown in Fig. 1. The tube is 30 cm long with an inside diameter of 3 cm. Two shocks are produced at both ends of the tube which collide at the center. This collision brings the plasma to rest and raises the thermal energy to ~ 100 ev/particle as deduced from the measured shock velocities. Streak camera photographs (see Fig. 2) show that the high injection velocities can be maintained by the external field as a result of the compression. Image rotators are used to observe the compression at two points along the tube. The shock velocity decreases after the time of maximum compression and peak current. The radius of the plasma column at the shock front is very nearly that of the tube radius. The radius of the plasma column behind the front decreases with increasing distance from the front.

HIGH COMPRESSION EXPERIMENTS

In order to generate strong magnetic fields in a long single-turn coil or in a tube surrounded by several single-turn coils connected in parallel, very high currents are required. For a tube 20 cm long and a current of 8×10^6 amp the field is $\sim 500,000$ gauss. Furthermore, the inductance of such coils is low (of the order $\sim .01 \mu$ h for a radius of ~ 4 cm) so that a very low inductance external circuit is required for efficient energy transfer to the electro-magnetic shock tube. To meet these requirements, a capacitor bank with a short circuit current capability of 15×10^6 amp has been constructed. The other electrical parameters are tabulated in Table 1.

The capacitors are connected by parallel copper plates with eleven switches. Each switch consists

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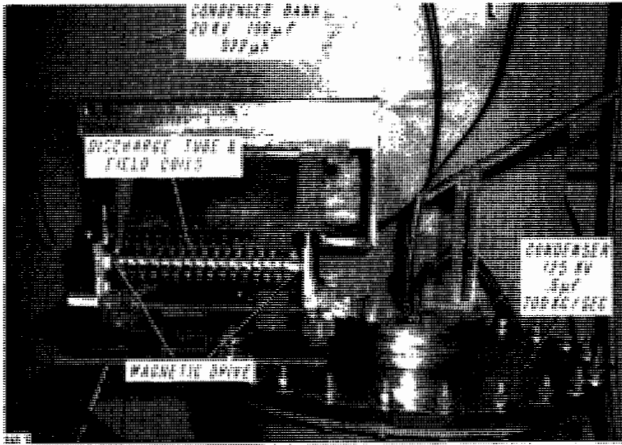


Figure 1. Shock tube and field coils

simply of a gap in the transmission line across which the current flows when triggered from an external spark. The parallel plate geometry is therefore maintained and the switch inductance is small ($\sim .001 \mu\text{h}$ for eleven switches). The external inductance is somewhat variable depending on the number of switches that are fired. The magnetic field of the plate for the return current drives the ionized gas generated by the switch discharge away from the dielectric insulation. This "magnetic insulation" is very effective and will be described more fully in a separate publication. The effective operation of this capacitor bank and switches has shown that discharges with tens of millions of amperes and magnetic fields in the megagauss range are entirely feasible for large systems by scaling up the present apparatus. One of the magnets used for the high field work is shown in Fig. 3. The coils are designed to withstand magnetic pressures up to 2×10^{10} dynes/cm². They are connected to a transmission line 2.5 m wide which passes through the floor and is connected to the cap-

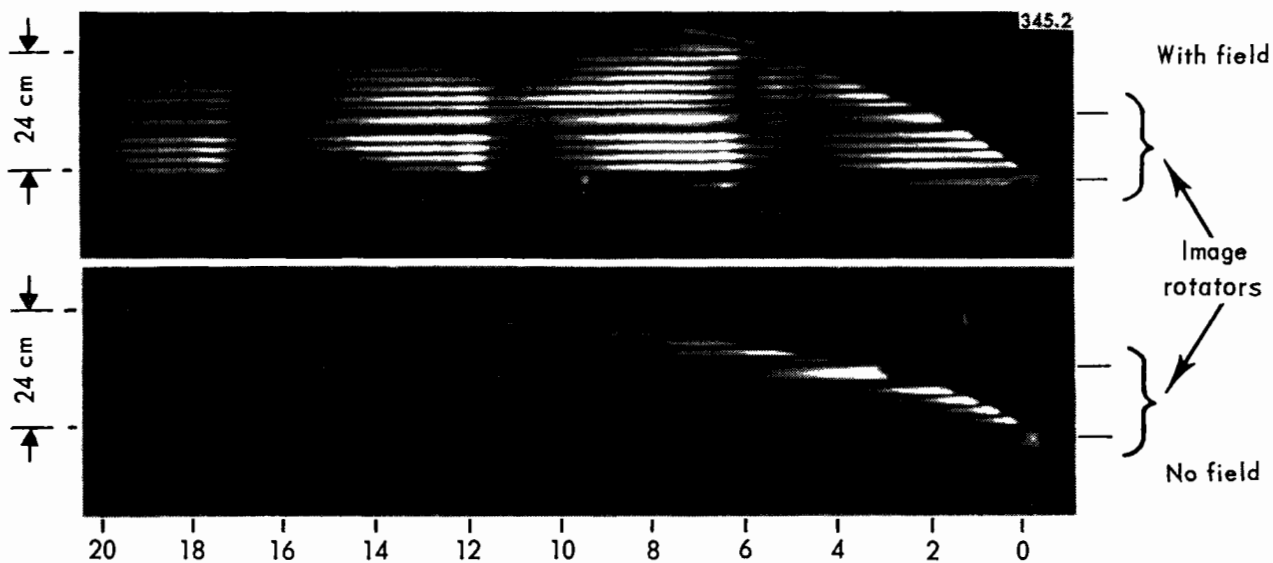
citor bank in the room below. The width and separation of the coils are one centimeter. The inside diameter here is 3.6 cm. A shock wave can be initiated at one or both ends of the tube. Notice also in Fig. 3 that there is no center turn. As a result there is a magnetic mirror geometry^{3,4} with a mirror ratio of 2.5:1. The usefulness of such mirror geometries for high β experiments at high densities for plasma trapping is one of the major problems now under study.

Table 1. Characteristics of Entire Bank

Voltage	20 kv
Number of capacitors	99
Total energy	285,000 joules
Total capacitance	1430 μf
External inductance	$\sim 0.003 \mu\text{h}$
$I_{\text{short}} = V (C/L)^{1/2}$	$\sim 15 \times 10^6$ amp
$(dI/dt)_{\text{short}} = V/L$	$\sim 8 \times 10^{12}$ amp/sec
Rise time, short	$\sim 2.8 \mu\text{sec}$
Load inductance (typical coils)	$\sim 0.007\text{--}0.05 \mu\text{h}$
Number of switches	11
I_{load}	up to 8×10^6 amps
H_{max}	$\sim 500,000$ gauss
Rise time with load	$\sim 5\text{--}10 \mu\text{sec}$

Equations for the Compression Cycle

In certain experiments, shock waves are sent into both ends of the coil. After the initial acceleration, the collision of the two shocks at the center of the tube will bring the plasma to rest in a rising magnetic field. Calculations of the compression cycle *after* a high-temperature plasma has been formed in the tube by shock heating have been carried out. In this analysis it is assumed that the compression is adiabatic since the velocity of radial compression is slow compared to the speed of sound in the hot plasma. Exact solutions of the equations of motion for the plasma boundary can be found for a $\gamma = 2$ gas initially at a uniform temperature, T_0 , and density N_0

Figure 2. Streak camera photograph with and without confining field. $H = 15,000$ gauss; $P = 400$ microns deuterium. Time scale is in microseconds

(ions and electrons)/cm³. The effective γ of the plasma could be greater than $5/3$ at high temperatures due to non-equilibrium velocity distributions so that taking $\gamma = 2$ is not too unrealistic for the present purpose. In any event, the qualitative results are not very sensitive to the value of γ , and assist in the selection of experimental parameters.

After the collision of the two primary shocks the current in the coil continues to rise until at a time t_0 the magnetic pressure at the surface of the plasma exceeds the gas pressure, $N_0 k T_0$ (the subscript zero refers to values of the various variables at the time t_0). At this time the plasma begins to contract.

The voltage V_L across the coils (load) is given by

$$V_L = \frac{d}{dt} (IL) = L_c \frac{d}{dt} [I (1 - (R_p/R_c)^2)], \quad (1)$$

where L_c is the inductance of the coil with no plasma inside the tube, R_p is the plasma radius and R_c is the coil radius.

The appropriate circuit equation is

$$V_L = V_0 - \frac{1}{C} \int_{t_0}^t I dt = L_c \frac{d}{dt} [(1 - R_p^2/R_c^2)I]. \quad (2)$$

In addition we have the relation for the pressure balance

$$H^2/8\pi = NkT, \quad (3)$$

the expression for the field in a solenoid with n turns/cm

$$H = 4\pi n I / cl, \quad (4)$$

the adiabatic relation

$$T/T_0 = (N/N_0)^{\gamma-1}, \quad (5)$$

and the density ratio after radial compression

$$N/N_0 = (R_0/R_p)^2, \quad (6)$$

where $R_0 = R_p(0)$ is the plasma radius at $t = 0$.

Equations (3), (4), (5), (6) yield

$$I = I_0 (R_0/R_p)^\gamma, \quad (7)$$

where I_0 is the current required to start the compression and is given by

$$I_0 = \frac{lc}{n} \left(\frac{N_0 k T_0}{2\pi} \right)^{1/2}. \quad (8)$$

Substituting (7) into (2) yields a differential equation for the plasma radius R_p . With $\gamma = 2$, the solution of this equation in terms of appropriate dimensionless variables is

$$I = V(C/L_0')^{1/2} \sin [(\tau + t_0/(L_c C)^{1/2})(L_c/L_0')^{1/2}] \quad (9)$$

for $0 \leq t \leq t_0$; and

$$I/I_0 = T/T_0 = N/N_0 = (R_0/R_p)^2 = \mathcal{J}_0^{-1} \sin \tau + \cos \tau \quad (10)$$

for $t \geq t_0$.

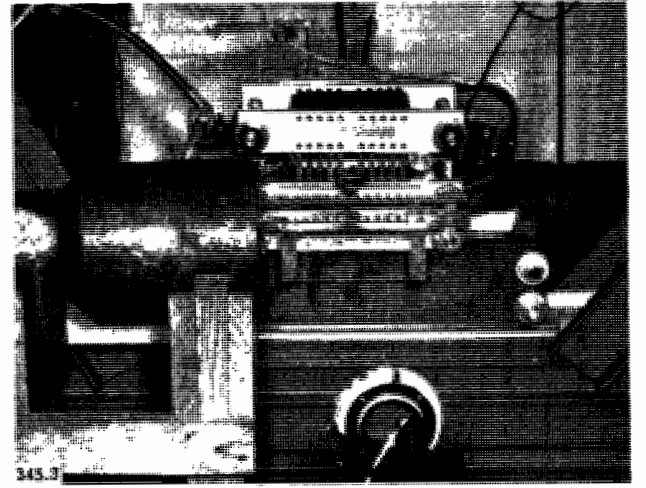


Figure 3. Coils system for high field experiments

where V , C , are the voltage and capacitance of the capacitor bank, respectively, and

$$\tau \equiv (L_c C)^{-1/2} (t - t_0)$$

$$\mathcal{J}_0 \equiv I_0/I_m$$

$$L_0 = L_c(R_c^2 - R_0^2)/R_c^2$$

$$L_0' \equiv L_0 + L_e$$

$$L_e \equiv \text{external inductance}$$

$$I_m \equiv V(C/L_e)^{1/2} (1 - I_0^2 L_0' / V^2 C)^{1/2}$$

$$t_0 = (L_0' C)^{1/2} \sin^{-1} (I_0^2 L_0' / V^2 C)^{1/2}.$$

The compression ratio, final density and temperature can be expressed in terms of the quantity $\mathcal{J}_0$:

$$\mathcal{J}_0 = \frac{I_0}{I_m} = \left[V \left(\frac{C}{L_e} \right)^{1/2} \left(1 - \frac{I_0^2 L_0'}{V^2 C} \right)^{1/2} \right]^{-1} \times \frac{lc}{n} \left(\frac{N_0 k T_0}{2\pi} \right)^{1/2}. \quad (12)$$

Using the expression $L_c = 4\pi^2 R_c^2 / c^2 l$, for a single turn coil ($n = 1$), gives

$$\mathcal{J}_0 = \left(\frac{\bar{N}_0 k T_0}{V_0 C / 2} \right)^{1/2} \frac{R_c}{R_0} \left(1 - \frac{I_0^2 L_0'}{V^2 C} \right)^{-1/2}, \quad (13)$$

where $\bar{N}_0 \equiv N_0 \pi R_0^2 l$ is the total number of atoms in the tube initially. Now for high compression ratios the energy stored in the capacitor bank is large compared to the magnetic energy in the load at the time the plasma first pulls away from the walls. This implies that $V^2 C \gg I_0^2 L_0$, so that

$$1/\mathcal{J}_0 = \left(\frac{\frac{1}{2} C V_0^2}{\bar{N}_0 k T_0} \right)^{1/2} \frac{R_0}{R_c} \quad (14)$$

$$= \left(\frac{\text{Energy stored initially}}{\text{Plasma energy at } t = t_0} \right)^{1/2} \frac{R_c}{R_0} \equiv \left(\frac{\epsilon_1}{\epsilon_0} \right)^{1/2} \frac{R_c}{R_0}.$$

With this result, one finds that at the peak of the compression cycle $\sin \tau \simeq 1$,

$$T_{\max} = T_0 \left(\frac{\epsilon_i}{\epsilon_0} \right)^{1/2} \frac{R_c}{R_0} \quad (15)$$

and the detachment time is

$$t_0 = I_0 L_0' / V \quad (16)$$

for initial conditions $N_0 = 10^{16} \text{ cm}^{-3}$; $T_0 = 10^6 \text{ }^\circ\text{K}$; $I_0 = 1.5 \times 10^4 \text{ amp}$, $t_0 = 1.5 \times 10^4 \text{ V}^{-1} L_0' \text{ } \mu\text{sec}$ and $\epsilon_0 = 0.14 \pi R_0^2 l \text{ joule}$ for L_0' in microhenries and V in volts.

PLASMAS PREHEATED WITH EXTERNALLY GENERATED SHOCK WAVES

Stability of the Compressed Plasma

In the coil shown in Fig. 1 the axial magnetic field has ripples near the wall of the tube due to the finite coil spacing. With this field configuration, the plasma formed behind the advancing shock front is compressed by the first half-cycle of rising magnetic field and it has been observed in streak camera photographs that the confined plasma appears to be radially stable (see Fig. 2). It is seen that the radius follows the field in an adiabatic fashion for the first half-cycle of the axial field. The compression generates a radially imploding shock wave on the next half-cycle after the shock pre-heated plasma has cooled by wall contact. A more detailed account of these particular experiments, using the apparatus of Fig. 1, can be found in Ref. 2.

In the coil shown in Fig. 3 the axial magnetic field is more uniform because of the large coil dimensions. It has been observed that there is a tendency for the plasma behind the advancing front to develop radial oscillations with an amplitude ranging from a few millimeters to a centimeter, so that the plasma may or may not be carried to the wall. Oscillations of a similar type are observed in the center section of the coil, between the magnetic mirrors, after the collision of two shock waves generated at opposite ends of the tube. Here again the amplitude of the radial oscillation is small (see Fig. 5B, 5C) or is large enough to carry the plasma to the walls as in Fig. 5A. The qualitative difference between the experimental results pertaining to the propagation of shock waves in axial magnetic fields obtained with the coil of Fig. 1 and the coil of Fig. 5, may be associated with ripples in the magnetic field near the wall, the field being quite uniform along the magnetic axis in both cases.

If there is no confining field the plasma fills the tube as seen in 5D. The time between compressions (half cycle) in Fig. 5A, B, C is six microseconds. (The expanded time axis in Fig. 5A resulted from operating the streak camera at a higher writing speed.) Because of the curvature of the field between the mirrors one would expect some instabilities to arise. However, these experimental observations suggest that for sufficiently large tube diameters such instabilities are not necessarily fatal in high compression

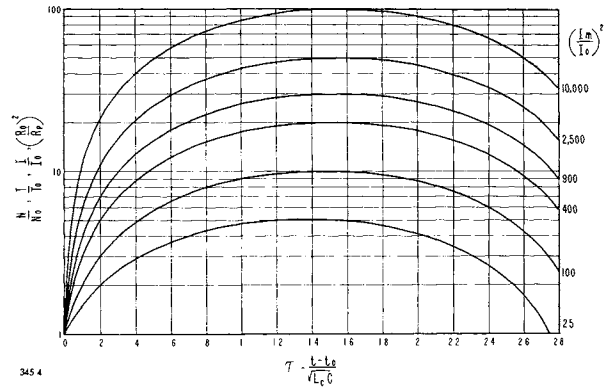


Figure 4. Adiabatic compression of a $\gamma = 2$ plasma

sion experiments of the type described. A final answer to these questions must await further studies under a variety of conditions.

Observations with a Plastic Scintillator

In Fig. 6 an oscillograph trace of signals from a plastic scintillator shielded by 2 cm of lead is compared with the background transient noise from the discharge. The neutron sensitivity is one scintillation per 4×10^3 neutrons. The bursts at 4 and 12 μsec correspond to two current maxima. These maxima arise from a programmed discharge in which part of the bank was fired just as the shock entered the coil and part was fired 8 μsec later. The current at the second maximum was $2 \times 10^6 \text{ amp}$ and $1.5 \times 10^8 \text{ amp}$ at the first. The peak field was 100,000 gauss and the ambient pressure was 100 microns. The pulse height to be expected for an individual neutron scintillation is about 1/10 of a division as checked by observing 0.66 Mev γ -ray signals from a cesium source which has about the same average pulse height. Streak camera photographs showed that the plasma column did not touch the tube walls in this experiment. In no case have we observed signals if there were severe oscillations of the plasma which carried the plasma to the walls. Scintillations have also been observed when dI/dt is zero. In one event pulses were observed for $\sim 30 \text{ } \mu\text{sec}$ over three half-cycles of the axial field. If the signals are due to neutrons, then a yield of the order 5×10^4 neutrons in the second and third oscillations is indicated and if they are due to neutrons of thermonuclear origin, then it appears that the impurities picked up at the wall when $H \simeq 0$ do not seriously contaminate the discharge for some time. Experiments to distinguish between neutrons and γ rays have not been completed.

With the magnet shown in Fig. 3, the scintillator output is sporadic from shot to shot, presumably due to the unpredictable plasma oscillations which appear in the streak camera photographs. Since these oscillations did not occur with the coil configurations shown in Fig. 1, it is hoped that these difficulties can be overcome in a properly designed coil.

HIGH-COMPRESSION EXPERIMENTS WITHOUT EXTERNAL AXIAL SHOCKS

Experimental Apparatus

The experimental arrangement is shown in Fig. 7. The magnetic fields are now generated in single turn steel coils with magnetic mirrors. The coils are split and bolted together to facilitate the insertion of quartz tubes in close proximity to the coil. This method of construction has been found to be suitable for generating magnetic fields up to approximately 300,000 gauss. At higher field strengths arcing in the area of the split causes difficulties. The use of inductively coupled mirrors offers one possibility for circumventing this trouble in future high field experiments; a schematic drawing is shown in Fig. 8. The mirrors are steel discs with a hole and slot, insulated from the main body of the coil by mica and plastic. The parallel plate transmission line leading to the coil is clamped together by large bonded fiber glass blocks so as not to distort the symmetry of the magnetic field unnecessarily (see Fig. 7).

The circuit diagram is shown in Fig. 9. The operation of the system is as follows: a 70 kv, .01 μf , 1 Mc capacitor is discharged to generate a high voltage pulse leading to 23 trigger electrodes in a magnetically insulated switch of the type described earlier. The 23 cables and preheater bank can be seen to the right of Fig. 7. The preheater bank is connected by copper plates directly to the main collector plate. This switch closes the preheater circuit and a voltage is applied across the coil. About 0.5 μsec after the preheater is switched, a high-voltage pulse arrives through a delay line to trigger a second 70 kv capacitor which, in turn, sends a high-voltage pulse through the 55 coaxial cables leading to trigger electrodes in

the switches on the main capacitor bank. These multiple triggers fire the switches in several places as can be seen in Fig. 10.

The large collector plate on the top of the main bank has an area of 7.5 m^2 and has a capacitance $C_p = 0.15 \mu\text{f}$. The collector plate inductance is small compared to the load inductance so that the collector plate and coil constitute a resonant circuit with a frequency $f = (L_c C_p)^{1/2}/2\pi$, in the 10 Mc range. The preheater bank is used to drive this resonant circuit at high power levels in order to preheat and ionize the molecular deuterium. Typical current and voltage characteristics, calculated on an analogue computer, for different load and preheater inductances, L_c and L_h , are shown in Fig. 11. With the preheater charged to 40 kv, the circuit on discharge rings at approximately 1 Mc and is modulated by an rf signal of approximately 10 Mc having an amplitude of approximately 20 kv. The average rf power from this discharge is approximately 4000 Mw. The maximum currents are several hundred kiloamperes. This discharge is damped in about 2 microseconds. The calculated and observed characteristics of the circuit are, during the early phases of the preheater discharge, in good agreement. The preheater bank consists of 9 specially designed low-inductance capacitors each with an inductance somewhat less than 10^{-8} h. The total capacitance, C_h , is one microfarad. Because of the low-inductance properties of the capacitors and switch, a high voltage appears across the coil. This combination of a high-frequency voltage pulse and high current (200 kiloamperes) very effectively pre-ionizes the plasma, so that the surface currents flow in times in the order of 0.5 microseconds and the plasma is closely coupled to the primary coil. In these experiments a 27 Mc, 1 kw rf signal is used to partially ionize the gas initially.

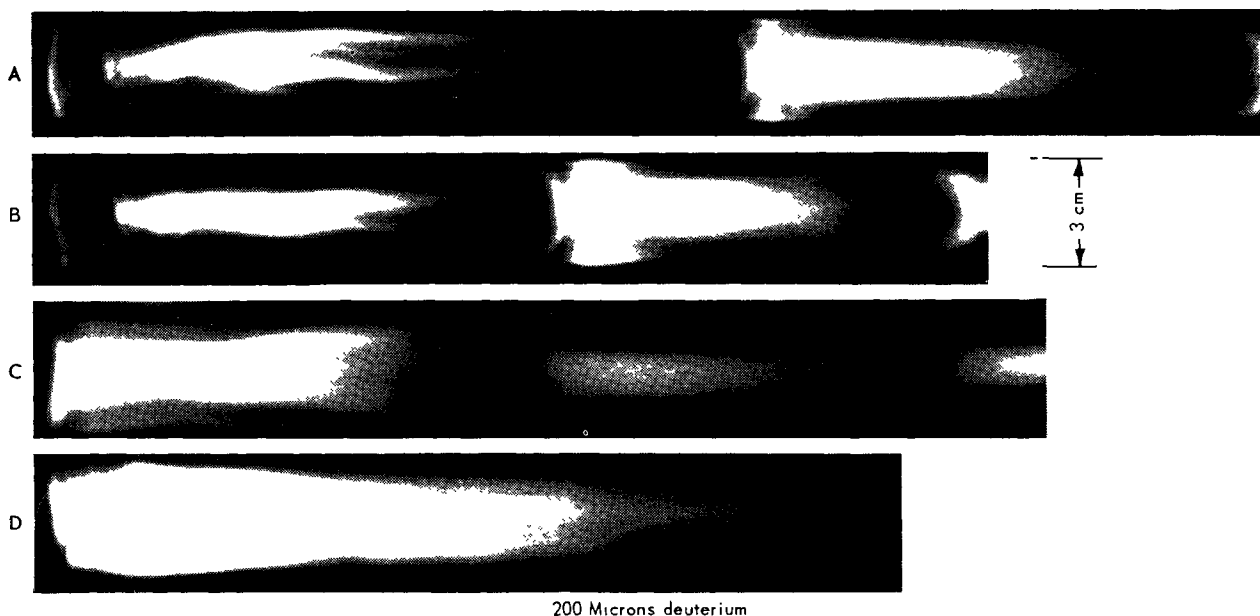


Figure 5. Streak photographs of the radial compression (A, B, C). In D there is no confining field. Preheating by externally produced axial shock waves

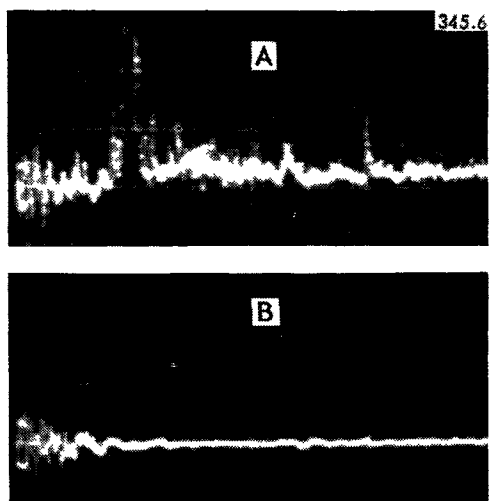


Figure 6. Observations with a plastic scintillator. Time scale 2 μ sec per division. A) Output of plastic scintillator, B) Background noise

Magnetic Compression without the Preheater Discharge

If the deuterium is weakly ionized with the rf excitation without the preheater discharge, the magnetic field penetrates into the plasma and is trapped in the ionized gas at later times. A weak imploding shock wave is generated on the first half-cycle and the plasma is finally compressed at low β to about one third of its original diameter. On the next half-cycle the plasma which was preheated on the first half-cycle implodes with high velocities ($\sim 10^7$ cm/sec). In addition, strong shock waves with about the same velocity are generated by the magnetic pressures in the magnetic mirrors. These shock waves move along the axis of the tube and collide at the center. This behavior is similar to the phenomena observed using an auxiliary capacitor for the axial shock generation. The shock propagation is also shown schematically in Fig. 8. Streak camera photographs of this behavior with an f/3 camera (writing speed here was 2.2 mm/ μ sec) are presented in Fig. 12 for two different experiments under the same conditions. The behavior of the plasma is reproducible and the oscillations of the type previously discussed (see Fig. 5) are not present.

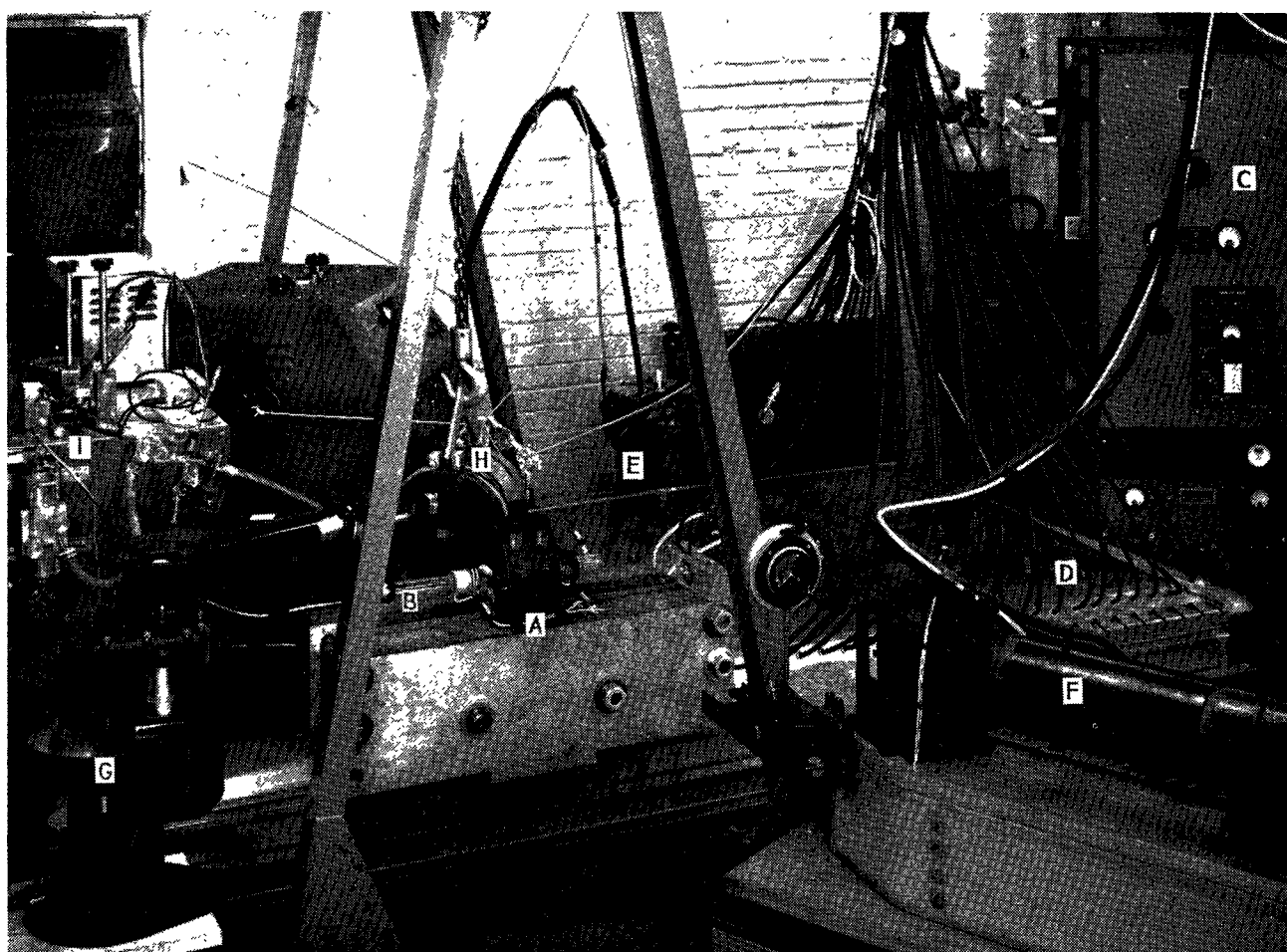


Figure 7. Experimental apparatus

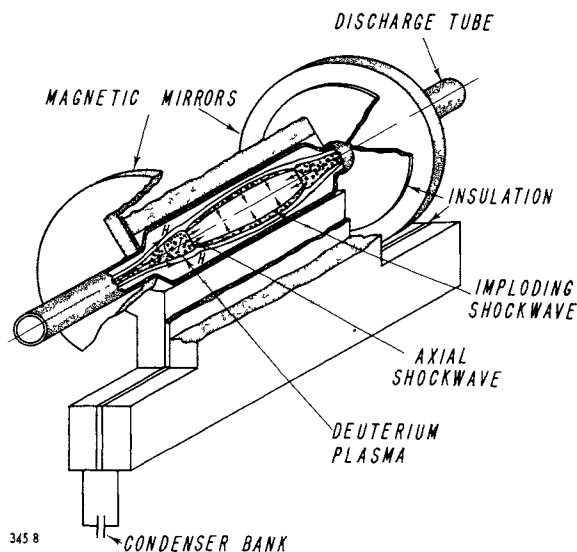


Figure 8. Electromagnetic shock tube with magnetic mirrors showing direction of shock propagation

The plasma radius as a function of time is given in Fig. 13. At $t = 0$ the plasma begins to contract and a shock wave labeled (A) is formed. The boundary between the plasma and the magnetic field is labeled (B). The shock wave is reflected from the axis and moves outward until at (C) the reflected shock meets the magnetic field-plasma interface and is brought to rest at (D) by the pressure of the rising axial magnetic field. The shock bouncing is then repeated as in the ordinary dynamic pinch until at (G) the axial shock waves collide at the center of the tube. The collision of the axial shocks raises the internal energy of the plasma so that it expands to about one-half the original radius. The rising magnetic field then compresses the plasma to about one-tenth its original diameter. Then at (I) the plasma begins to undergo large amplitude collective radial oscillations. The mechanism responsible for these oscillations is not yet understood completely. A series of experiments are in progress to determine their origin. These radial oscillations are not observed on the first half-cycle of the discharge even though the peak current is twice as great as on the second half-cycle.

Spectroscopic observations with an $f/2$ spectrograph with a time resolution of 10^{-7} sec (Fig. 14) show that

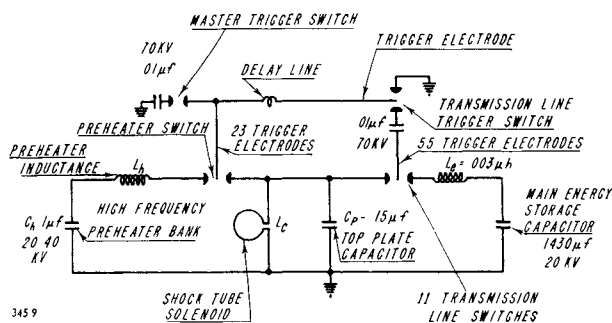


Figure 9. Schematic diagram of magnetic compression system

the luminosity on the first half is due to deuterium-line radiation. During the second half-cycle, after the imploding shock waves preheat the plasma, the Balmer lines disappear and only a continuum can be seen when the axial shock waves hit at the center of the tube. These observations are consistent with low temperatures on the first-cycle because of the poor inductive coupling of the capacitor discharge to the rf excited plasma. For comparison, a spectrum obtained when the plasma touches the wall of the tube due to oscillations of the type discussed in the preceding section of this paper is presented. In that case the impurity spectrum dominates.

The next question is: What is the nature of the radial oscillations that occur near peak compression? A streak camera photograph showing the compression at several position along the tube (Fig. 15) shows that these oscillations occur uniformly along the confined column of plasma. As remarked earlier, a complete understanding of the origin of these oscillations must await further investigation.

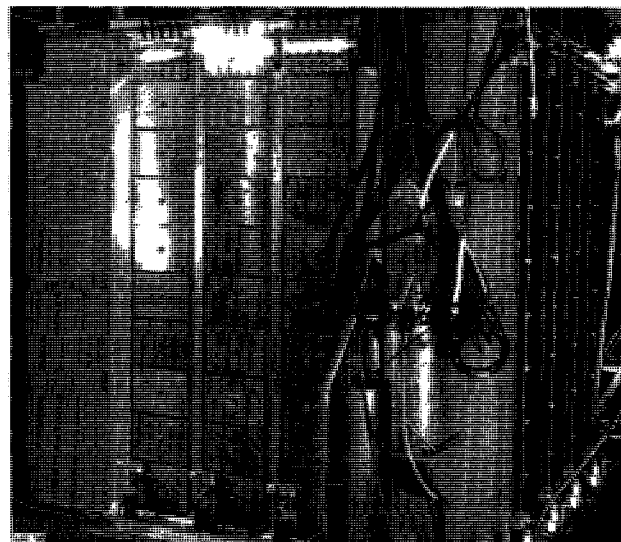


Figure 10. 285,000 joule condenser bank with switches firing at 13 points. Trigger capacitor and cables in foreground

The voltage, dI/dt , and current are shown in Fig. 16. The spikes on the second half-cycle are caused by the rapid contraction and expansion of the plasma column. These spikes are not observed without the rf pre-excitation as shown in Fig. 17. With a streak camera one also observes that no radial shock waves are produced.

To summarize these experiments without a pre-heater discharge, we observe that high velocity radial and axial shock waves can be generated on the second half-cycle of a high power capacitor discharge by passing the high current through a single turn coil with magnetic mirrors. In addition, the plasma is spectroscopically pure and in a high state of ionization as can be seen from the broadening of the Balmer lines during the first half-cycle and by observing the continuum during the second half-cycle.

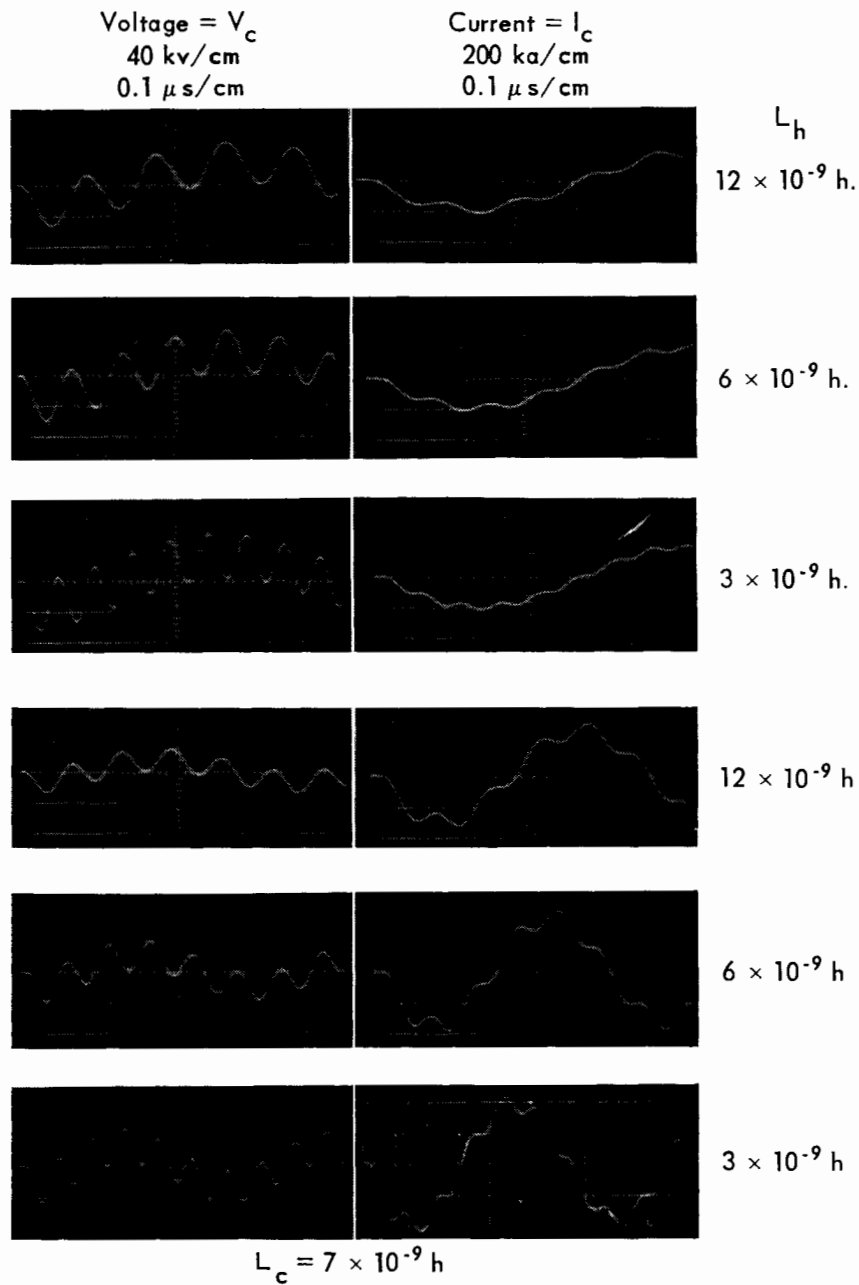


Figure 11. Preheater voltage and current for various values of the inductance of the external circuit, L_h , and coil, L_c . Analogue computation

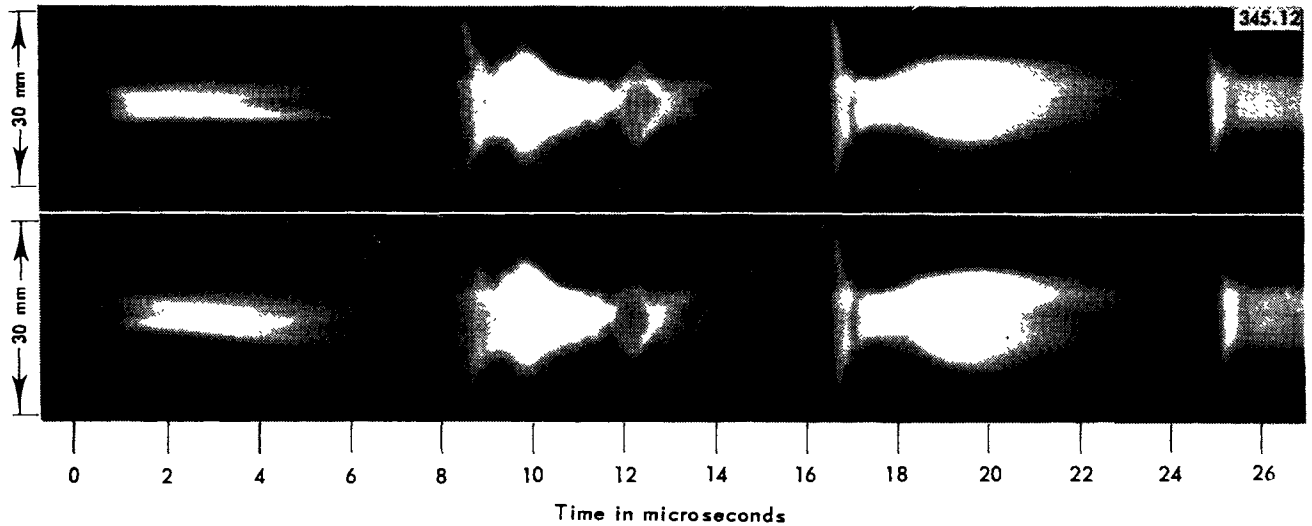


Figure 12. Smear-camera photograph of radial compression without the preheater for two experiments under the same conditions: $p = 0.3$ mm deuterium; $V = 15$ kv; $C = 392 \mu\text{f}$; $R = .003 \Omega$; $l = 20$ cm; mirror ratio 2/1; $I_1 = 1580$ ka; $I_2 = 790$ ka, where I_1 and I_2 are the current maxima on the first and second half-cycles

Magnetic Compression with Preheater

Because the main discharge is damped, only one-half the maximum current can be utilized for magnetic compression on the second half-cycle after 75% of the energy in the bank has been dissipated.

If the preheater bank is fired before the main bank is discharged, one observes the shock bounces, the burst of continuous radiation when the axial shocks collide and the radial oscillations at high compression on the first half-cycle of the discharge. Streak camera photographs showing this behavior are shown in Fig. 18. The coil here was 10 cm long with a mirror ratio of 3/1. The first flash of light is due to the discharge of the preheater bank. At $t = 0$, the main bank is switched and the preheated plasma implodes with a high velocity (again of the order 10^{-7} cm/sec) depending on the circuit parameters and ambient gas pressure. At times the radial oscillations do not appear (Fig. 18). The behavior on the first half-cycle is similar to that observed previously without the preheater on the second half-cycle.

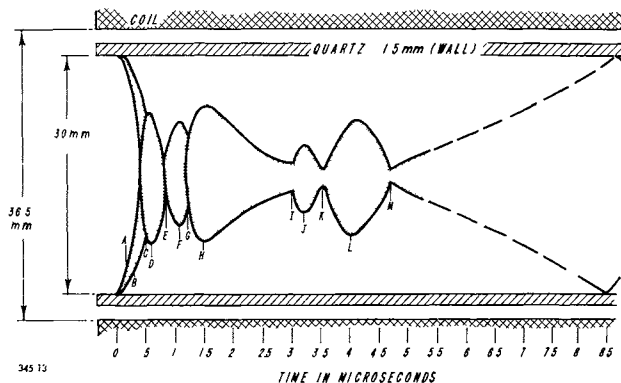


Figure 13. Radial motion of the compressed plasma on the second half-cycle without the preheater. Same conditions as in Fig. 12

We have made a theoretical analysis of the shock implosion (Fig. 19) using a model in which the plasma is carried toward the axis on a thin shell and the magnetic forces balance the inertial forces. The calculation is similar to the calculation by Rosenbluth⁵ for the pinch effect. The solid curve is the plasma radius and the dashed curve is the velocity, both in dimensionless units, as a function of an appropriate dimensionless time. The velocity and compression are given by the relations

$$R_p/R_0 = y$$

$$\frac{dR_p}{dt} = \left[\frac{\pi}{25} \left(\frac{V}{L_0} \right)^2 \left(\frac{R_0}{l} \right)^2 \frac{1}{\rho} \right]^{1/4} \frac{dy}{dT}$$

$$T = \left[\frac{\pi}{25} \left(\frac{V}{L_0} \right)^2 \left(\frac{R_0}{l} \right)^2 \frac{1}{\rho} \right]^{1/4} t, \quad (17)$$

where the symbols have the meaning as in the previous calculation: with V in volts, L_0 in henries, ρ in gm/cm³, R_0 , R_p and l in centimeters and t in seconds. The curves are for various ratios

$$(L_0 + L_e)/[L_e(R_0/R_c)^2]$$

of the external to internal inductance. This parameter is essentially the ratio of the magnetic energy stored outside the electromagnetic shock tube to that stored inside.

The theoretical time for the implosion phase in our present experiments is of the order $0.2 \mu\text{sec}$, which agrees closely with the observed times. Therefore, it is possible to predict with some accuracy the shock velocity and plasma energy during the initial dynamic stages of the compression. For a discharge with $V = 15$ kv, using 1/8 the energy of the large bank and for an ambient deuterium pressure of 0.1 mm, the shock heating raises the plasma temperature to about 80–90 ev. This is also consistent with a pressure balance calculation just after the axial shock collision. The

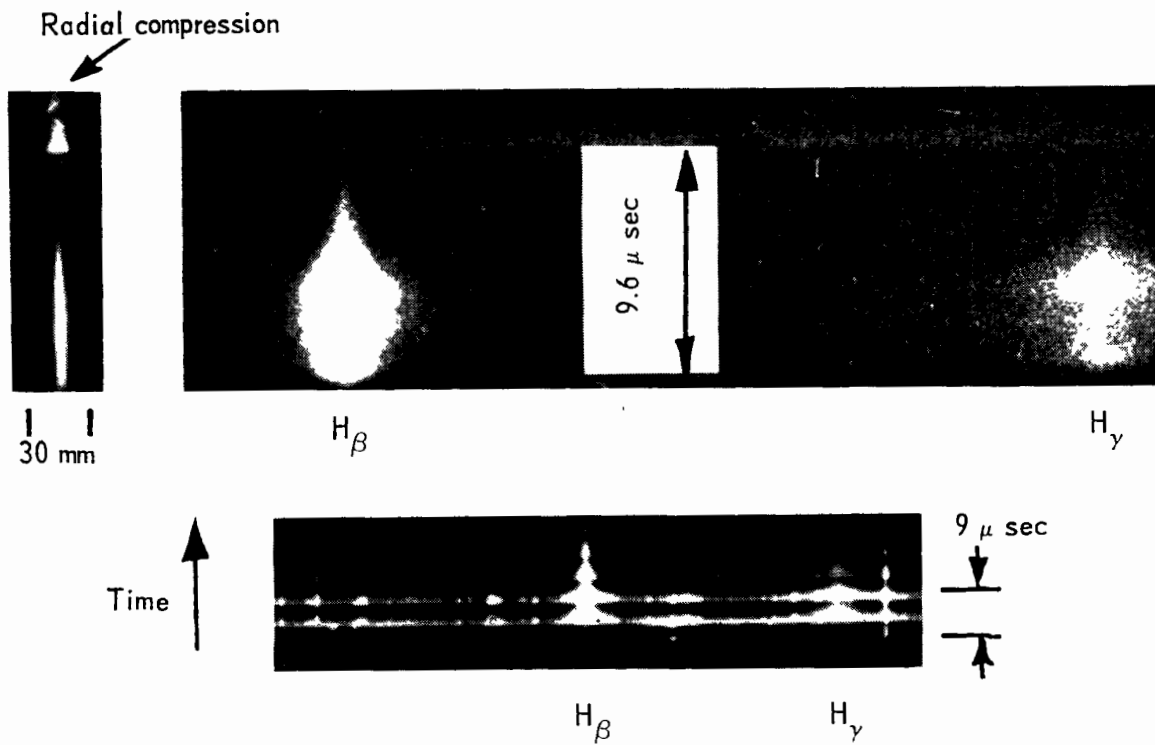


Figure 14. Spectrograms of the radiation with (top) and without (bottom) instabilities that throw the plasma to the walls. The radial compression is shown on the same time-scale as the time resolved spectrogram at the top

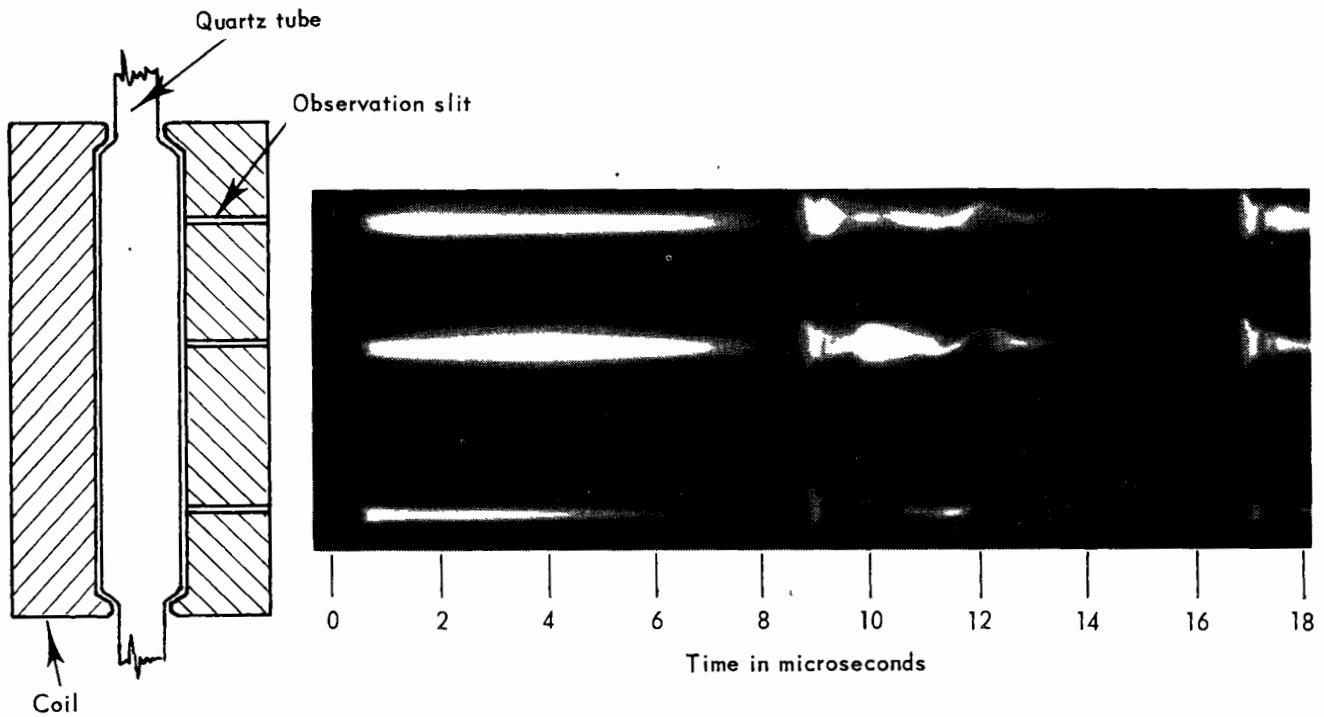


Figure 15. Smeared camera photographs of the radial motion at three points along the tube. $V = 15 \text{ kv}$; $C = 392 \mu\text{f}$; $l = 10 \text{ cm}$; mirror ratio 3/1

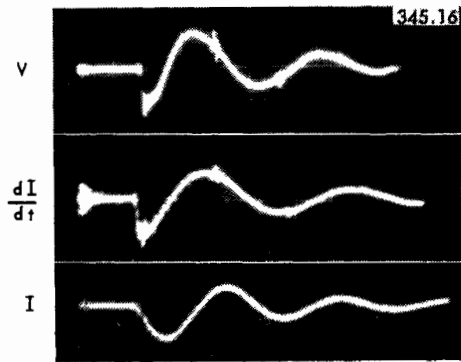
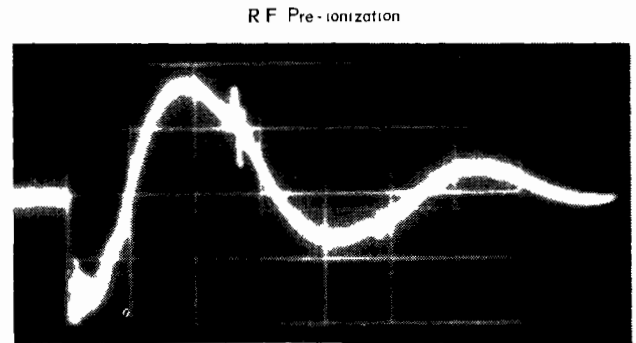


Figure 16. Typical electrical characteristics of the main discharge
Same conditions as in Fig. 12

adiabatic compression constricts the plasma by additional factor of 5, so that for $\gamma = 5/3$ a temperature of 7.3×10^6 °K is indicated for a final volume of about 0.7 cm^3 . The field in the central region is about 75,000 gauss and the calculated final density is 10^{17} ions/cm³. At full bank power with a central field of the order 200,000 gauss, temperatures of several kev are energetically possible.

Observations with a plastic neutron counter in our early experiments at higher power levels (Fig. 6) are consistent with a plasma energy of about one kev because the final field strength and initial temperature from the shock preheating are higher than in the previous example. The relatively quiescent behavior of the plasma for several microseconds and the correlation of the scintillations with maximum current



5 μsec division

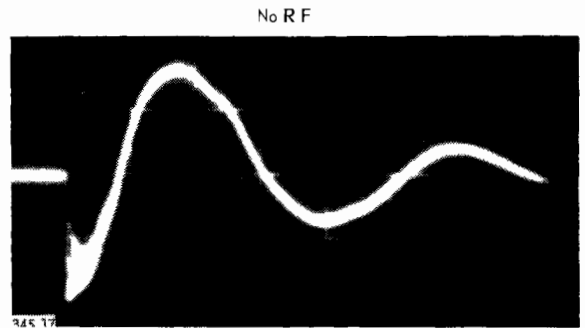


Figure 17. Voltage across the coil with (top) and without (bottom) RF pre-excitation

support this view. Independent measurements of the temperature and electron densities from X ray and optical continuous radiation have not yet been com-

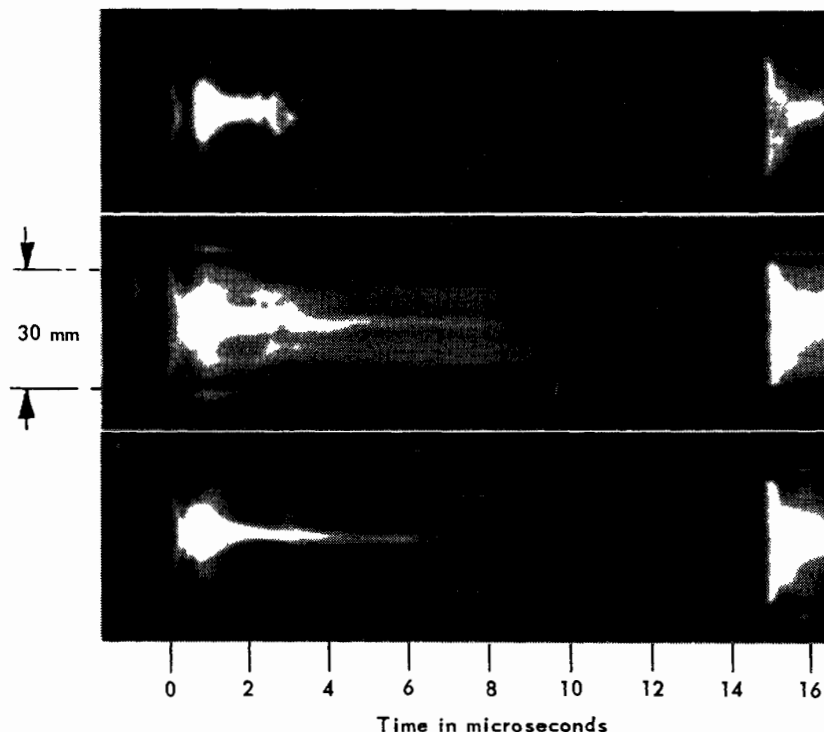


Figure 18. Radial motion of the plasma with preheating. $V = 15 \text{ kv}$; $C = 392 \mu\text{f}$; $p = 0.1 \text{ mm}$ deuterium, preheater voltage 25 kv. vs. time.

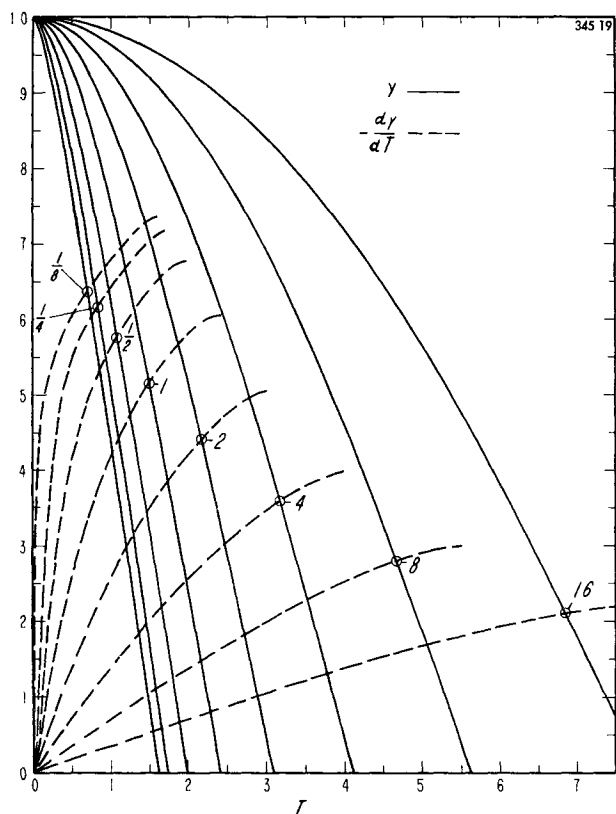


Figure 19. Radius and velocity of the imploding plasma vs. time as discussed in the text

pleted but it is hoped that they will yield an independent estimate of densities and temperatures. However, because there may be large accelerating voltages resulting from the radial oscillations, there is a good possibility that hard radiations are caused by this mechanism. Situations of this kind have been encountered in many other plasma experiments and it would be no surprise if this were the case here.

One is encouraged, however, by the fact that the plasma can be kept away from the tube walls for many microseconds in a high state of compression and that the radial oscillations show a certain degree of symmetry, reproducibility and appear to be of a

collective nature. Further study will undoubtedly disclose their origin and it may be possible to eliminate them or at least to determine their effect on the state of the plasma. They may in fact provide a heating mechanism.

Experiments in a toroidal geometry should help to clarify the situation (Fig. 20). This will show whether the radial oscillations are due to the presence of the magnetic mirrors. The torus consists of large steel plates connected to the transmission line. A photograph of one plate with a circular groove to contain the discharge tube is shown in Fig. 21. The major diameter here is 30 cm and the bore is 6 cm. With full power of the 285,000 joule capacitor bank, a field of 100,000 gauss can be generated in this device. The experiment should also give some basic

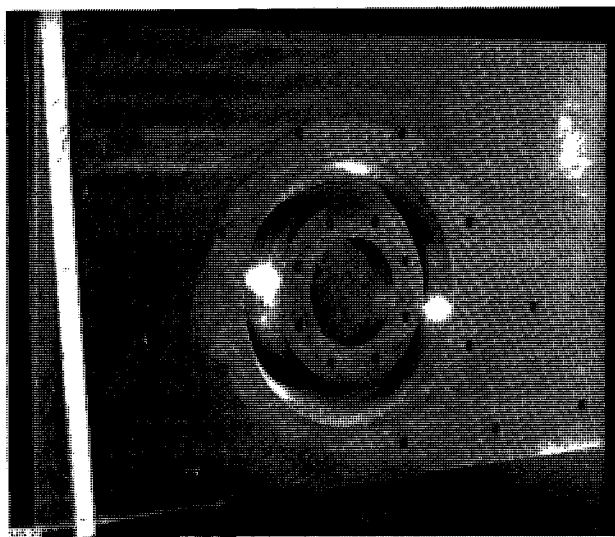


Figure 21. One-half of the steel torus

information on the magnitude of the drift velocity of a plasma in a known field gradient.

ACKNOWLEDGEMENTS

The author would like to thank the many persons who gave assistance during the course of these investigations. The assistance and encouragement of Drs. W. R. Faust and H. Griem have been of particular value. Their critical comments and suggestions helped to clarify many of the points discussed in this paper. Most of the experimental apparatus was constructed by T. H. DeRieux and L. J. Melhart. A number of electrical engineering problems were solved in collaboration with C. B. Dobbie, particularly problems associated with the pre-heater circuit. The forming of complex quartz chambers is due to the inspired work of L. B. Clark. J. M. Frame and C. C. Martin designed the coils and worked out the mechanical design of the capacitor banks. The numerical integrations of the equations for the radial motion of an accelerated plasma were performed by A. D. Anderson. Finally, the assistance of C. E. Thompson on photographic techniques is gratefully acknowledged.

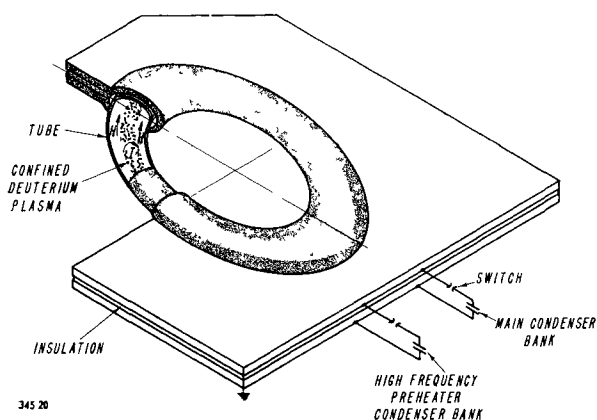


Figure 20. Schematic drawing of the torus

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Acceleration of Plasma into Vacuum

By John Marshall *

The first part of this paper is a discussion of the magnetic acceleration of plasma. The second part contains a description of some experiments which have been performed.

Work similar in some respects to that described here has been reported by Thonemann, Cowling and Davenport,¹ by Bostick² and by Kolb.^{3, 4} Thonemann and his collaborators studied the effect on a plasma of a radio-frequency traveling wave in a capacitor-loaded helical delay line. They found a net negative electric current to flow in the direction of the traveling magnetic waves, presumably because the electrons in the plasma were dragged along more efficiently than were the ions. Bostick has developed a source capable of propelling puffs of plasma into vacuum at speeds of the order of 10^7 cm/sec. His plasma gun employs a vacuum spark between gas-loaded titanium electrodes. The plasma, which consists of a mixture of ions of titanium and the gas with which the source is loaded, is driven away from the source by the magnetic field of the current flowing through it. Kolb has used programmed external magnetic fields to accelerate shock waves. Since his shock waves move through a gas which is initially at low temperature, dissipative processes are involved and the result is the heating of plasma rather than its acceleration to high velocities.

In the work reported here the intention is :

1. To produce a burst of gas *in vacuo*.
2. To ionize the gas and heat it to such an extent that it becomes a good electrical conductor.
3. To accelerate the plasma thus produced into vacuum by the use of external time-varying magnetic fields.

MAGNETIC ACCELERATION OF PLASMA

Let us consider a cylindrical tube surrounded by a solenoid, carrying current so as to provide an axial magnetic field, and containing a hot plasma. The plasma might form a cylindrical body down the axis of the tube excluding by its diamagnetism and surface eddy currents, the magnetic field produced by the

solenoid. Such a situation actually can exist just after the passage of a sufficiently strong shock wave in an axial direction along the tube.⁵ Now let us imagine a wave of increased axial magnetic field to move along the tube. The magnetic lines of force, as the wave approaches a given position, are pushed inward toward the axis of the system, and expand to their original position again as the wave passes. The lines of force ahead of the magnetic wave diverge so as to produce a forward component of force on the plasma ahead. Inside the magnetic wave the magnetic pressure is high and, in the case of equilibrium, so would be the plasma pressure. If the magnetic field is high enough it will exert enough pressure on the plasma trapped at the center to drive the plasma forward ahead of the field, and thus to carry along a considerable mass of plasma at the speed of the magnetic wave. The situation is superficially quite similar to peristalsis in the gastro-intestinal tract of an animal. In the same way that a contractile wave along the esophagus of a giraffe can propel a gulp of water from ground level up to its stomach, so can a wave of increased magnetic field propel a mass of plasma along the axis of our tube.

As an example of the pressure which can be exerted on a plasma by a magnetic field let us consider a field of 30,000 gauss. This is approximately the strength of the magnetic piston employed in some of the apparatus to be described below. $B^2/8\pi$ for this field strength is 3.58×10^7 dynes/cm² or about 35.4 atmospheres. Ordinarily one might expect to accelerate rather small masses of plasma so that a 35-atmosphere pressure, if exerted over a suitable distance, should lead to respectable velocities.

The most convenient method of producing a moving magnetic piston appears to be the discharge of a capacitor bank into one end of an artificial delay line composed of a solenoid loaded with capacitors at intervals along its length. The velocity of such a line is given by $v = 1/\sqrt{L'C'}$ and its characteristic impedance by $Z_0 = \sqrt{L'/C'}$. Here the velocity will be in cm/sec and the impedance in ohms if L' and C' , the series inductance and shunt capacitance per unit length, are expressed respectively in henries/cm and farads/cm. It is quite plain that the parameters L' and C' can vary as a function of distance along the line, so as to produce an accelerating magnetic piston, exerting a constant force over a considerable distance.

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It is not inconceivable that really high plasma velocities could thus be achieved.

Let us again use our example of a 30,000-gauss piston, and compute the velocity which could, in principle, be given to a lump of plasma of reasonable mass. Assume that the piston velocity is so adjusted as to exert the full pressure of a 30,000-gauss field against the plasma over a distance of 50 cm. Assume that the plasma to be driven corresponds to 0.02 cm³ of deuterium at normal temperature and pressure for every square centimeter of area. This would amount to about 3.5 μ g of plasma per cm². These particular numbers are chosen because they correspond to an amount of plasma which has been driven in this work, although not under these ideal conditions. Remembering that the velocity attained under constant acceleration a through a distance s is given by $v = \sqrt{2as}$, and substituting an acceleration of 3.6×10^7 (dynes/cm²)/ 3.5×10^{-6} (g/cm³) = 10^{13} cm/sec², we compute a velocity of approximately 3×10^7 cm/sec. The kinetic energy of the plasma jet would be approximately 180 joule/cm², and its momentum about 100 g cm/sec per cm² of area.

The system of a magnetic piston driving a lump of plasma has phase stability similar to that which a nuclear particle accelerator, such as a synchrotron or a linac, must have in order to operate successfully. The plasma rides on the front of a wave as does a surf board. If the plasma at some point should be moving more slowly than the wave, it will drift backward relative to it and find itself riding higher where there is more force available to drive it. If it should move faster than the wave, it will get ahead; the force will decrease and the wave will catch up again. Thus the speed of the piston does not have to be precisely adjusted. The plasma will tend to adjust its speed and to stay ahead of the piston. However if the accelerating process relies on phase stability, the full pressure of the magnetic piston cannot be used. The plasma must necessarily ride part way up the wave. If it should pass over the crest and find itself on the back of the wave, it will be decelerated somewhat and left behind.

Somewhat more latitude for error in the matching of piston and plasma speed can be provided by the use of a piston with a flat top. This can reduce the danger of the slipping of the plasma on to the back slope of the wave. The resistance of the lump of plasma to the penetration of magnetic field implies that if the piston tends to catch up on the plasma, the plasma will make a hole in the magnetic field and thus automatically adjust its rear boundary to move at the maximum possible acceleration. A flat-top piston, of course, requires more energy than does a peaked one of equal strength.

If the back end of the plasma mass rides imbedded in the front of the piston, however, it will be compressed sideways by the magnetic pressure in the piston and its cross-sectional area will be reduced, as for instance has been observed in the Los Alamos experiments on shock channeling.⁵ This will tend to limit the amount of plasma which can be driven.

With such an arrangement, really appreciable fractions of the energy of a magnetic piston might be transferred to the jet of plasma. Of course as energy is taken out of the piston, its strength will decrease and, to produce the effect described, means would have to be found to feed more energy into the piston as it drives the plasma.

From the point of view of efficiency of energy transfer, the traveling wave magnetic piston is nearly ideal. The magnetic wave carries all the energy of the driving capacitor bank (assuming negligible resistive losses) and in principle should be able to transfer a large fraction of it to the plasma. The traveling wave, however, is not the only method of producing such a piston. Another method would be to have a large number of separate coils arranged along the acceleration system and excited from separate capacitor banks in a programmed manner. Also it would be possible in principle to use a long coil, tapered in such a way as to produce a magnetic field larger at one end than at the other. If the current through the coil is raised slowly the effect is, to some extent, that of a moving piston. In both of these alternate methods, the energy required to produce the magnetic field is much larger than in the traveling wave method.

In order to drive plasma electromagnetically it must be highly ionized and heated to a high enough temperature so that there will not be serious penetration of field during the driving process. What appears to be a satisfactory method of heating and ionizing is the passage of an intense shock wave along the tube just before the piston circuit is energized. A convenient method of generating such a shock wave is to discharge a capacitor through a one-turn coil looped around the acceleration tube.⁶ The shock wave, as it passes through the gas, ionizes it and incidentally imparts to it an initial velocity in the direction in which it is to be accelerated.

Quick-acting Valve

A mechanical valve has been developed for producing the burst of gas at the input end of the plasma acceleration system (Fig. 1). Basically what the valve has to do is to admit something of the order of 1 cm³ of gas to the system in a time of the order of 100 μ sec. In 100 μ sec, deuterium, which is the gas used in most of the experiments described here, moves approximately 10 cm at the molecular velocities characteristic of room temperature. Thus it becomes possible to have a pressure of deuterium gas of about one millimeter of mercury in the input end of the system while the rest of the system remains temporarily at high vacuum. Under these conditions the plasma produced by the ionization of the gas can be accelerated unimpeded through the system.

The valve is very simple and differs very little from an ordinary mechanical valve such as might be used as a water faucet. The main differences are that it is held shut by a spring, that it uses a teflon (polyperfluoroethylene) gasket, and that it is opened by a hammer blow. The spring tension on the valve

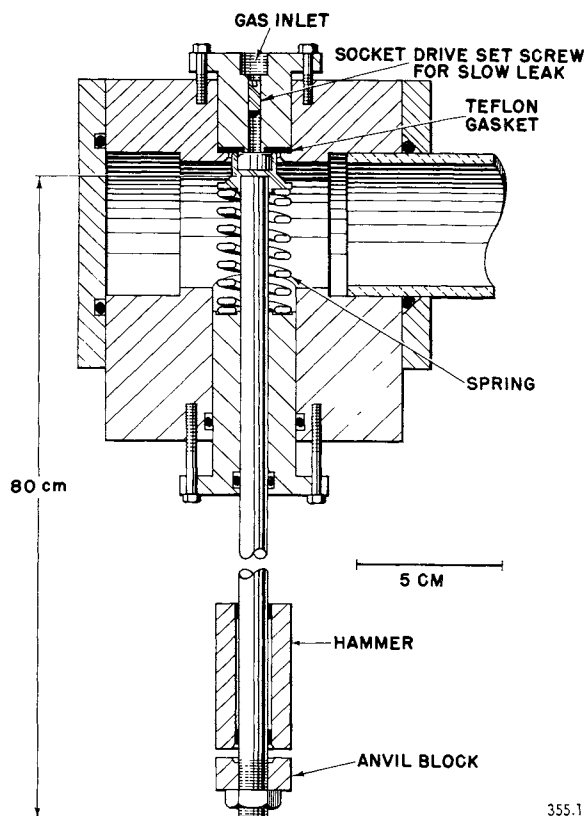


Figure 1. Quick-acting valve. One of the many possible arrangements for such a valve. Gas flowing through the leak into the small plenum in the valve stopper can be admitted to the system by allowing the hammer to fall and strike the anvil block

stopper is used so that the hammer blow can be absorbed non-destructively by an elastic system. Teflon is used as a gasket material because of its non-sticky properties. In such a valve it is essential that opening can be achieved with a very small though sudden motion of the stopper. Any gasket stickiness would tend to increase the stopper motion necessary to open the valve, and almost certainly would reduce the reproducibility of performance from one shot to the next. For this reason teflon was chosen and worked very well in the first model. No other material has been tried, and no experimental information has been obtained that it is actually better than any other material. A hammer blow was chosen to open the valve simply because it provides a sudden motion that is easily controlled in strength up to and beyond the elastic limits of the materials used. The original hammer consisted simply of a brass cylinder sliding under the action of gravity along a length of drill rod. The cylinder can be dropped from a variable height so as to strike, with an adjustable impulse, against an anvil block attached to the end of the rod. The impulsive wave produced moves back along the rod at the speed of sound in steel (4500 m/sec) and serves to open the valve. With this arrangement all other functions of the apparatus are timed to occur subsequent to the hammer blow and are controlled by an

electronic delay unit started by an electric contact made by the hammer in striking the anvil.

In some experiments it is inconvenient to have all functions of the apparatus timed to start after the hammer blow, and it becomes expedient to use some substitute for the falling weight which can be timed precisely after other events. For this purpose an electromagnetic hammer has been built (Fig. 2). It consists of a small coil in close proximity to an aluminum anvil block. A capacitor is discharged through the coil by means of an ignitron, and the magnetic pressure of the field generated by the low inductance coil produces an impulse similar to that produced by a hammer blow. The design of the coil is such that most of its inductance is due to the high concentration of magnetic field where it is squeezed between the coil and the anvil block. Since this gap is originally very small, the inductance is also small at the beginning of the current pulse and the complete impulse can be delivered in less than 50 μ sec.

Three methods are readily available for regulating the amount of gas admitted by the valve. They are control of (1) the volume emptied by the valve (this implies that the valve is opened far enough to empty the volume involved), (2) the pressure of the gas behind the valve and (3) the impulse delivered to the valve stopper and, therefore, the degree to which it opens. The first method, which involves an adjustable plenum just behind the stopper, is an attractive one in that the valve may bounce and open more than once without affecting the amount of gas admitted. For most purposes, however, a combination of the second and third methods is easier and appears to be capable of giving highly reproducible results from shot to shot. In most cases, it turns out to be unimportant

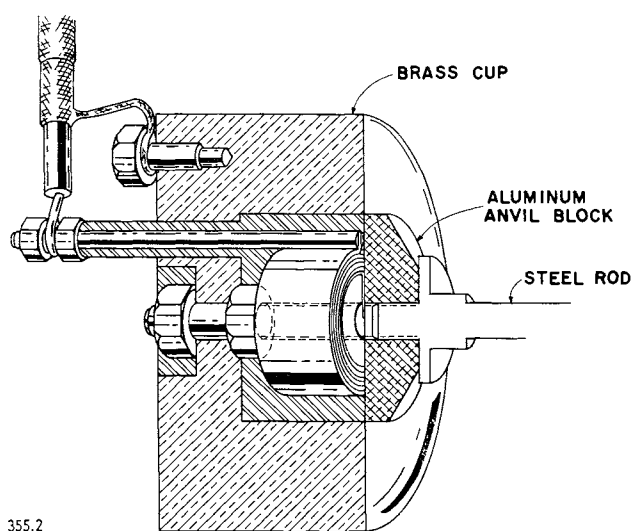


Figure 2. Electromagnetic hammer. Spiral coil wound of copper ribbon is potted solidly in epoxy resin so as not to be distorted by its own blows. Magnetic lines of force are crowded by skin effect so as to pass through 0.3 mm gap between coil face and anvil block. Hammer blow effect in steel rod is produced by discharging 2-3 kv capacitor through coil by means of an ignitron

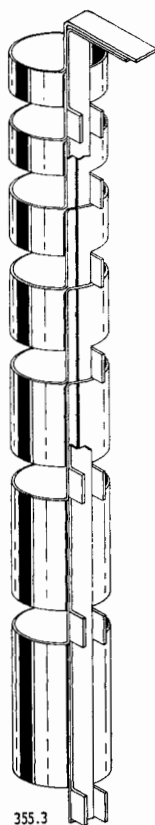


Figure 3. Typical construction of piston coil. In this case the solenoid is tapered in pitch so as to produce a line with increasing velocity. Capacitors are connected at the pairs of tabs between turns by coaxial cable paralleled in groups of four. Ground return conductor is held close to axial transition sections of solenoid so as to cancel asymmetrical field and reduce inductance. Coil is insulated with teflon and potted in glass-reinforced epoxy resin to give it strength to withstand magnetic forces

whether or not the valve opens more than once. Everything of interest for one shot is over long before the valve has a chance to bounce open again.

Plasma Jets from Shock Wave Blow-off

It has been mentioned above that an intense shock wave provides a convenient method of preheating and ionizing the gas. Actually the passage of a shock wave through a gas which decreases in density, so that the shock runs into vacuum, is in itself a method of introducing a jet of plasma into vacuum. Presumably the plasma jet consists of the particles more or less within the last mean free path of the gas. Since these particles have no more particles in front with which to collide, they simply keep on going. If the shock wave and the blow-off plasma jet are arranged to move parallel to a magnetic field, along the axis of a current-carrying solenoid for instance, any sideways motion of the ions is averaged out and the jet tends to move along the lines of force. Neutral gas, however, which may accompany the jet or be produced therein by recombination, is unaffected by the magnetic field and consequently is lost through the sides, leaving the jet fully ionized.

The apparatus used in studying this phenomenon (Fig. 4) has been a glass tube approximately 125-cm long connected at one end to a vacuum system and at the other to the quick-acting valve. Over most of its length, the tube was of approximately 10-cm outside diameter and was wound with a solenoid which could be connected to a capacitor bank through an ignitron so as to produce an axial magnetic field. At each end the tube was reduced through conical transition sections to a 5-cm diameter, and a two-turn shock driving coil was located about halfway along the transition at the end near the quick-acting valve. The shock driving coil could be connected by means of a spark gap to a 22.5 μf capacitor bank, charged to 20 kv. The timing of this spark gap and of the ignitron on the axial field (B_z) bank could be accurately controlled by a variable electronic delay circuit triggered by the hammer blow of the quick-acting valve.

The shock wave is launched from the shock driving coil described above by magnetic interaction of the primary current in the coil and secondary currents induced in the low pressure gas inside it. Experimentally it is observed that the gas will break down only at times at which the magnetic field inside the coil is passing through zero. At such a time the gas can

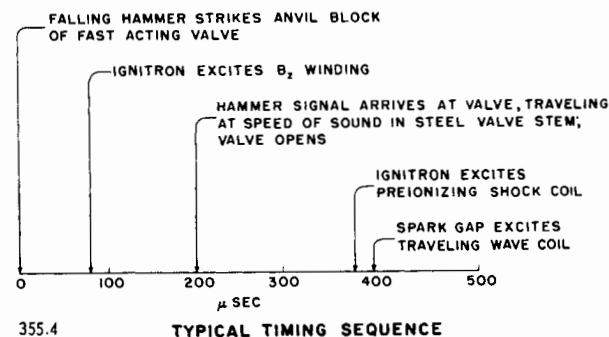
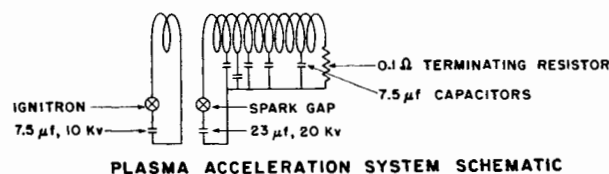
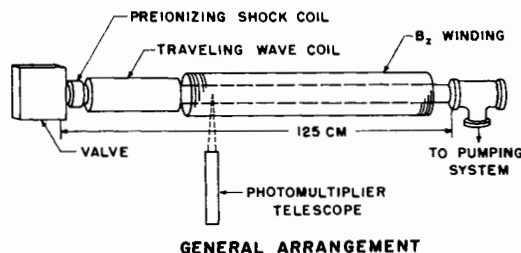


Figure 4. Arrangement of equipment for accelerating plasma with a moving magnetic piston. The arrangement depicted here has been superseded by one employing much higher voltages on the shock driving coil. Note that here the magnetic piston is produced in a line which is tapered by varying the intervals at which capacitors are connected

be ionized by the back emf produced by the rapidly changing flux inside the coil. A ring gas current results, of such magnitude and direction as to shield the regions inside the discharge from the field of the coil. The magnetic pressure external to the discharge drives the plasma toward the axis of the system, picking up gas ahead of it, pinches it off and squirts it out axially in both directions away from the coil. The effect is as if there had been an explosion, and shock waves are produced traveling along the tube in both directions.

The shock driving coil and the capacitor bank which drives it constitute a resonant system which rings so as to produce a train of damped oscillations. Generally a new shock wave is generated each time the current in the coil passes through zero; at least this is true so long as there is gas present to be driven. At very high voltages applied to the coil, the gas appears to be swept out completely by the magnetic field and only one shock wave is generated, but with the arrangement described here, a long series of shock waves is produced. In order to make the apparatus generate a shock wave on the first half cycle of the train of oscillations, it is necessary to pre-ionize the gas. This is done with an electrode brought into the tube in the vicinity of the shock driving coil and connected to a small capacitor through a current limiting resistor, the capacitor being charged to a potential of 10 kv relative to ground. Without pre-ionization the first shock wave appears after one half cycle and on a number of successive half cycles thereafter.

The propagation of the shock wave and of the plasma jet which it produces is conveniently observed by photomultiplier telescopes connected to a multibeam oscilloscope. Each photomultiplier is arranged with a lens and a slit so as to be exposed to light from just one point along the axis of the tube. With such an arrangement, it is observed that the shock wave accelerates as it moves into gas of lower density and that it becomes less steep. Presumably the sharpness of the shock front is related to the mean free path of the particles under the conditions prevailing at that point and with that particular shock intensity. When the shock runs out into vacuum, the jet that it produces begins to separate into slow and fast plasma; the fast particles moving out in front, leaving the slow particles behind. In the absence of particle-particle collisions, this would have the effect of reducing the temperature in the forward and backward degree of freedom, while leaving the sideways degrees of freedom unaffected.

Unfortunately very little data have been taken which can contribute to a quantitative study of the phenomenon. This is partly the result of pressing onward toward the problem of acceleration with a moving magnetic piston, and partly because of the non-existence of instrumentation for the measurement of the rapidly changing gas densities and velocities involved.

Systems identical with that described in this section have been adapted or are being adapted as plasma injectors for thermonuclear machines. One has been

in operation for some months on Ixion, the crossed electric and magnetic field spinning plasma machines under development at Los Alamos.⁷ Another is being installed on a large linear pinch machine which is being studied currently.⁸

Magnetic Piston

As mentioned above, the most convenient method of producing a moving magnetic piston appears to be the use of an artificial delay line consisting of a coarse solenoid loaded at intervals with capacitors. The line currently in use consists of a solenoid of 2.7-cm radius, with the turns spaced 3.8 cm. It is loaded with one 7.5 μf capacitor for each turn. Such a line has an inductance of 2×10^{-8} henry/cm, a characteristic impedance of 0.1 ohm and a phase velocity for low frequencies of 5×10^6 cm/sec. A low impedance source of emf V applied suddenly to one end of this line should produce a magnetic field rising in a step to a value $B = 3.30 V$. This would mean that a 20 kv capacitor discharged into the line would produce a 6.6×10^4 gauss field falling off exponentially after a sharp rise with a time constant given by $\tau = RC$, where R is the 0.1 ohm line impedance and C is the capacity of the bank. Actually the rise time of the signal is lengthened by the imperfections of the line, so that a 22.5 μf capacitor bank produces only about half the magnetic field in the wave front that one would compute.

In order to simplify the effects observed and also to lengthen the life of capacitors and spark gap, the line is terminated at its output end with a 0.1 ohm resistor. This reduces reflections from the end of the line and makes it so that the current in the line can be described as a single pulse moving at line velocity from one end to the other. Some difficulty was encountered in developing a resistor suitable for this service. It must be capable of carrying pulse currents of at least 10^5 amp, have low inductance and absorb the full energy of the capacitor bank each time the equipment is pulsed, 1500 joules for the present arrangement. Also it should be of thin enough material so that skin effect does not materially affect its resistance for the important frequency components of the pulse. A successful design consists of nichrome ribbon clamped between brass plates with teflon insulation. The teflon appears to be able to stand the momentary high temperatures and the brass plates provide good mechanical support as well as a low inductance return current path.

Each capacitor as well as the terminating resistor is connected to the line through 4 parallel 30 ohm coaxial cables about 70 cm in length so as to achieve low inductance. The central wires of each group of four cables connect to one turn of the solenoid while the outer conductors connect to a ground return strap. The return strap is as close as is practical to the connection in the axial direction between solenoid turns so as to reduce inductance and avoid distortion of the field. Insulation between turns and from the turns to the ground return strap is of teflon and the whole coil is potted in epoxy resin with glass tape

reinforcement so that it will be strong enough to withstand the electromagnetic forces. The coil in use at present has 10 turns and has a total length of about 38 cm.

Shock Driving Circuit

The shock wave which pre-ionizes and heats the gas to be driven by the magnetic piston is, in the present arrangement, produced by a one turn coil connected through a spark gap to a $0.88 \mu\text{f}$ capacitor which has been operated as high as 60 kv. The inductance of the coil is approximately $0.065 \mu\text{h}$ while that of the entire circuit is computed from the ringing frequency to be $0.295 \mu\text{h}$. Thus only 22% of the voltage applied to the capacitor appears across the coil. A good fraction of the parasitic inductance of the circuit is in the spark gap, which is of the four electrode type and is operated in air at atmospheric pressure. The spark gap is triggered from a $0.1 \mu\text{f}$ capacitor through 3 ignitrons connected in series so as to be capable of withstanding 50 kv.

PERFORMANCE

The apparatus in its present form appears to be capable of driving appreciable masses of plasma at the speed of the magnetic piston. The speed is measured by photomultiplier telescopes focused at intervals along the axis of the drift tube into which the piston drives the plasma. An accurate measure of speed is somewhat difficult using this technique because of the diffuse nature of the plasma jet. However, gross features of the luminosity vs. time curve can be followed from telescope to telescope (Fig. 5), and it appears that within 20% or so the plasma moves with piston speed.

The momentum acquired by the plasma has been observed by two methods. One of them employs a capacitor microphone and the other a ballistic pendulum. The capacitor microphone is placed on the axis

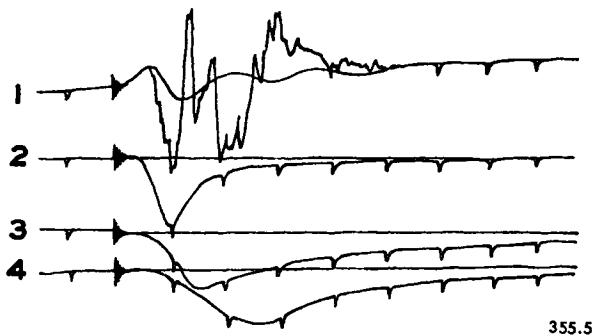


Figure 5. 4-beam oscilloscope record of data from typical shot with magnetic piston. This shot used 9 kv two-turn shock coil preheating with piston line tapered by varying interval between capacitor connection positions. Trace 1 shows B_z beyond end of guide field solenoid. Traces 2, 3 and 4 are photomultiplier telescope records at positions 53.7, 77.4 and 97.3 cm from the shock driving coil. The small pips are $10 \mu\text{sec}$ time markers. With each trace is a null trace, obtained in the same way as the traces carrying the signals, but with no gas admitted to the system

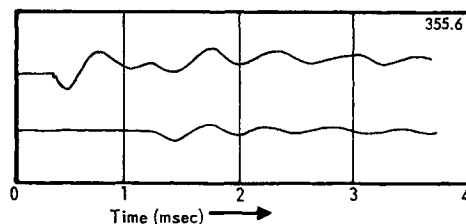


Figure 6. Tracing of oscilloscope record of microphone signal. Upper trace shows early ringing signal due to plasma impact on diaphragm. Lower trace shows neutral gas signal alone

of the system on a probe extending well into the solenoid which provides the axial magnetic field in the drift space. A grounded aluminum foil, stretched in front of a high impedance pick-up electrode charged to 300 v, is exposed to the plasma and the electrode voltage is registered on an oscilloscope by means of a cathode follower. Since the ringing period of the microphone foil is long compared to the blow given it by the plasma, its behavior is really that of a ballistic pendulum, the initial ringing velocity being proportional to the impulse. However, its period is short compared to the time for un-ionized gas to reach it from the quick acting valve, and so it can separate the two impulses.

With the capacitor microphone (Fig. 6) it is observed that the neutral gas admitted by the valve arrives more or less with a sharp front at the other end of the tube, and that it moves at a velocity of approximately 1250 m/sec. The plasma is observable as a considerably larger impulse arriving long before the gas. The capacitor microphone is somewhat difficult to calibrate accurately because of the complicated response of its diaphragm to the plasma.

A ballistic pendulum method has been developed to measure momentum transfer from the plasma. The pendulum is suspended in vacuum with a bifilar support beyond the end of the B_z solenoid, but with a disc on an extension arm inside the solenoid to receive the impulse from the plasma. The pendulum has approximately a 1.6 second period and a mass of 17.5 g. The disc which receives the impulse is 4 cm^2 in area and is non-conducting so as not to be affected by the electromagnetic forces of pulsed magnetic fields.

Impulses of up to 80 g cm/sec have been delivered to each unit area of the pendulum by a shot. The valve alone, with no electrical apparatus triggered, delivers about 16 g cm/sec per unit area of the pendulum disc. This is very nearly equal to the momentum which would be carried if all of the 2.25 cm^3 atmos of gas admitted in one shot were to move at the 1250 m/sec observed gas front velocity. Actually, of course, all of the gas will not move at this speed but, on the other hand, the molecules will recoil from the disc and deliver probably twice their axial component of momentum to it.

Subtracting the effect of the neutral gas we are left with an impulse of 64 g cm/sec delivered by the plasma to each unit area of the disc. At the observed

speed of 5×10^6 cm/sec (which is also the speed of the magnetic piston) this would be the momentum carried by about $13 \mu\text{g}$ of plasma or a little more than $3 \mu\text{g}/\text{cm}^2$. If the plasma jet were of the same cross-sectional area as the tube in which it moves, it would have a total mass of about $50 \mu\text{g}$ and would correspond to somewhat more than 0.25 cm^3 of deuterium gas at normal temperature and pressure.

It has been found necessary to use rather high voltage on the preheating shock driving coil. Initial attempts with a low voltage system, like that used in the shock wave blow-off experiments described above, were not very successful in driving large amounts of plasma. A considerable amount of momentum is imparted to the plasma by the high-voltage shock alone and, if the present state of the apparatus were the final one, there might be no point in the use of the magnetic piston. This report is written, however, at an intermediate stage in the development of the technique and it is hoped that it may be possible, in the future, to accelerate plasma jets in this way to considerably higher velocities.

Preliminary observations have been made of a phenomenon which may be quite interesting. A

search coil was inserted into the end of the system on a movable probe so as to generate an emf proportional to the time derivative of the z component of the magnetic field along the axis of the system. With the output from the search coil integrated electrically so as to display B_z on an oscilloscope, very complicated but highly reproducible patterns were obtained. The effect was strongest just outside the end of the B_z solenoid, but was still observable approximately 20 cm farther out. It appears to be due to some sort of magnetohydrodynamic oscillation induced when the plasma jet crosses the diverging lines of magnetic force at the end of the solenoid.

ACKNOWLEDGEMENTS

The writer is most grateful to William Basmann who has co-operated closely in this work and who is responsible for the design and construction of a large fraction of the apparatus. Thanks are also due to other members of the laboratory for many helpful discussions, in particular Keith Boyer, W. C. Elmore and James Tuck.

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High-current Pulse Discharges

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Pulsed gas discharges may be called "fast" if the inertial forces arising during the accelerated motion of the gas are important. In these discharges the time for the current to rise to the first peak or break in the curve $i(t)$ is comparable with the time of gas contraction towards the axis of the discharge chamber. Several papers dealing with the discharges of this kind have been published.¹⁻⁹ This paper presents results of the study of the phenomena that occur when such a powerful pulse discharge passes through gas. The experiments were conducted under the following conditions: capacitor voltage was 20 to 120 kv; maximum current was 200 to 1600 ka; rate of current rise at the beginning of the process was from 10^{11} to 10^{12} amp/sec; energy stored in the capacitor bank was up to 5×10^5 joules. Mainly discharges in deuterium were studied. Some experiments were performed with discharges in other gases.

At large energy densities, where more than 10 joules are given off per square centimeter of chamber surface during the process, the subsequent development of the process may be effected by the gas given off by the ceramic walls even during the first stages of the powerful pulse discharge. It was found in the experiments that metal-walled chambers give off less gas, but the character of the pulse gas discharge in them differs greatly from that in chambers with non-conducting walls. The development of magnetic and electric phenomena inside the column of the gas discharge will be described for chambers having both non-conducting and conducting walls. Detailed studies of neutron radiation and high-energy electrons in chambers with conducting walls are presented in the last two parts of this paper, by separate authors given there.

DISCHARGES IN CHAMBERS WITH NON-CONDUCTING WALLS

Current and Voltage Oscillograms

The general character of the gas discharge can be derived from the current and voltage oscillograms. If the experiment is carried out under sufficiently pure

conditions, contamination of the gas by particles from the chamber walls being prevented, then two or more transients are found in the voltage oscillogram at certain values of the initial pressure and voltage, while breaks are seen in the current oscillogram at the corresponding moments.¹ These oscillograms indicate that the inductance of the gaseous gap changes periodically; that is, the plasma column contracts periodically. The inductance of the gaseous discharge gap can be determined from the current and voltage oscillograms, and from this inductance the effective column radius can be found, as a function of time and the rate of contraction. The data obtained in this way agree with that obtained by other methods and will be given below

More detailed information concerning the development of the discharge can be obtained by measuring the magnetic field inside the discharge by means of minute coil-type magnetic probes. If the magnetic field distribution is found then the current density distribution along the section of the chamber, the magnetic field energy, and the inductance of the gaseous gap can be determined as a function of time. In our experiments the magnetic field was measured simultaneously at four points inside the discharge. Current, voltage and the neutron pulse (for discharges in deuterium) were also registered. Symmetry of the gaseous column was checked beforehand, and all calculations were fulfilled up to the moment when the symmetry still existed.

Figure 1 shows curves for the inductance of the gaseous gap during a typical discharge. They were calculated in two ways: from the current and voltage traces $I(t)$, $V(t)$ (neglecting resistance)

$$L^*(t) = \int_0^t V(\tau) d\tau / I(t); \quad (1)$$

and from the values of the magnetic field inside the discharge

$$L(t) = l \times 10^{-8} \int_0^R H_\phi(r, t) dr / I(t). \quad (2)$$

Thus, the assumption that the resistance may be ignored in our process is verified because these two curves coincide well enough.

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* Academy of Sciences of the USSR.

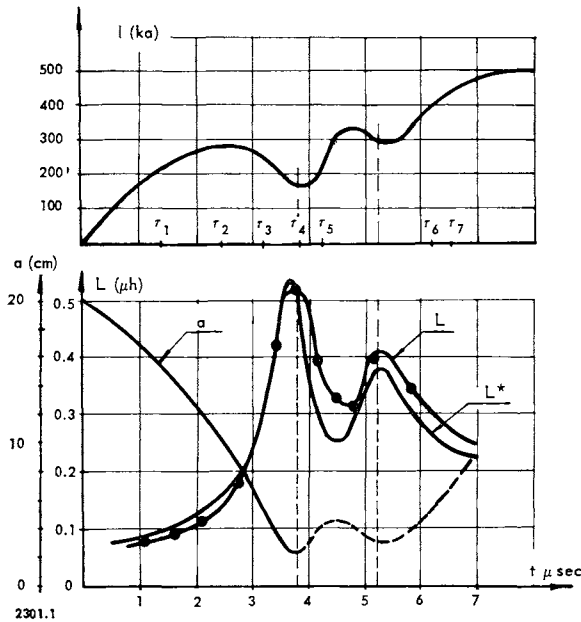


Figure 1. Inductance of the gas discharge (calculated in two ways) and the effective radius a of the current column as a function of time a discharge in deuterium at $p_0 = 0.05$ mm Hg and $V_0 = 40$ kv. The discharge chamber is a cylinder 90 cm high and 40 cm in diameter. The capacitance of the bank is $80 \mu\text{f}$. The current curve is shown on top

The effective plasma column radius variation is given in this figure as well. This plot was calculated from the values of inductance. Figure 2 gives the results of calculation of the energy introduced into the discharge and the magnetic field energy. The current density distribution over the cross-section of the discharge column at different moments at the same discharge conditions is shown in Fig. 3.

Magnetic field measurements show that after the plasma column has separated from the wall (this occurs some time after the process has begun), a considerable part of the pinched current is concentrated within a cylindrical sheath 3 to 5 cm thick. If we assume that the thickness of this sheath is determined by skin-effect, the gas conductivity at this stage of the discharge must be taken equal to 10^{13} – 10^{14} cgse units. This conductivity corresponds to an electron temperature of not greater than 10 ev. The rate of contraction, computed from the moments when the magnetic field begins to be registered by the probes located at various distances from the axis of the chamber, rises from 3.5×10^6 cm/sec at the beginning of the process to 7 – 8×10^6 cm/sec at the end of the first contraction. The same rates can be obtained from the time variation of the inductance. These figures are for a discharge in deuterium at $p_0 = 0.05$ mm Hg and $V_0 = 40$ kv. At other initial pressures $p_0 = 0.01$ mm Hg and $p_0 = 3$ mm Hg, and the same initial voltage $V_0 = 40$ kv, the maximum rate of contraction is equal to 10^7 and 2×10^6 cm/sec, respectively. (At initial pressures above 0.3–0.4 mm Hg the current reaches the axis of the chamber during the second quadrant of the period, while at 10 mm Hg

no magnetic field in the central zone is registered at all. The initial voltages here are equal to 40 kv.)

By the time of the first voltage transient the current reaches the center of the discharge chamber. Then, the current begins to rise rather sharply near the walls of the chamber while it continues to increase in the central zone. This redistribution of current results in a lower inductance of the gap and in a corresponding drop in voltage and rise in total current.

A similar redistribution of current occurs at the time of the second voltage break. During the time interval between the first and second voltage transients, the current in the central zone rises somewhat at first, then falls, and rises again. Some time after the second or third voltage transients the current in the central zone vanishes and does not reappear there until the very end of the process. By the time of the second voltage transient, the instabilities characteristic of this type of gas discharge become essential.

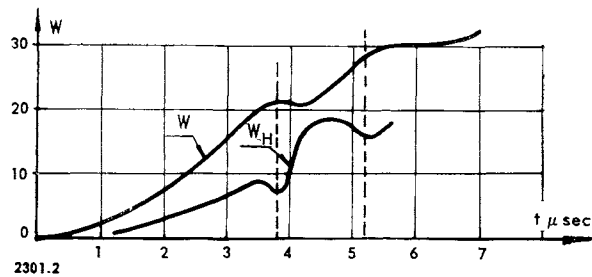


Figure 2. The total discharge energy W and the magnetic field energy W_H in kilojoules as a function of time for the conditions given in Fig. 1

The part of the total energy that increases the energy of the particles can be found from the difference between the curves of Fig. 2 at $p_0 = 0.05$ mm Hg and $V_0 = 40$ kv. Consideration of the possible energy losses to the walls during this stage of the process shows that there is no evidence that they are large, although there is no direct experimental proof of this point.

Two cases must be considered in estimating the possible temperature, $T_e \ll T_i$ and $T_e \approx T_i$. If equilibrium between the electrons and the ion gas is

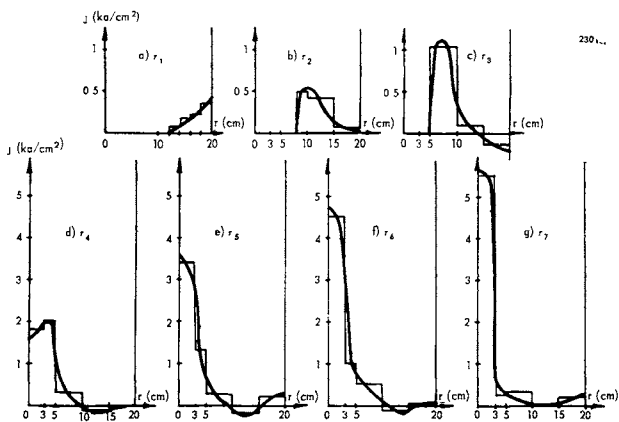


Figure 3. Current density distribution along a cross section of the discharge for the times indicated in the current curve of Fig. 1

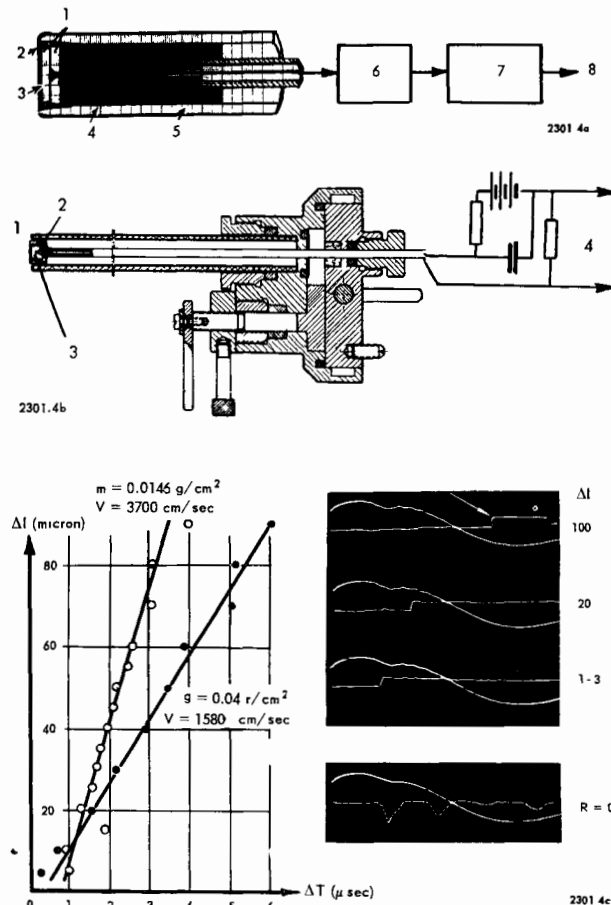


Figure 4. Design and calibration of the piezoelectric pickup. (a) Diagram of piezoelectric pickup: 1, 0.5 mm of piezoceramic material; 2, insulation; 3, silver foil; 4, zinc rod; 5, porcelain tube 8×12 mm; 6, cathode follower; 7, amplifier; 8, to oscillograph. (b) Diaphragm pickup: 1, diaphragm; 2, contact; 2, 3, gap; 4, to oscillograph. (c) Left: time-position curves for two diaphragms of different mass; right: three traces from diaphragm pickup and pulse shape from piezoelectric pickup. $U_0 = 20 \text{ kv}$, $p_0 = 0.1 \text{ mm Hg}$ deuterium

not as yet established, practically all of the energy stored in the gas is that of heavy particles. In this case the average energy of the deuterons is 200 ev at the instant of maximum contraction, assuming that all of the particles were involved in the pinch process. If equilibrium between the electrons and the ion gas has already been established by the time of the maximum contraction, the average energy per particle amounts to 100 ev, which corresponds to a temperature of 70 ev. Non-uniform contraction of the plasma column and cumulative effects may lead to local heating. Therefore, the above estimates are not in contradiction with the temperature of one million degrees that was obtained from the spectroscopic measurements.¹²

Pressure Measurements

The compression of the particles at the central axis and their temperature rise should be accompanied by a sharp increase in the pressure at the central zone. This pressure rise was studied by means of piezo-

ceramic pickups set at various points inside the discharge. The design of the pickup is schematically shown in Fig. 4a. A diaphragm detector shown in Fig. 4b was designed to determine the sensitivity of the pickup. It enables the pickup to be calibrated under conditions similar to those in the gas discharge. Time-position curves for two diaphragms of different mass are plotted in Fig. 4c; these were drawn up after the treatment of the experimental data. The scale of the pressures recorded by the pickup can be obtained by comparing the area under a pulse on the trace of the pickup signal with that of the pulse received with the diaphragm.

Oscillograms of pressure pulses, which are synchronized with the full current flowing through the chamber, are shown in Fig. 5 for different positions of the pickup. With the pickups set radially the pressure rise from the outside of the cylinder to the center can be seen as the plasma column contracts towards the axis of the discharge, and then the pressure wave spreads in the opposite direction. Pickups set axially record only a small pressure rise in the outer regions of the discharge chamber during the contraction. A sharp pressure rise was recorded near the axis during the last phase of contraction, the pulses received by the axial and radial pickups being about the same. As the discharge column expands, the axial pickups near the perimeter also record a marked pressure rise. These experiments show that during the first stage of contraction the radial velocities of the particles directed inwards to the chamber axis dominate. At the instant of maximum contraction and when the plasma column expands, the other velocity components are essential as well.

When a magnetic probe, which measured the time derivative of the magnetic field at a fixed point, was set near the piezoceramic pickup (Fig. 5), it was found that the pressure pulse appeared somewhat later than the time derivative of the magnetic field during the contraction of the discharge column.

The concentration of particles at the pickup can be calculated from the measured values of the pressure pulse, the particle velocity and the time during which the pressure acted on the pickup. For the experimental conditions listed in Fig. 5 the calculated concentration of particles in the region near the chamber axis is about 100 times greater than the initial one. Knowing the pressure and particle density in the compressed state, it is possible to estimate the temperature by means of the formula $p = 2nkT$. In the experiment we now consider the initial density of atoms was equal to 7×10^{15} particles/cm³; the final density was found to be about 7×10^{17} particles/cm³. At this moment, the pickup registered a pressure of about 100 atmospheres. The temperature obtained from these values of pressure and particle density is a very rough estimate, $T = 50 - 100 \text{ ev}$.

Heavy Gases

Let us briefly consider some characteristic features of the development of the discharge in heavy gases such as xenon, for example. By comparing results of

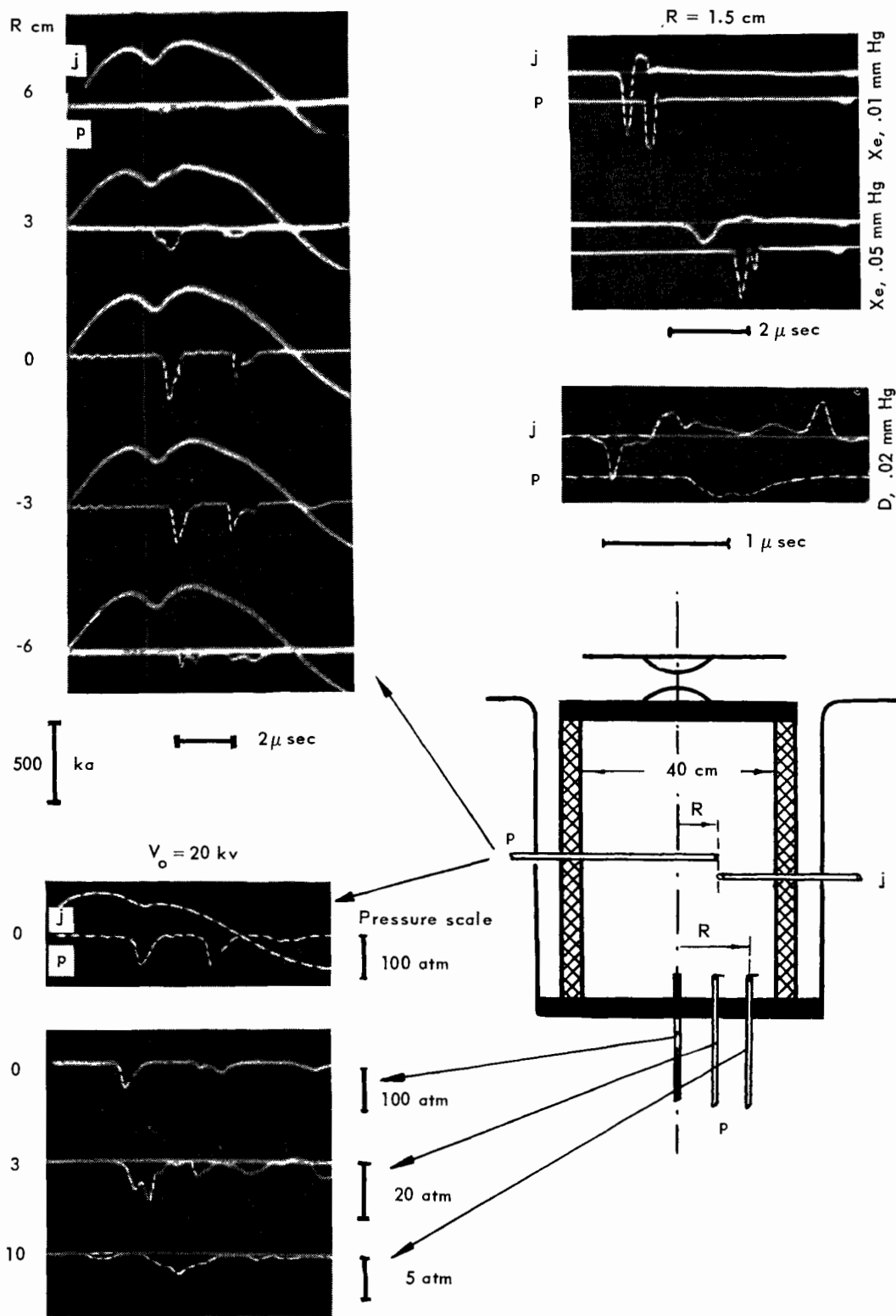


Figure 5. Diagram and results of measurements with a piezoceramic pickup

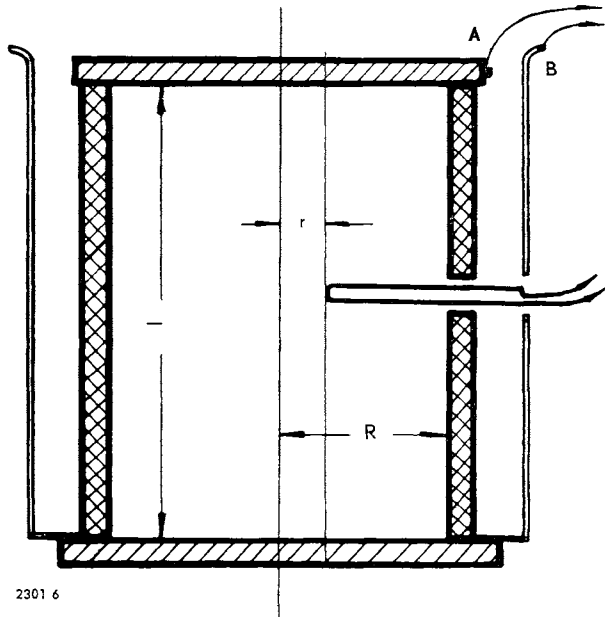


Figure 6. Discharge chamber with a probe for measuring the longitudinal electric field. A voltage divider is connected to the points A and B

measurements at pressures for which the masses of xenon and deuterium within the chamber are the same we find the general character of the current density distribution along the discharge cross section to be similar. The rate of contraction for xenon, however, is somewhat larger than for deuterium. Moreover, for discharges in deuterium the maximum magnetic field in the central zone is 3 to 4 times larger than that near the walls, while for discharges in xenon the magnetic field amplitude is almost the same along the entire discharge cross-section. Therefore the current concentration at the chamber axis is much less for discharges in xenon than in deuterium.

It is interesting to note, for an initial pressure of the xenon of 0.1 mm Hg, that during the second half-period the current sheath formed at the walls is also seen to move towards the center just as in the first half-period, provided cylindrical symmetry is not disturbed throughout the entire process.

All oscillograms for discharges in xenon-deuterium mixtures containing 1% of xenon atoms are similar to oscillograms for pure xenon. This character of the discharge in pure xenon and in mixtures agrees with data on the X radiation which accompanies the discharges in these gases.¹⁰

Longitudinal Electric Field

In addition to the magnetic field inside the discharge, the longitudinal electric field E_z was measured by means of magnetic probes of a somewhat different construction. The idea of this method of measurement is as follows. Let us consider a contour ABCDEA in Fig. 6. We can write

$$\oint E_z dl = - \frac{1}{c} \int \int \dot{H}_\phi dS. \quad (3)$$

Neglecting the voltage drop across the parts of the contour running along metal conductors, and assuming that the column has cylindrical symmetry, and that E_z and E_r do not depend on z , we obtain

$$E_z = \frac{V}{l} - \frac{1}{c} \frac{\partial}{\partial t} \int_r^R H_\phi dr \quad (4)$$

where V is the voltage applied to the divider.

The time derivative of the magnetic flux in the right-hand side of Eq. (4) was measured by an elongated loop set along the radius (Fig. 6). The longitudinal electric field was determined simultaneously at four points at different distances from the chamber axis. The results for the most typical discharge are given in Fig. 7. The electric field in the central region of the discharge is equal to zero during the first stages of the process, then it rises and attains a value of 500 to 700 v/cm by the time the voltage transient occurs. At the time of the voltage transient, the electric field becomes negative over the entire cross-section of the discharge chamber, and its value in the central region reaches 1000 v/cm. By the

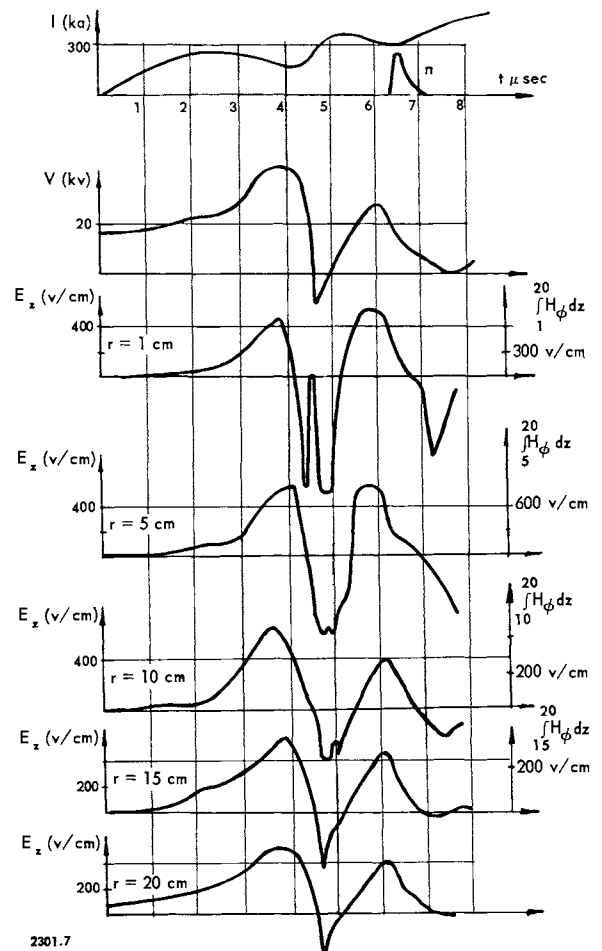


Figure 7. Longitudinal electric field at different distances from the axis of the discharge chamber. The operating conditions are given in Fig. 1

time the neutron pulse appears, the electric field recorded lies between 100 and 600 v/cm. Our method does not permit the measurement of local electric fields, which may attain large values due to the development of various forms of instability.

The ohmic voltage drop across the discharge column may be calculated from the longitudinal electric field. Calculations show it is valid to neglect the ohmic voltage drop as we have done.

The longitudinal electric field is associated with the current density and the magnetic field intensity by

$$j_z = \sigma \left(E_z + \frac{V_r H_\phi}{c} \right). \quad (5)$$

If the ohmic voltage drop can be neglected,

$$V_r = - \frac{c E_z}{H_\phi}. \quad (6)$$

The radial velocity of the particles during the contraction calculated in this way agrees well with the velocity of the internal current boundary given above.

At the moment of the voltage transients E_z is negative, while H_ϕ and j_z are positive. Therefore, V_r is positive, that is, the particles fly out from the central zone. The corresponding radial velocity is greater than $|cE_z/H_\phi|$. During a discharge at $p_0 = 0.05$ mm Hg and $V_0 = 40$ kv, the radial velocity of the particles is greater or equal to 1.5×10^7 cm/sec in the central zone and drops off in going away from the center of the discharge chamber.

Increased Rate of Current Rise

Let us pass to the question of how the discharge process changes when the initial rate of current rise is increased. For this purpose experiments were carried out on an apparatus having a chamber of 40 cm in diameter and 50 cm in height. The 64 μ f capacitor bank was charged to a voltage from 40 kv up to 120 kv. The initial rate of current rise varied from 3×10^{11} amp/sec to 10^{12} amp/sec, the discharge energy being much larger than that in the apparatus described before.

The current and voltage oscillograms with $V_0 = 40$ kv do not differ from those described above. At voltages above 80 kv, only one peak remains on the voltage oscillogram. A sample oscillogram is shown in Fig. 8. The same series of measurements were made as described above for voltages across the chamber to 100–120 kv. Only a few measurements were made with magnetic probes at maximum voltages due to the experimental difficulties. Some characteristic features for discharges at voltages of 80 to 120 kv are listed below:

1. The current at the walls of the discharge chamber comprised a considerable part of the full current (about 50%) during all stages of the discharge process. This greatly influences the discharge inductance. Therefore, the rate of contraction found from the inductance curves are 1.5–2 times less than that found by means of magnetic probes.

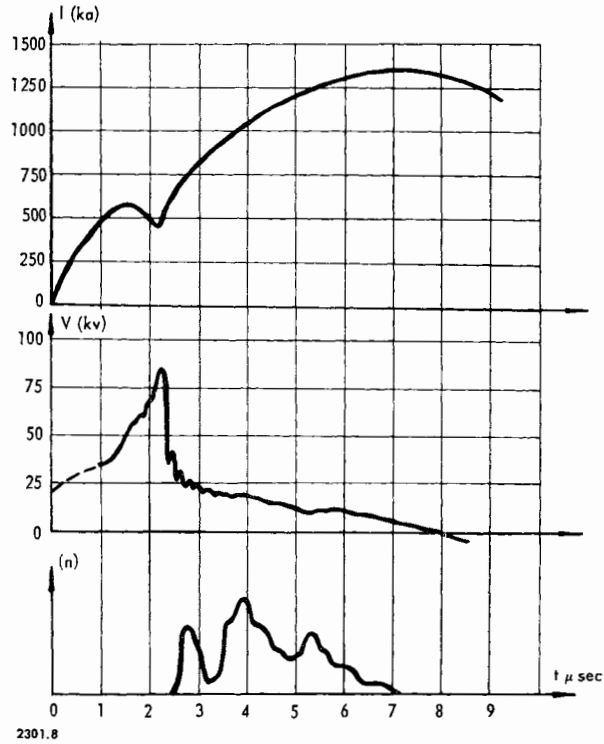


Figure 8. Oscillograms of current, voltage and the neutron pulse recorded with a scintillation counter, for a deuterium discharge at $p_0 = 0.05$ mm Hg and $V_0 = 100$ kv. The discharge chamber is a cylinder 50 cm high and 40 cm in diameter. The capacitor bank is 64 μ f

2. Since the current at the walls is large, it is difficult to prevent contamination of the chamber with the wall material.

3. The magnetic field in the central zone continues to rise at the time of the voltage transients, and exists for several microseconds, attaining 14 kilo-oersteds at 5 cm from the discharge chamber axis ($V_0 = 100$ kv).

4. The energy of the particles at the highest voltages (120 kv) cannot be estimated by the method described above since the magnetic field could not be measured because of the experimental difficulties. Taking the total energy introduced into the discharge up to the time of the first transient and subtracting the 35% from it that is allotted for the magnetic field according to theoretical calculations, we obtain 500 ev per particle (including electrons).

Hard Radiation

We shall now describe the hard radiation that accompanies a powerful pulse discharge in deuterium.³

At voltages not larger than 60 kv neutron radiation appears at the instant of the voltage transients (this is usually after the second peak, when instabilities develop). Simultaneously a hard X-ray pulse appears (of the same duration as the neutron pulse); the discussion of this pulse will be given later on. Neutron radiation at voltages above 80 kv is recorded as long as the compressed state exists, that is, from 2 to 3 microseconds. A typical oscillogram of this

neutron pulse, which was obtained by means of a scintillation counter, is given in Fig. 8. Discharges in hydrogen with the same initial mass of the gas were accompanied by hard X radiation; however, this radiation lasted for not more than 0.3 microseconds.

The appearance of an X-ray pulse in the region of the first current break indicates that there is an acceleration process which can result in the neutron radiation. However, since the X-ray pulse is much shorter in time than the neutron pulse, it is suggested that some neutrons may emerge due to a non-accelerated process as well.

It is not difficult to estimate the possible contribution of the thermonuclear reactions to the neutron effect observed by making use of the experimental data available and assuming a 50-fold compression of the plasma column (this follows directly from optical data¹² and measurements of the compression density by the piezo-ceramic pickup). This estimate gives a neutron yield of 5×10^5 neutrons (at a temperature of 330 eV). This calculation, however, is very sensitive to the number of particles over which the energy is distributed. If we assume that the energy is not imparted to all of the particles in the discharge chamber, it must be assumed that only 40% of the total number of particles should be heated up in order to explain the entire neutron yield (5×10^7) by thermonuclear processes. These considerations indicate that a part of the neutron yield observed may be due to thermonuclear reactions.

For an initial voltage of 60 kv the temperature in the compressed state at the time of the first maximum contraction does not exceed 100 eV. Neutron radiation is noticed at the time of the second contraction and may be explained by the presence of deuterons accelerated to high energies in the discharge. These deuterons (having energies up to 200 keV) were found in direct experiments (see below), and also in recording the neutrons using the recoil protons obtained in nuclear emulsions (the method described in Ref. 6).

STUDY OF FAST CHARGED PARTICLES

In 1953, after detecting hard X radiation⁵ accompanying powerful discharges in hydrogen and deuterium, the upper bound of the energy spectrum was estimated. It was established that there exist in the spectrum quanta having energies of several hundred keV at a discharge tube voltage of 15 to 20 kv. However, the gradual falling off of the spectrum made it difficult to establish the energy limit of the radiation under study.

Cloud Chamber Studies

A cloud chamber method of measuring the energy of quanta was employed to determine more accurately the energy of the X-ray quanta. The main advantage of using a cloud chamber in studying short bursts of X radiation is that the device has a comparatively short sensitive time and electromagnetic obstacles do not effect its operation.

In order to determine the electron energy, the lengths of the electron tracks formed in the cloud

chamber were measured. The well-known method for determining the electron energy from the track curvature in the magnetic field was not suitable for the conditions of the experiment, since at electron energies of about 200 keV the mean dispersion angle greatly exceeds the maximum value at which it would be still possible to measure the track curvature in the magnetic field.

In determining the energy of the X-ray quanta using the electron energy, it is necessary to determine whether these electrons are formed as a result of the photo-electric effect or the Compton effect. The Compton mechanism of electron formation by quanta having energies above 200 keV was found to be more probable under the conditions of the experiment. The calibration experiments in which a stream of X rays of known energy was directed into the cloud chamber from an X-ray tube proved this estimation to be correct.

All these experiments were conducted with a porcelain discharge chamber 100 cm long and 17.6 cm in diameter at an initial hydrogen pressure of 0.06 mm Hg (corresponding to maximum hard X-ray yield), a discharge voltage of 40 kv and a maximum discharge current of about 200 ka. A 36 μ f capacitor bank was used.

It is difficult to follow on the photographs the long tracks formed in the cloud chamber, which correspond to high-energy electrons, because of the large number of short tracks there. To determine the character of the spectrum in the high-energy range the Wilson chamber was shielded from the discharge tube with lead 15 mm thick (which greatly absorbed the soft part of the X-ray spectrum). Oscillograms of the scintillation counter pulses were obtained at the same time as the tracks were photographed. If there was no pulse on the oscillogram, the corresponding photograph of the tracks was not developed.

Since one camera was used to photograph the tracks, the picture on the photographs is the projection of the electron traces onto a plane perpendicular to the optical axis of the camera. When projecting a sufficiently winding track on a plane, the length of its projection is $\pi/4$ of its actual length. Winding electron tracks were studied in Ref. 11, where it was shown that the above factor could be used with high enough accuracy.

The electron energy was determined from the range-energy curve drawn up on the basis of known experimental data.

In order to take into account the background contribution, the cloud chamber was expanded after each time the traces of the electrons formed by the X-ray pulse were photographed, and then the tracks were photographed without any discharge. A histogram of the energy spectrum for the electrons formed by X-ray pulses accompanying a discharge in hydrogen is given in Fig. 9. The spectrum of the electrons formed by the background is given in the same figure. The same number of photographs were treated to construct the histograms for the effect studied and for the background. Out of a total of 500 photographs,

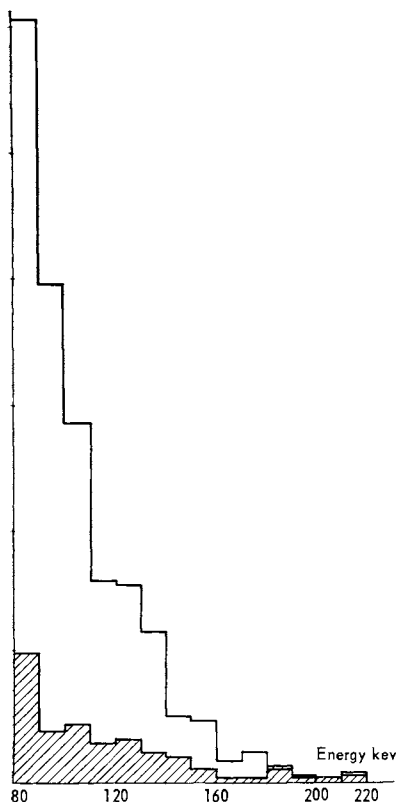


Figure 9. Energy distribution of the recoil electrons obtained in a cloud chamber when observing X radiation of the gas discharge

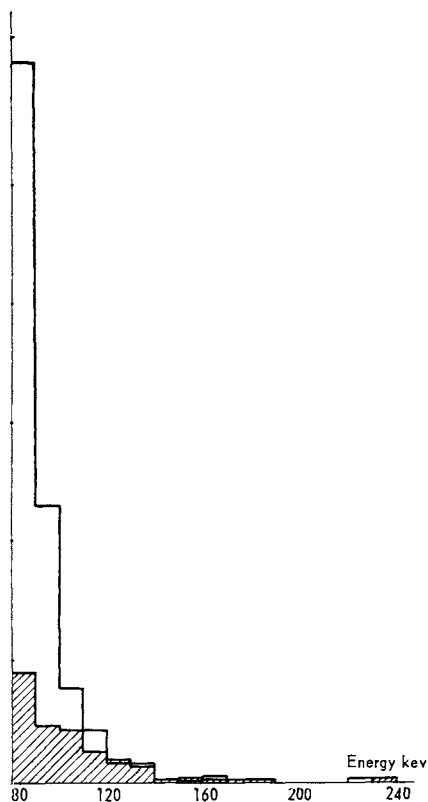


Figure 10. Energy distribution of the electrons formed by X radiation from the X-ray pulse tube, $V = 245$ kv

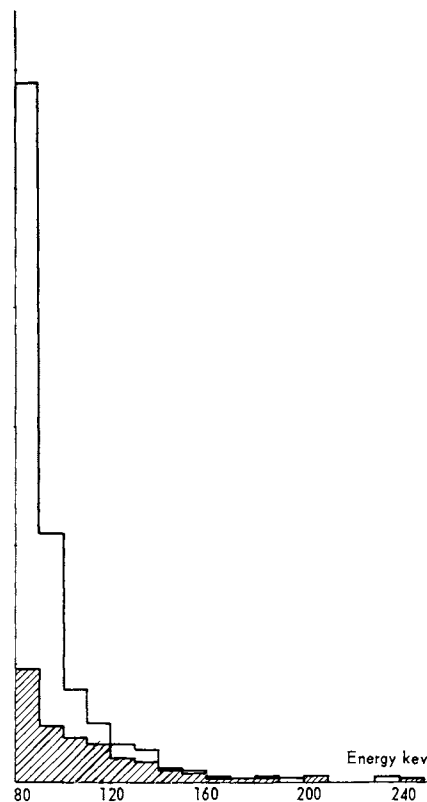


Figure 11. Energy distribution of the electrons formed by X radiation from the X-ray pulse tube, $V = 285$ kv

189 were chosen to be treated in accordance with the criterion indicated above.

Calibration Measurements

Figures 10 and 11 show the calibration histograms for electron comparison spectra. They were obtained while X irradiating the cloud chamber from an X-ray pulse generator. The voltage pulses on the X-ray tube amounted to 240 and 285 kv. As seen from the histograms, the maximum energies of the electron spectra in these cases amounted to 120 kev and 140 kev, respectively.

As follows from the elementary theory of the Compton effect, the maximum energy of the Compton electrons corresponding to X-ray quantum energies of 240 kev and 285 kev are equal to 116 kev and 145 kev, respectively. These values agree with the experimental results within measurement accuracy. The maximum energies of photoelectrons knocked out from the glass at the same X-ray quantum energies amount to 200 kev and 265 kev (the highest ionization potential of the K shell is about 20 kv).

Thus, the data presented show that under the conditions of the experiment at energies above 200 kev it is much more probable for the Compton effect to occur than for the photo-effect.

According to the histogram of Fig. 9, the energy spectrum limit for the electrons in the cloud chamber

is estimated at 180 kev for discharges in hydrogen. Assuming that the electrons in the high energy region of the spectrum are formed due to the Compton effect we can find the energy of the X-ray quanta.

Results of Cloud Chamber Studies

Calculations show that a powerful discharge in hydrogen at an initial capacitor bank voltage of 40 kv is the source of quanta of radiation having an energy of 320 kev. Analysis of this experimental data permits us to assert that the number of X-ray quanta with an energy above 320 kev does not exceed 0.1% of the total number of quanta in the discharge, on the average.

The measurement error on the calibration tube does not exceed 5%. However, it should be remembered that the value of the maximum energy of the spectrum was obtained by averaging the spectra of a large number of pulses. Therefore, it is possible that several quanta may exist in the spectrum with a higher energy.

Thompson Parabola Studies

Thompson's parabola method was used for direct measurements of the deuterium ion velocities. By emitting a narrow beam of charged particles from the discharge gap (a porcelain tube 80 cm long and

17.2 cm in diameter) into a special chamber, the ions may be analyzed according to their velocities and values of e/m . The width of the beam was set by two inlet diaphragms, $d_1 = 1$ mm and $d_2 = 0.5$ mm. In order to maintain a high vacuum in the chamber, the first diaphragm was shut with a high-speed electromagnetic valve.

Each discharge took place in pure deuterium at a pressure of 0.02 to 0.06 mm Hg, a discharge voltage of 40 kv and a discharge current peak of $1.5-2 \times 10^5$ amps.

The photograph plates were exposed for not more than several microseconds during one pulse discharge (about one half-period, which was equal to 8 microseconds). All plates were exposed when neutron radiation was present (on the average there were 10^8 neutrons per pulse).

The velocities of the deuterons emerging from the plasma (perpendicular to the discharge axis) were measured. Just noticeable traces of parabolas were obtained after the plate had been exposed for 80-100 pulses.

The stream of particles was much more intense in experiments with charged particle beams emitted along the discharge axis through an outlet in the cathode (Figs. 12, 13). Faintly noticeable parabola traces appeared in the plates after one pulse discharge.

Exposures of 3-5 pulses resulted in parabolas that were quite distinct. In the sections of the parabolas from 80 kev and higher on some photo plates there are black spots of 1 mm in diameter with clear-cut boundaries. These spots are evidence of good focusing in a very narrow velocity band for small groups of deuterons.

There is much scatter in the experiment results at the same operating discharge parameters. The maximum energies of the deuterons usually lie between 80 and 170 kev, and more rarely from 170 to 200 kev.

Upon increasing the exposure to 10-15 pulses

there appear parabolas of negative deuterium ions with energies from 4 to 120 kev. These fast negative ions are evidently formed by charge exchange with fast positive ions; that is by the appearance of fast neutral particles and subsequent electron capture. Here the direction of the fast-particle velocity remains almost the same. The maximum velocities for the negative ions of deuterium have a sharp boundary but they are less than those of the deuterons on the same plate. The direction of the negative deuterium ion velocity coincides with that of the deuterons; therefore, they move towards the cathode in the retarding electric field, losing part of their energy.

Upon increasing the exposure (to 25 and more pulses), in addition to D^+ and D^- parabolas, parabolas corresponding to the ions of D_2^+ , D_2^- and electrode and tube materials appeared on the photo plates. Parabolas of H^+ and H^- were obtained in experiments with hydrogen. From 5 to 10% of helium was added to the deuterium to obtain a reference parabola.

By charging the capacitors at reversed polarity, it is possible to analyze the charged particle beam emitted through the outlet in the anode. No signs of parabolas were discovered on photographic plates exposed for 50 pulses. Therefore, fast deuterons arise only during the first half-period of the current discharge. Furthermore, the electric fields in which the deuterons are accelerated to high energies are of the same polarity as that from the applied discharge voltage. The stream of fast deuterons is directed along the axis of the discharge tube towards the cathode with slight dispersion toward the walls.

Thus, applying the parabola method for measuring ion velocities in a powerful pulse discharge in deuterium, it was found that deuterons exist having energies right up to 200 kev. The existence of deuterons with energies five times greater than the discharge voltage indicates that acceleration processes develop in the discharge.

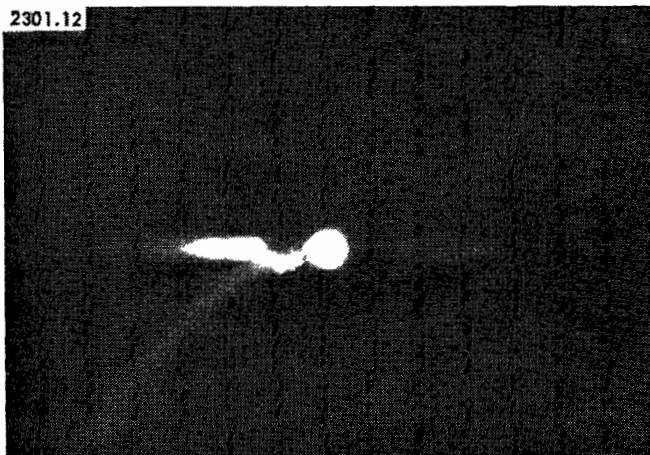


Figure 12. Ion energy analysis by Thomson parabola method. $B = 900$ gauss; $E = 950$ v/cm; $V_p = 40$ kv; $P_{D_2} = 0.04$ mm Hg. 6 pulse discharges; D^+ parabola to the left and downwards; D^- parabola to the right and upwards. Horizontal line: $E = 0$



Figure 13. Ion energy analysis by Thomson method. $B = 1500$ g; $E = 1850$ v/cm; $V_p = 40$ kv; $P_{D_2} = 0.04$ mm Hg; photographs for two different directions of H^- . 7 pulse discharges, D^+ parabola upwards (slightly blackened), D^- parabola downwards

Summary

The basic experimental facts known to date regarding the hard radiation from a gas discharge in porcelain chambers at pulse voltage up to 50 kv show:

1. The neutron and X radiation during the discharge process appear at the same time.
2. Deuterons responsible for the appearance of neutrons during discharges in deuterium are accelerated towards the cathode, while the maximum intensity of X radiation appears in the region near the anode.⁵

3. Both types of radiation always appear in the same well-defined range of gas pressure in the discharge tube.

4. The energy of the X-ray quanta (and, therefore, of the electrons which create them) was experimentally determined to be 300 kev, while the deuteron energy determined was 200 kev. These values coincide well with each other within the limits of experimental error.

Experimental facts prove that hard X radiation from powerful pulse discharges is of the same character as the neutron radiation which arises in deuterium discharges. The radiation is most likely due to the electric fields directed along the tube axis which accelerate the charged particles (electrons and ions). As has been pointed out previously,⁵ these electric fields may arise from a redistribution of currents while the radius of the discharge column is changing. An electric field of the order of a few hundred volts per centimeter arises at the time of the X-ray pulse. This corresponds to a potential difference of tens of kilovolts. Local electric fields that appear in this process may attain extremely large values if certain types of instabilities develop.

DISCHARGES IN A METAL-WALLED CHAMBER

Experiments were carried out with metal-walled chambers in the discharge circuits, which were similar to those with ceramic-walled chambers. (Some of this work is discussed in Ref. 13, vol. 4, p. 170, 182, 201.) A schematic view of the metal-walled chamber used in most of the experiments is shown in Fig. 14. Experiments showed that the discharge process in metal-walled chambers develops differently than in ceramic ones.

Evolution of the Discharge

Measurements with magnetic probes have shown that the discharge process begins with a breakdown from the electrodes to the metal wall of the chamber. Here, the discharge process, which is symmetric about the longitudinal axis of the chamber, only takes place near the positive electrode. The discharge subsequently develops as shown in Fig. 14. The dashed lines show how the current gradually spreads toward the chamber axis. The zone of maximum compression is situated near the positive electrode surface, and becomes smaller in extent along the chamber axis with larger initial gas pressures.

This type of metal chamber may be used to obtain a highly pinched discharge at relatively large initial pressures (up to 4–30 mm Hg). This phenomenon

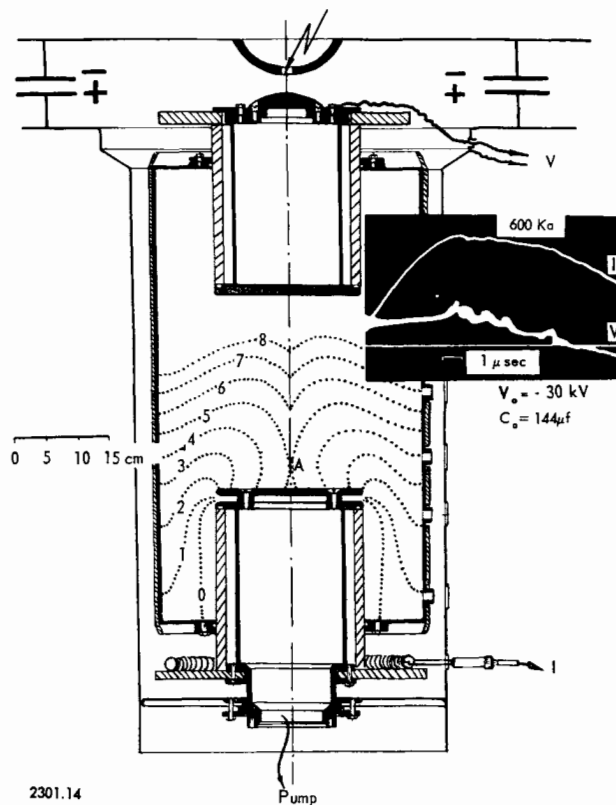


Figure 14. Metal-walled chamber. 0-8: Boundaries of current propagation within the chamber at 1 μsec intervals. A: Point of earliest compression of the current to the axis

was not observed in ceramic chambers with discharge circuits of similar parameters.

The high contraction may be obtained because only a relatively small part of the gas filling the chamber participates in the discharge. The larger the initial pressure, the smaller this amount of gas.

Since it is possible to work at high initial pressures and since the metal walls do not give off much gas, a large number of discharges can be carried out and well-reproducible results may be obtained without changing the gas in the chamber.

Neutron Production

Just as in the case of ceramic chambers, the oscillograms for the full discharge current through the chamber and the voltage across its electrodes indicate that the inductance of the discharge gap increases when the discharge column pinched (see Fig. 14). At the time of the maximum gas contraction to the chamber axis, intense neutron radiation is observed over a wide range of pressures. Localization of neutron radiation was studied in two ways by means of collimator devices.

(a) A scintillation detector with a naphthalene-anthracene crystal was placed in a paraffin block with a collimator slit (Fig. 15). By moving this device longitudinally and transversely with respect to the chamber axis, the sources of neutron radiation were discovered. They were located in the region of

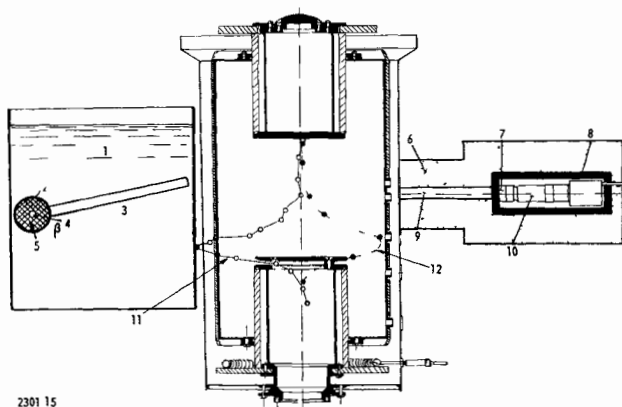


Figure 15. Experiments on neutron collimation. Apparatus on the left: 1, water; 2, cadmium; 3, slit; 4, β counter; 5, paraffin. Apparatus on the right: 6, paraffin; 7, scintillator; 8, lead; 9, slit; 10, photomultiplier. Experimental curves: 11, ratio of readings of collimated and non-collimated counters minus the background at corresponding positions of the slit; 12, relative difference in amplitudes of the scintillator pulses with the slit open and closed at corresponding positions of the collimator

maximum discharge contraction, which was determined by means of magnetic probes (point A on Fig. 14).

(b) Considerable intensity of neutron radiation (up to 3×10^9 neutrons per discharge) made it possible to carry out quantitative measurements by means of another collimator device, which is shown in the left side of Fig. 15. A paraffin block in a cadmium jacket was placed in a tank of water. This block slowed the neutrons that fell on it through a flat slit formed in the water shield by a hermetically sealed tin box. The slowed-down neutrons activated the silver foil (Ag^{109}) which enclosed a β counter, whose pulses were counted by an ordinary scale device. The collimator slit was moved inside the tank and pointed at the selected region in the chamber. The readings of the collimator counter were compared with those of an unprotected monitor placed near the discharge chamber. The data of a large number of experiments for determining the longitudinal and transverse distribution of neutron radiation were treated and the results are given in the drawing of the chamber (Fig. 15).

Experiments with nuclear emulsions, prior to those of a subsequent part of this paper, also showed that the source of neutron radiation is located close to the positive electrode near the chamber axis. After treating a large number of proton tracks in the emulsions (which were placed at the cathode and anode of the chamber), it may be seen that less anisotropy is observed here than in the similar case with ceramic chambers at voltages not exceeding 40 kv. The energy spectrum obtained for the neutrons is given in Fig. 16. The location of the cassettes with the nuclear emulsions is shown in this figure as well.

Even when adding large amounts of xenon to deuterium (500–600% by mass), the intensity of neutron radiation, in contradiction to the case with ceramic chambers, was reduced only by several times,

the time of current maximum contraction to the chamber axis being a little more comparing with a discharge in pure deuterium. However, the characteristic pinch of the discharge was not observed when the chamber filled with the same amount of pure xenon.

THE THEORY OF PLASMA COLUMN CONTRACTION AND OSCILLATION AT HIGH RATES OF CURRENT RISE

Due to magnetic forces the plasma column formed by passing a powerful current pulse through a cylindrical discharge chamber is pinched with a rate which is determined by the compressing forces and the plasma inertia.

The qualitative theory for the contraction process was developed by Leontovich and Osovets.² Later on Braginskii and Migdal¹³ qualitatively considered the problems connected with increasing plasma conductivity, ionization, the neutral gas sweep by the charged particles, the shock waves, cumulation effects and the growing mass of the moving gas with time. It is extremely difficult to calculate the contraction and pulsation of a plasma column in detail taking into account all of these factors. Only a magnetohydrodynamic approximation was used, in which the plasma was considered to be a monatomic ideal gas having a constant conductivity σ . This model does not account for ionization and the energy loss through radiation, as well as unfully sweeping of the neutral particles by charged ones which are directly effected by electromagnetic forces. The free path of the particles is considered to be small. Let the pressure be $P_0 = 0.2$ mm Hg for example. For a cross section of 3×10^{-16} cm² the mean-free path is equal to 0.2 cm. Assuming a compression ratio of

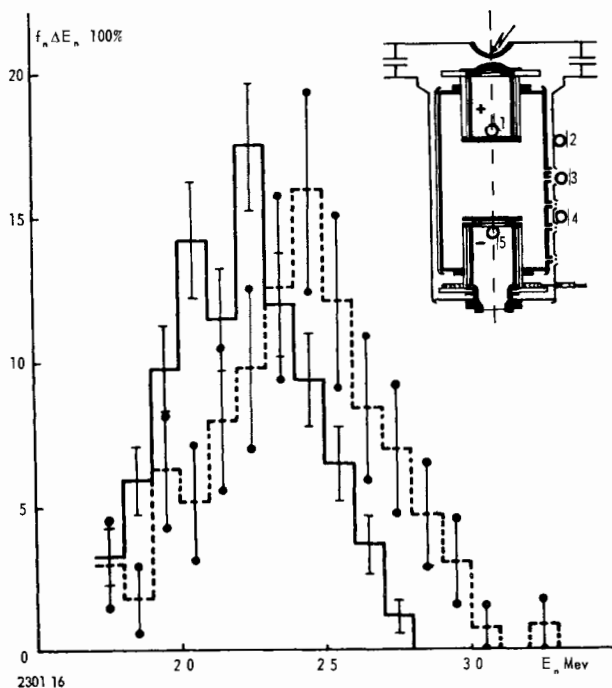


Figure 16. The neutron energy spectrum $p_0 = 2$ mm Hg; $V_0 = 30$ kv; solid curve for anode (1); broken curve for cathode (5)

30, full ionization, and ion temperature of 100 ev the mean-free path of the ion amounts to approximately 0.1 cm., and the time of heating the electrons by the ions to approximately 10^{-7} sec. It is believed that the adopted model correctly accounts for the main mechanical and electrodynamical effects, e.g., inertia and gas motion, heating in the shock wave and skin effect.

The constant value for the electrical conductivity may be justified in part by the fact that the electron temperature is almost constant (several ev) as long as the ionization is incomplete. Even this schematic formulation of the problem leads to a complex set of non-linear partial differential equations. This set of equations was solved numerically on an electronic computer under the supervision of I. M. Gelfand and R. P. Fedorenko.

The equations for the region inside the plasma in dimensionless variables are as follows:

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \frac{\partial v}{\partial r} = 0, \quad (8a)$$

$$\rho \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} \right) = - \frac{\partial p}{\partial r} - H \frac{\partial r H}{r \partial r}, \quad (8b)$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} + \frac{\rho v^2}{2} \right) + \frac{1}{r} \frac{\partial}{\partial r} \times \left[r v \left(\frac{\gamma}{\gamma - 1} p + \frac{\rho v^2}{2} \right) \right] = \frac{\partial r H}{r \partial r}, \quad (8c)$$

$$\frac{\partial H}{\partial t} = \frac{\partial E}{\partial r}, \quad (8d)$$

$$\frac{\partial r H}{r \partial r} = K_\sigma (E + v H), \quad (8e)$$

$$T = \frac{p}{\rho}, \quad (8f)$$

where ρ = density, v = velocity, p = pressure, T = temperature, E = electric field along the axis, H = magnetic field. The density of the undisturbed gas is taken as unit of density and the radius of the discharge chamber a_0 is taken as unit of length. The units of measurement of the other quantities are as follows (t_1 is the unit of time):

$$v_1 = \frac{a_0}{t_1}; \quad p_1 = \rho_0 v_1^2; \quad E_1 = \frac{v_1}{c} H_1; \\ H_1 = \sqrt{4\pi \rho_0} v_1; \quad T_1 = M v_1^2, \quad (9)$$

where M is the average particle mass. The following dimensionless quantity is also introduced:

$$K_\sigma = \frac{4\pi a_0^2 \sigma}{c^2 t_1}. \quad (10)$$

The initial and boundary conditions are as follows:

$$\rho = 1, v = 0, p = 0, H = 0 \quad \text{at } t = 0, \quad (11a)$$

$$v = 0, \quad H = 0 \quad \text{at } r = 0, \quad (11b)$$

$$p = 0 \quad \text{at } r = a(t), \quad (11c)$$

where $a(t)$ is the coordinate of the edge of the column determined in the process of integration. Outside the column $\sigma = 0$; therefore, instead of Eq. (8f) we may write the following for the magnetic field:

$$H(t, r) = \frac{1}{r} H(t, 1) \quad a < r < 1. \quad (8g)$$

The current is excited by a capacitor C which is charged to an initial voltage of V_0 , and is discharged through a gap of length l in series with some external inductance L_0 .

The corresponding electrical equations may be used as the boundary conditions for Eqs. (8). In dimensionless form they are

$$K_C \frac{dV}{dt} = -H(t, 1); \quad E(t, 1) + K_L \frac{dH(t, 1)}{dt} = V \quad (12)$$

where the following quantities are chosen as units:

$$V_1 = V_0; \quad E_1 = V_0/l; \quad H_1 = (ct_1/a_0) E_1 \\ = (ct_1/a_0) (V_0/l), \quad (13)$$

$$t_1 = \left(\frac{\sqrt{4\pi \rho_0} a_0^2 l}{c V_0} \right)^{\frac{1}{2}} = 3.48 \times 10^{-6} p^{\frac{1}{2}}_{[\text{mm Hg}]} \\ \times a_0 \left(\frac{l}{V_0 [\text{volts}]} \right)^{\frac{1}{2}} \quad (14)$$

and the following designations are adopted:

$$K_C = \frac{2lc}{c^2 t_1^2}; \quad K_L = \frac{L_0}{2l}. \quad (15)$$

When $t = 0$ we have

$$V = 1. \quad (11d)$$

Using Eqs. (8) and (12) we derive the law of conservation of energy in dimensionless form:

$$\int_0^a \left(\frac{P}{\gamma - 1} + \frac{\rho v^2}{2} + \frac{H^2}{2} \right) 2r dr \\ + \left(\ln \frac{1}{a} + K_L \right) H^2(t_1, 1) + K_C V^2 = \text{const.} \quad (16)$$

The following values of the dimensionless variables were adopted for purposes of integration:

$$\gamma = 1.66; \quad K_L = 1; \quad 1/K_C = 0.46; \quad K_\sigma = 8; 64; 512$$

Figures 17, 18 and 19 give some results of numerical integration for $K_\sigma = 64$ and also corresponding experimental curves for a discharge chamber 40 cm in diameter and 90 cm long at $V = 40$ kv and $C = 83$ μf . The scales for quantities having dimensions relate to a pressure of $p_0 = 0.2$ mm Hg. The ratio $K_L = L_0/2l$ for this chamber is near unity; $1/K_C = 0.46$ at pressure of $p_0 = 0.2$ mm Hg. Under these conditions the units of time, temperature and pressure are equal to $t_1 = 2.62 \times 10^{-6}$ sec, $T_1 = 120$ ev and $p_1 = 2.8$ kg/cm² respectively. The value of $K_\sigma = 64$ corresponds to an electric conductivity $\sigma = 3 \times 10^{13}$ absolute units.

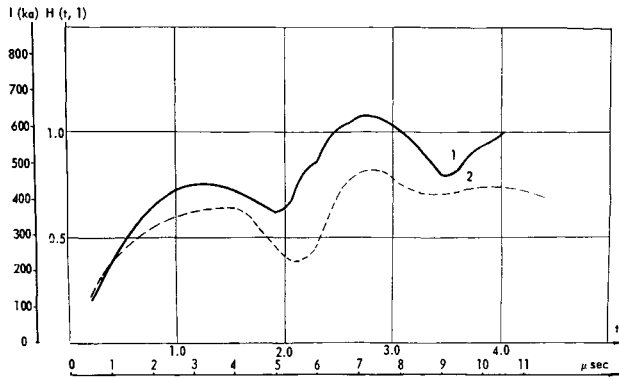


Figure 17. Current vs. time curves. Solid curve: theoretical. Broken curve: experimental

Note that for a pressure $p_0 = 0.05$ mm Hg, the units will be as follows: $t_1 = 1.85 \times 10^{-6}$ sec, $T_1 = 240$ ev, $p_1 = 1.4$ kg/cm² and $1/k_C = 0.23$. In calculating T_1 the average mass of a particle M is taken equal to the mass of the deuterium atom, which means neglecting the electrons.

The first contraction of the string occurs when $t = 1.9$ (dimensionless quantity); the shock wave reaches the center at 1.7. The density is increased up to 50 times during contraction, the average increase taken over the area of the cross section is about 30 times. The dimensionless pressure reaches 21 during the contraction, which corresponds to 60 kg/cm² for $p_0 = 0.2$ mm Hg, and 30 kg/cm² for $p_0 = 0.05$ mm Hg. The dimensionless temperature averaged over the area of the cross section reaches 0.54 during the first contraction. This corresponds to 65 ev for $p_0 = 0.2$ and 130 ev for $p_0 = 0.05$ mm Hg, when neglecting the electrons. If the electron and ion temperatures are the same, the corresponding figures are 32 ev and 65 ev. The temperature is 20% higher during the second contraction. The temperature greatly increases ("cumulative effects") near the axis when the second shock wave impinges. The central part of the gas, comprising one-tenth of the total mass, has a temperature 1.5 times the mean temperature.

It is convenient to relate the mean temperature to the current at the instant of maximum contraction by means of an equation that is similar to the usual equilibrium relationship between temperature, current and number of ions N per unit length:

$$T = \frac{\eta I^2}{2c^2 N} = 3.12 \times 10^{12} \frac{\eta I_{ka}^2}{N} [\text{kev}]. \quad (17)$$

The coefficient $\eta \approx 2.7$ during the first contraction and $\eta \approx 2.1$ during the second.

The control sum in Fig. 19 deviates from a constant value due to the errors introduced when solving the equations numerically.

The results of integration are similar for $K_\sigma = 512$, but the skin effect is much more pronounced, and the column is pinched somewhat more greatly. For $K_\sigma = 8$ the string expands up to the "wall" ($r = 1$) after the first contraction.

The following discrepancies between the theoretical and experimental results should be pointed out:

The compression is somewhat greater and the current somewhat less in the experiment than in the calculations. The expansion observed after the first contraction is less than that following from the calculations and afterwards the oscillations are observed to decay quickly. This is an indication that the elasticity of the plasma column in reality is less than the value adopted for the magnetohydrodynamic model, which does not account for energies expended on ionization, excitation and radiation, as well as the influence of various kinds of instability.

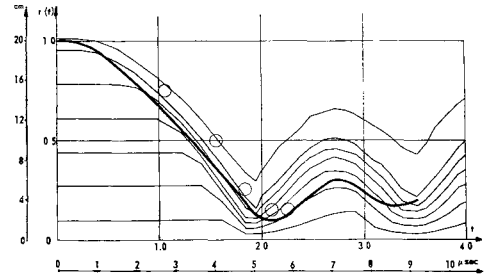


Figure 18. Curve of trajectories. Heavy curve, radius vs. time according to inductance measurements (from oscillograms). Circles, position of the maximum magnetic field according to measurements with magnetic probes

The magnetic field measurements by means of magnetic probes indicate that the peripheral current rises after contraction. This phenomenon is most likely connected with the fact that part of the gas is not involved in the pinch by the ions and remains outside; moreover, the neutral particles may fly out (evaporate from the periphery of the string) and, finally, gas may be given off by the chamber walls. The quick current rise in the peripheral region of small

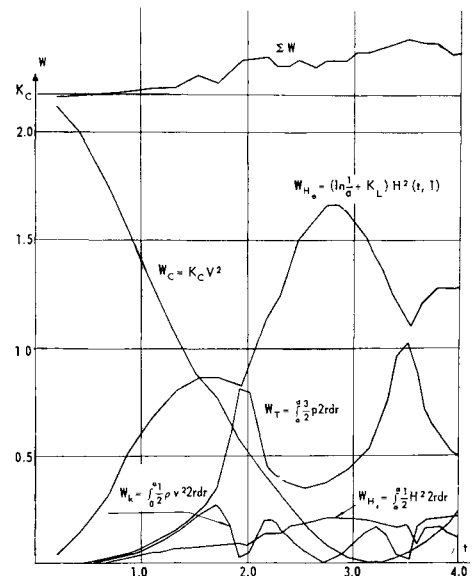


Figure 19. The energy distribution vs. time

inductance causes the current curve on the oscillogram to rise more rapidly after passing through its minimum value than would be expected on the basis of calculations.

The calculations, of course, do not account for a certain initial delay of the current near the chamber walls, which is connected with the formation of plasma dense enough to sweep the neutral gas.

NUCLEAR EMULSION STUDIES †

Neutron radiation in high-current pulse discharges in deuterium has been detected in experiments both in chambers with non-conducting walls³ and in chambers with conducting walls.¹⁴ The energy spectrum of neutron radiation in discharges in chambers with non-conducting walls has been studied by a number of authors.^{6, 15, 16} In the present work the neutron radiation accompanying pulse discharges in deuterium in a chamber with conducting walls was investigated by means of nuclear emulsions. All the investigations were carried out for the two polarities of the voltage applied to the electrodes of the chamber. It was shown that the sources of the neutrons are mainly concentrated in a small region near the anode. As a result of a study of the angular distribution of the neutrons, a second (appreciably weaker) source near the cathode was also hypothesized. On the basis of studies of neutron spectra obtained on five plates placed in different positions in relation to the chamber, a conclusion was reached about the accelerator mechanism producing the main body of the neutron radiation. The maximum energy of the deuterons was approximately 200 kev. The results of the present work are compared with data obtained in chambers with non-conducting walls.

Description of Experiments

In the experiments, a capacitor bank of 145 microfarad capacity was discharged through the chamber filled with deuterium at a pressure of 2 mm Hg. The initial voltage V_0 of the bank was 30 kv.

Soviet nuclear plates, NIKFI [Cinematographic and Photographic Scientific Research Institute] type K, with an emulsion thickness of 100 microns, were used in the work. The emulsions used were calibrated by irradiation with 2.52 Mev. The location of the plates in relation to the discharge chamber is shown in Fig. 20. The plates were exposed to about 50 discharges through the chamber giving a total neutron intensity of $\sim 10^{10}$. They were also exposed for both polarities of the chamber electrodes. In all, about 8000 tracks of recoil protons were scanned. Simultaneously, unexposed controls from the same batch of emulsion were examined to determine the number of background tracks. It appeared that there were practically no background protons.

Results

After the plates had been scanned it became clear that for each plate there was a clearly defined preferential angle of incidence of the neutrons. In Fig. 20, ϕ denotes the angle between the direction from which the neutrons from the discharge chamber strike the plate and the axis of the chamber, measured clockwise from the cathode ($\phi = 0^\circ$) to the anode ($\phi = 180^\circ$). It follows from consideration of the values of ϕ that the sources of the neutron radiation are located on the axis of the chamber, chiefly in the neighborhood of the anode, which agrees with the results obtained by the method of collimating counters.¹⁴

The "center of gravity" of the sources lies at a point 3-4 cm from the anode. It should be observed that all the plates were completely scanned for each of the two polarities of the voltage across the chamber electrodes. Here, results are given only for one polarity, indicated on Fig. 20 (the anode is the upper electrode). Similar results were obtained when the polarity was reversed (allowing for the spatial change in the direction in which the angle ϕ is read off; i.e., ϕ is measured from the cathode to the anode in both cases).

Figure 21 shows the energy spectra of the neutrons obtained for all five plates with various values of ϕ . For ease of comparison, the spectra are standardized: values of f_n , the percentage ratio of the number of neutrons in a given energy interval to the total number

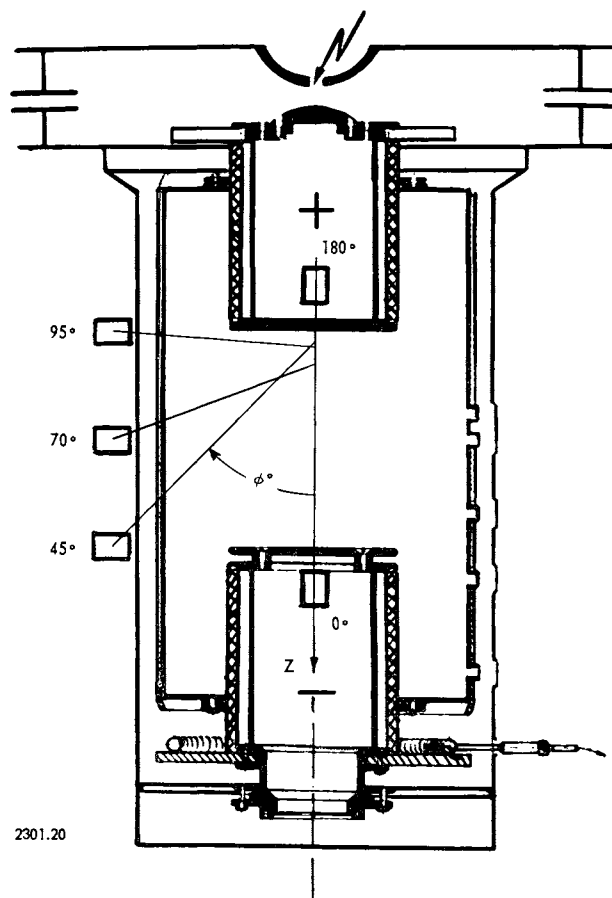


Figure 20. Metal chamber and arrangement of nuclear plates

† Work performed in 1958. This part is by M. M. Sul'kovskaya, N. V. Filippov and V. A. Khabrov.

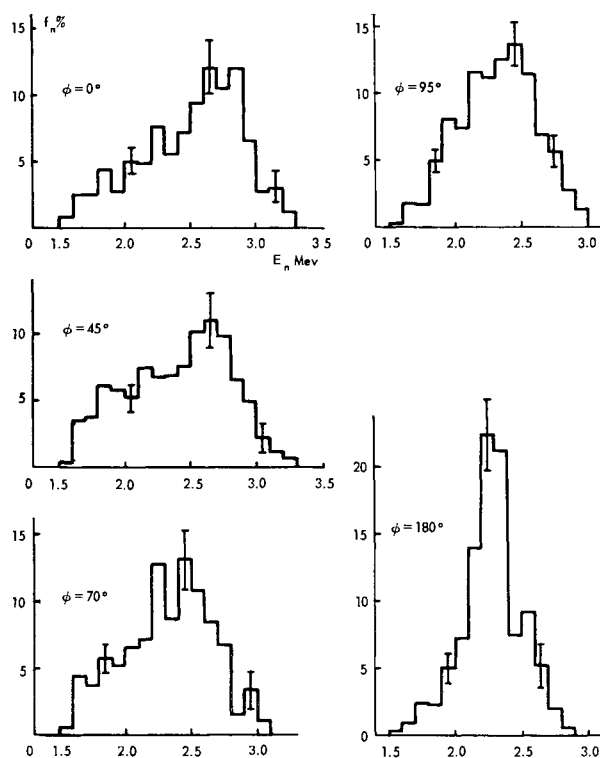


Figure 21. Neutron energy spectra

of neutrons, are plotted as the ordinates. The energy of the neutrons is determined to an accuracy of ± 50 keV. The maxima of the cathode and anode spectra are displaced by approximately 200 keV towards the end corresponding respectively to energies larger and smaller than those of "thermonuclear" neutrons. It must be pointed out here that in the work described in Ref. 14 and briefly above a neutron energy spectrum was obtained (Fig. 16 above and Fig. 5 in Ref. 14) that had been obtained in the course of preliminary experiments where no special steps were taken to reduce the effect of the scattering of the neutrons. As a result, the cathode spectrum maximum was somewhat displaced towards the low-energy end; this, as was explained subsequently, led to the erroneous conclusion that the shift between the maxima of the cathode and anode spectra obtained in a chamber with conducting walls is less (see above) than that obtained in a chamber with non-conducting walls.^{6, 15, 16} In the present work, the neutron energy spectrum is the result of experiments carried out under conditions which eliminate the effect of scattering so far as possible, although some 20–30% of the neutrons are still scattered.

The form of the spectra obtained indicate the existence of an accelerating mechanism producing neutron radiation in chambers with conducting walls (just as in the case of chambers with non-conducting walls).^{6, 15, 16} The acceleration of the deuterons occurs in a small region near the anode; and the velocity of the main body of fast deuterons is directed along the axis of the discharge chamber from anode to cathode.

From the spectra obtained it is possible to calculate (allowing for straggling) that the maximum energy of the accelerated deuterons must be about 200 keV.

In examining the spectra of neutrons leaving the chamber at different angles ϕ (Fig. 21), a shift of their maxima towards the lower-energy end as ϕ increases is noticeable. Figure 22 shows the relationship between the angle ϕ and the neutron energy at the maxima of the neutron spectra (the calculated curves are represented by broken lines). It is clear from Fig. 22 that deuteron energies of 25–50 keV correspond to the maxima of the neutron spectra.

Assuming, as a first approximation, that the neutron source is a point situated 3.5 cm from the anode, we plotted the angular distribution of the neutrons. This showed that, for the lateral and anode plates ($\phi = 45^\circ$ – 180°), the experimental results are in satisfactory agreement with the angular distribution of neutrons corresponding to a deuteron energy of ~ 50 keV. In the case of the cathode plate ($\phi = 0^\circ$) it was found that the intensity of neutrons leaving the chamber in the direction of the cathode was ~ 3 times that of neutrons radiated in the direction of the anode (whereas, even at the maximum deuteron energy of 200 keV, it should have been only 1.5 times as great). The ratio could come out at ~ 3 only if the energy of the accelerated deuterons was taken as 1.5–2 MeV, which completely contradicts the results of other experiments. This contradiction can be disposed of if we assume the existence of a second neutron point-source not more than 5 cm from the cathode with an intensity not exceeding 10% of that of the main source. This second source can make a

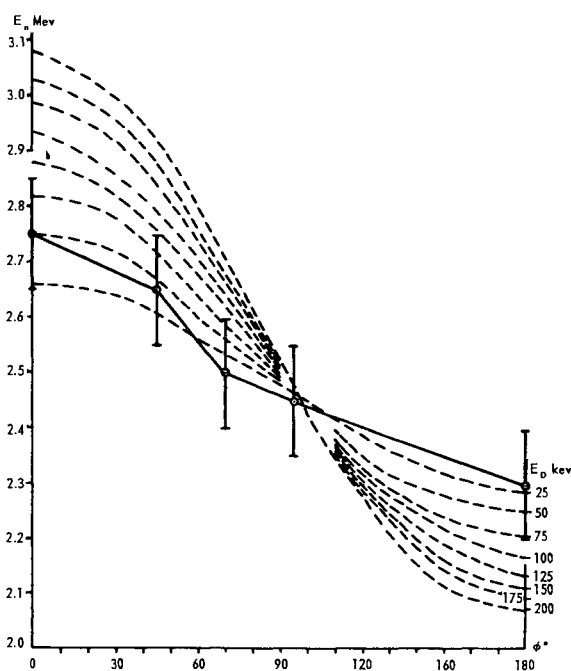


Figure 22. Measurements of neutron anisotropy. Energy at which the maximum in the neutron energy spectrum occurs, as a function of angle of observation

substantial contribution *only* to the neutron flux falling on the nearest (cathode) plate.

The cathode and anode neutron spectra obtained in the present work were also compared with the corresponding spectra for chambers with non-conducting walls. In Fig. 23, 1-4 are neutron spectra $\dagger$ obtained respectively in the present work and in that described in Refs. 15 and 16. In this diagram, the cathode spectra of the neutrons are shown in dotted lines and the anode spectra in full lines.

It is noticeable that in anode spectra 1 and 4 there is a relatively large quantity of neutrons with energies approaching the "thermonuclear". Nevertheless, all four spectra are extremely similar to one another despite the fact that they were obtained for neutron radiation produced under different initial conditions (type of wall, dimensions of chamber, pressure, voltage, etc.) and have a different longitudinal distribution of the sources of the radiation. However, we do not have sufficient grounds for giving a definite answer to the question: Is there, or is there not, the same accelerating mechanism for chambers with conducting and those with non-conducting walls?

Conclusions

The following deductions can be made from the results obtained here:

(1) The sources of the neutron radiation are concentrated along the axis of the chamber, chiefly ($\sim 90\%$ of the total intensity) in a small area not more than 6-7 cm long in front of the anode. The center of gravity of the sources is situated 3-4 cm from the anode.

(2) On the basis of the observed angular distribution of the neutron radiation, the existence was assumed of a second neutron source, with an intensity not exceeding 10% of the total intensity, situated not more than 5 cm from the cathode.

(3) The main body of the neutron radiation is dependent upon the existence of some sort of mechanism for accelerating deuterons. The maximum energy of the latter is 200 keV, and the deuteron energy corresponding to the maximum of the neutron energy spectra is 20-50 keV.

(4) Comparison with the results of other experiments ^{6, 15, 16} shows that the general form of the energy spectra of neutron radiation is identical for chambers with conducting and for those with non-conducting walls, and is affected to only a small extent by changes in the discharge conditions (shape and dimensions of the chamber, initial voltage, pressure of the deuterium, etc.).

ELECTRONS CAUSING THE HARD X RAYS§

The work here described was devoted to the measurement of the energy of the electrons responsible for the emission of hard X-ray quanta. The experiments were

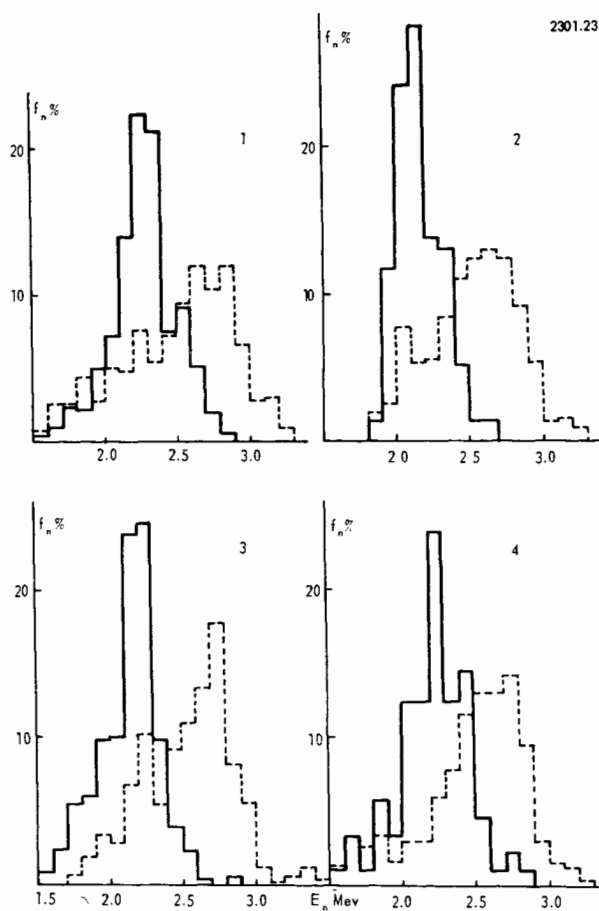


Figure 23. Neutron spectra: 1, from present work; 2, Demichev and Prokhorov; ¹⁵ 3, Anderson et al.; ⁶ 4, Hagerman and Mather ¹⁶

carried out under the conditions described in Part 3. During the work the presence of X-ray pulses was checked by means of a scintillation counter connected to a pulsed oscilloscope.

In accordance with the hypothesis that the electrons are accelerated along the axis of the discharge chamber, a hole was made in the center of the electrode which is positive at the moment at which the voltage is applied; this hole was covered with aluminum foil 6 μ thick. Behind the aluminum window was mounted a flat vacuum chamber situated between the pole-pieces of an electromagnet run off a pile of batteries. Hypersensitized X-ray plates, prepared at NIKFI [Cinematographic and Photographic Scientific Research Institute], were used to monitor the electrons.

When a slit diaphragm was interposed between the aluminum foil and the plate in the absence of a magnetic field, a sharp image of the slit field appeared when the plate was exposed to 5-10 discharges. When a weak magnetic field ($H = 20$ gauss) was applied, one end of the image of the slit became diffuse. This diffuseness could only be explained by negatively charged particles impinging upon the plate. The blackening observed could not have been caused by penetrating electrons of energy $W = 40$ keV originating at the start of the discharge,¹⁷ since when

$\dagger$ To facilitate comparisons we have reduced spectra 2, 3 and 4 to the form $f_n(E_n)$ used in the present work.

§ This part is by I. M. Podgorny, N. G. Kovalsky and V. E. Palchikov

the initial pressure of the hydrogen was reduced to 1.5×10^{-2} mm Hg, the intensity of the initial X rays increased sharply but no pulse of (hard) X rays were recorded by the oscillograph, and there was no blackening of the plate. These facts show beyond doubt that the electrons responsible for the generation of hard X rays are accelerated along the axis of discharge.

To determine the energy of these electrons, the plate was exposed in the absence of the magnetic field and then, without being changed, was subjected to several tens of discharges under a magnetic field ($H = 230$ gauss). This method enabled us to obtain on the same plate blackening due both to the energy spectrum of the electrons and to the line corresponding to the undeflected beam of electrons. This line was used for determining the position of the plate in the spectrograph. The plates thus obtained were scanned with an MF-4 || microphotometer.

To obtain the spectrograph dispersion curve, the electron tracks were plotted graphically. This called for thorough study of the topography of the magnetic field. Using the dispersion curve, and measuring on the microphotograph the distance between the base line and the boundary of the blackening caused by the electrons deflected by the magnetic field, it is easy to obtain the value of the maximum energy acquired by the electrons while moving along the axis of discharge.

As has been indicated,¹⁸ the curve of the energy spectrum of hard X rays falls sharply on the high-energy side. Hence the position of the boundary of the energy spectrum of electrons responsible for hard

X rays might be expected to vary with the exposure. To confirm this, a series of experiments was carried out in which the exposure varied from 20 to 90 discharges. Comparison of the corresponding microphotograms showed that, beginning with an exposure of 50 discharges, the boundary of the blackening ceases to be displaced directionally towards the base line (i.e., towards the high-energy side). It should, however, be pointed out that as the exposure increases the curve of blackening near the boundary becomes less steep, which makes it easier to determine the maximum energy. Processing of the experimental results obtained showed the maximum energy of the electrons to be 300 kev.

A 100-kev electron beam was used to confirm the accuracy of the plotting of the electron tracks. The electron source was placed inside the porcelain chamber; in addition, the geometry of the test corresponded in every detail to that used in the experiments described in this paper. The energy of the electrons as determined by measuring the potential difference agrees to within 3% with that obtained by plotting the dispersion curve. Allowing for the spread in the results of individual experiments, the authors consider that the results obtained are accurate to 5-6%.

Thus it has been shown by direct experiment that the electrons responsible for the generation of hard X rays accompanying high-current pulse discharges in hydrogen are accelerated along the axis of the discharge chamber. The maximum recorded energy of the electrons was 300 ± 20 kev, a figure which agrees well with the results obtained by measuring the boundary energy of the X-ray spectrum.

|| Translated designation.

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Studies of High-current Gas Discharges at High Rates of Build Up

By S. C. Curran, K. W. Allen, H. A. B. Bodin, R. A. Fitch,
N. J. Peacock and J. A. Reynolds *

INTRODUCTION

One of the principal aims of the AWRE work in the controlled thermonuclear field has been to study rapid methods of heating plasmas, such as shock-heating or fast adiabatic compression. It is felt that these methods may have advantages over joule heating which becomes inefficient at high temperatures because of the decrease in plasma resistivity.

The experimental work so far has been done with discharges between electrodes in straight tubes since this facilitates the work in several ways. Preliminary accounts of some of the results have been given by Allen *et al.*¹ and Bodin and Reynolds.²

The theoretical works of Shafranov,³ Taylor⁴ and Rosenbluth⁵ predict that in order to achieve stability with pinch containment, it is essential to confine at large radius, i.e., with small compression. This means that it is very desirable to raise the temperature to within about a factor ten of the final value before the plasma is adiabatically compressed, and shock heating recommends itself as a good method of achieving a rapid initial temperature rise. Two other factors in favour of rapid heating, which has been discussed in more detail by Bickerton,⁶ are as follows. First, if the plasma confinement time is limited by inter-diffusion of fields, this time should be used to best advantage by heating in a time short compared with the "stable time". Secondly, as the stable time depends on T_e , where T_e is the electron temperature, a rapid initial rise of temperature is desirable to minimise the early mixing of the crossed fields.

In order to shock-heat a plasma, the circumferential magnetic field, H_e , which compresses the plasma in a pinched discharge, must build up to a very high value in a time short compared with the collapse time; this implies a very high initial rate of rise of current, which is determined largely by the capacitor bank inductance and voltage. A high voltage and low inductance are mutually incompatible; therefore, in practice, a convenient voltage is chosen and the cap-

acitor bank built to have the lowest possible inductance.

Much of the effort so far has been concentrated on the design and construction of capacitor banks of extremely low inductance. In conventional bank design most of the inductance arises in the switch. The inductance contribution from the capacitor bank itself can be almost arbitrarily reduced by increasing the number of capacitors and by paralleling leads. At AWRE the switch is subdivided, and a "multiple parallel switch system", using one switch per capacitor, is used. The inductance due to the switches is reduced by a factor $1/n$, where n is the number of capacitors and associated switches.

Two such banks have been built and are in operation. These will be briefly described below. Although the first bank was small and intended largely as a prototype for the larger machine, when discharged into deuterium at low pressure it gave some interesting results, in particular regarding the neutron emission. These results had several features not found in the work of Artsimovitch *et al.*,⁷ first reported by Kurchatov⁸ or by Anderson *et al.*⁹ Recently published work, for example by Berglund *et al.*,¹⁰ by Hagerman and Mather¹¹ and by Burkhardt and Lovberg,¹² have confirmed the growing view that the phenomena which occur in fast linear pinches are very much more complex than had been thought earlier. It is hoped that the work here reported on discharges using a very low inductance bank, particularly as regards the effect of changing various parameters, will help to build up a consistent picture of the linear pinch—both from the point of view of the physics of the discharge, and from the point of view of possible applications.

The main results which will be discussed are as follows:

1. General observations of the discharge, such as the waveforms for voltage, V , current, I , and rate of change of current, dI/dt .
2. Neutron emission: (a) general properties of the neutron emission, including the yield, the length of the neutron burst and its occurrence relative to the

* U.K. Atomic Energy Authority, AWRE, Aldermaston.

pinch bounces; (b) the effect of impurities; (c) the effect of an axial magnetic field on the discharge and the neutron production; (d) the variation of neutron yield with pressure; (e) the variation of the yield with the current at neutron time; and (f) the effect on the neutron production of raising the bank inductance.

EQUIPMENT

Triggering System

Prior to discussing technical features of parallel switching, two additional and rather important advantages should be mentioned. First, in a conventional bank, with all the capacitors connected in parallel to a single switch, if one capacitor breaks down, the others will discharge into it. For banks of energy $\sim 10^6$ or more joules this is likely to result in a dangerous explosion. With parallel switching, all the capacitors are isolated until the moment of firing and the explosion hazard is greatly reduced. Secondly, the bank becomes a much more flexible device, as it may be subdivided into a number of smaller units, each fired at any desired instant during a discharge. A further advantage is that the construction of the gaps is simplified, since each gap has to carry only a small fraction of the total current.

The main problem in multiple parallel switching is to trigger the large number of parallel spark gaps, used as switches, in a time short compared to the time in which the discharge can change appreciably while in its early stages of development. Since the voltage stress in the gaps which break down late is progressively reduced by the voltage build-up from the early gaps, triggering of parallel gaps can present considerable difficulty. However, the gaps are not, in effect, paralleled until after the time required for a pulse to travel from one gap to another via the common connection at the collector. The aim is, therefore, to produce breakdown in all the gaps within this time, which is $\sim 15 \times 10^{-9}$ sec for the equipment described here.

The triggering system consists basically of a high voltage capacitor discharged by a single spark gap, G, into the paralleled trigger cables. This is shown in Fig. 1a. The rise time of the trigger pulses must be $\lesssim 10^{-8}$ sec, and this necessitates a very low inductance spark gap; the requirements for this gap are, however, much less severe than those for a single main-bank gap, both because of the inductance required and the energy involved. In the banks so far constructed, a short gap (~ 2 mm) pressurised to several hundred psi has been used; the trigger capacitors are pulse-charged from a Marx Generator or a pulse transformer, to reduce the variation of breakdown voltage of the pressurised gap.

The main-bank spark gaps work on the direct overvoltage principle. The trigger pulse is applied to a small probe recessed in one of the main electrodes, and initially appears across a small annular gap. This produces electrons by field emission, and the statistical lag is reduced to $\lesssim 10^{-9}$ sec. When the annular gap breaks down, the pulse appears across the main gap,

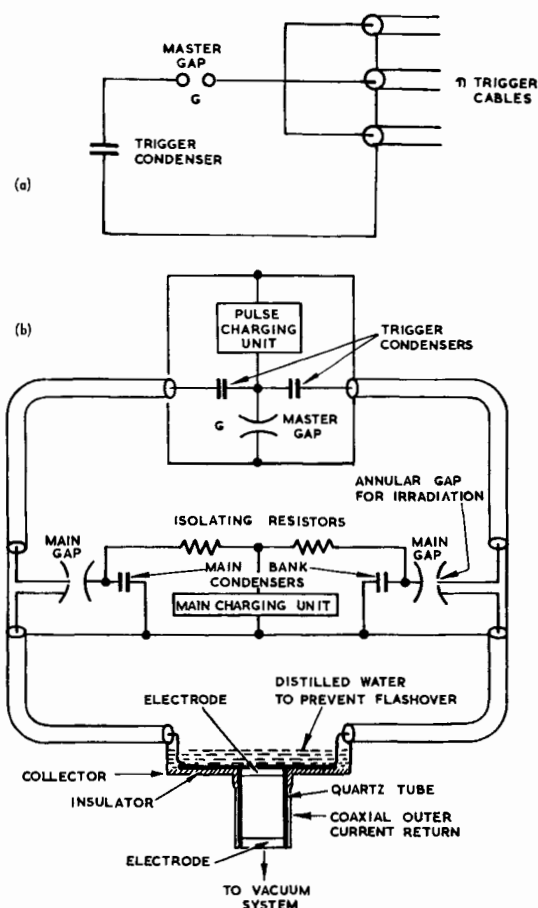


Figure 1. (a) The basic triggering system. (b) Schematic diagram of the triggering system showing two of the paralleled units (consisting of a bank condenser and spark gap, the connecting cable and a trigger condenser). The master gap, the pulse charging unit, main charging unit and the collector are common to the whole system

which is irradiated by the annular discharge, and the sum of the main-bank voltage and the trigger-pulse voltage is arranged to give a peak overvoltage $\sim 100\%$ to effect breakdown in the required time. This type of gap has the advantage that it can operate for any bank voltage between zero and the D.C. breakdown, without changing the position of the electrodes.

Figure 1b is a schematic diagram of the triggering system, showing two of the main-bank units and the principal components which are common to all units. Table 1 shows some of the details and performance figures of the two banks.

In all experiments carried out so far, the tube is pre-ionised by a very low energy discharge; the main bank is switched on about 8×10^{-6} sec after the tube breaks down.

Discharge Tube and Monitoring Equipment

Discharge Tube Assembly

The discharge tube used for most of the experiments on the small bank was an opaque tube of 28 cm length and 15 cm bore. The initial inductance was ~ 11

$\times 10^{-9}$ h. Plane copper electrodes were used; it was found after some hundreds of discharges that these were covered with a hard black non-conducting layer, about 0.01–0.02 in. thick, which was shown to be silicon. This layer, somewhat different in appearance on the anode and cathode, was pierced with many pinholes, spaced some millimeters apart, through which the copper could be seen.

The tube was evacuated with a mercury diffusion pump, usually to a base pressure $\sim 2 \times 10^{-6}$ mm Hg. It could either be sealed off, and filled with a fixed charge of gas, or a continuous flow system could be used. The purity for most of the work was $\sim 0.1\%$, but measurements were carried out down to $\sim 0.02\%$ with no significant change in neutron yield.

Monitoring Equipment

The voltage was measured by a high-resistance potential divider. The current was found by electrically integrating the voltage induced in a pickup coil which linked all the currents at the collector. The rate of change of current was obtained directly from this coil. The neutron pulse was recorded by a liquid scintillation counter; the total yield was found by counting the number of delayed pulses due to γ rays produced by slow neutron capture in hydrogen and cadmium. The large liquid scintillation counter (~ 20 cm radius) was surrounded by cadmium.

EXPERIMENTAL OBSERVATIONS WITH THE 6000 JOULE BANK

General Properties of the Discharge – V, I and dI/dt Waveforms

Figure 2 shows typical waveforms, together with a neutron pulse. It is noteworthy that, because of the very low inductance of the bank, the pinching is not shown nearly so markedly on the voltage curve as, for example, it is in the work of Artsimovitch *et al.*,⁷ Anderson *et al.*⁹ or Berglund *et al.*¹⁰ The inductive behaviour of the waveform is shown most clearly by the dI/dt waveform. The initial value (off the graph) is $\sim 4.5 \times 10^{11}$ amp/sec. The first reversal of the current sheath is seen to be very rapid from the high peak in the waveform; this is consistent with the current sheath's meeting the reflected shock wave from the axis of the tube. The

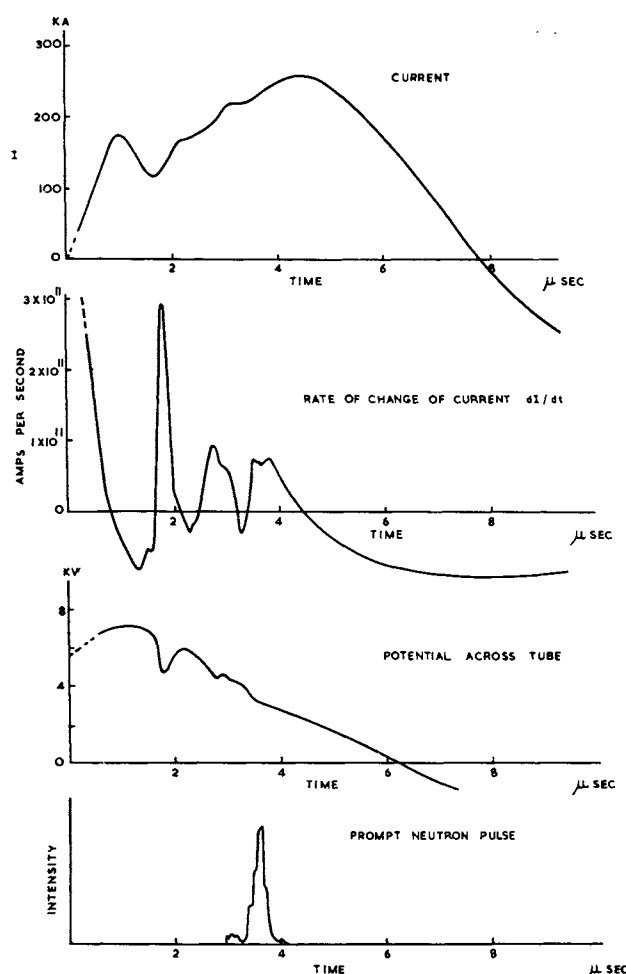


Figure 2. Typical waveforms of the discharge, including the neutron pulse. Bank: 118 μ f, 9 kv. Initial deuterium gas pressure 44 microns Hg. The initial value of dI/dt , 4.5×10^{11} amp per sec, is off the graph

general shape of the early part of the waveform is usually repeatable from shot to shot, under the same conditions, but the later part is variable. Three clear pinches are usually evident but up to five had been observed. The maximum current obtainable with discharges in deuterium is just under 300 k-amp.

Measurements have been made of the time to pinch, t , for various values of initial gap pressure, p ; the results for hydrogen are shown in Fig. 3 where t is plotted against $(p/E^2)^{1/2}$ where E is the average electric field across the tube up to the first pinch. Similar curves were obtained for other gases.

Neutron Production

General Properties

The yield generally was in the range of 10^4 – 10^5 neutrons. The maximum yield was $\sim 10^6$, but bursts of this magnitude were only observed in about 1% of firings. There was no evidence, even under conditions of the highest purity, for large yields $\sim 10^7$ – 10^8 , as reported by Artsimovitch *et al.*,⁷ Anderson *et al.*⁹ and others; the maximum currents

Table 1

	6 kj bank	45 kj bank
Nominal capacity, μ f	121	100
Number of units	109	200
Nominal working voltage, kv	10	30
Inductance (without tube), h	8.50×10^{-9}	5.5×10^{-9}
Inductance of capacitor and spark gaps, h	1.31×10^{-9}	1.8×10^{-9}
Inductance of cables, h	3.64×10^{-9}	2.0×10^{-9}
Inductance of collector plates, h	3.55×10^{-9}	1.7×10^{-9}
Short circuit dI/dt , amp/sec	1.2×10^{12}	5.5×10^{12}
Short circuit current, amp	1.1×10^6	3.6×10^6
Peak trigger voltage, kv.	20	30
Trigger voltage rise-time, sec	15×10^{-9}	15×10^{-9}

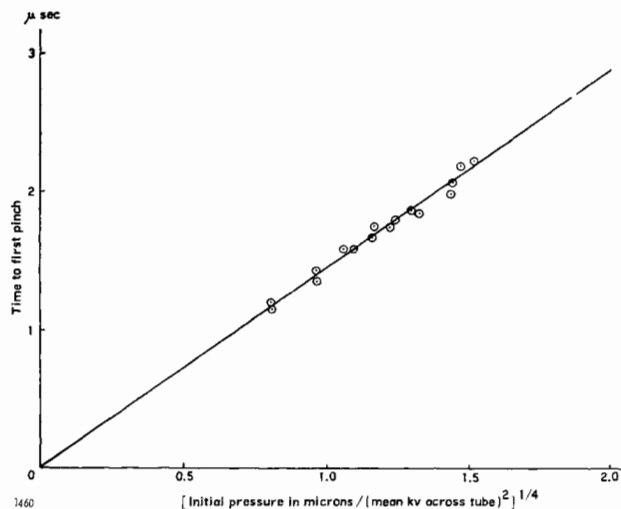


Figure 3. The experimental variation of the time to pinch in hydrogen, as a function of the fourth root of the initial gas pressure (voltage constant). Theory predicts a linear relationship. Bank: 118 μ f, 9 kv

in the present experiments were nevertheless as high as the values reported by these workers, although the bank voltage was lower.

A feature of the neutron production which differs from the work of Artsimovitch *et al.*⁷ and Berglund *et al.*¹⁰ using porcelain tubes, is that at the start of a series of experiments the neutron yield rises to its average value in about 6 shots, and a good burst is sometimes observed on the first shot. No prolonged firing is necessary to achieve the normal yield. Between experiments the pressure sometimes rises to ~ 10 –100 microns of mercury. If initial firings are in a sealed-off system, a pressure rise of some tens of microns is observed and this appears to be due to outgassing. This rise decreases with each successive shot, and becomes negligible at the same time as the neutron production reaches a steady value. The phenomenon is schematically illustrated in Fig. 4.

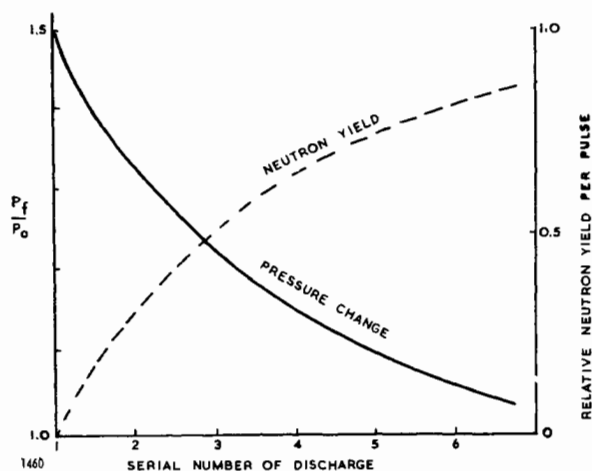


Figure 4. A schematic diagram showing the variation, with firing serial number, of the ratio p_f/p_0 , where p_f is the pressure in a sealed off system after firing and p_0 that before. The neutron yield, expressed as a fraction of the normal maximum, is also shown

The length of a neutron burst is generally in the range 0.5–0.7 μ sec, although extremes of 0.1 μ sec and 1.5 μ sec can be observed. The neutrons are usually emitted about the third pinch (Fig. 2 shows a typical example), but they could appear occasionally ($\sim 10\%$) at the second or fourth bounces. The neutron emission ceases before the current maximum. In about 1% of shots, a very sharp spike on the voltage waveform is observed (see Fig. 5). Such a spike is always accompanied by a large yield $\sim 10^6$, although sometimes big neutron bursts appear with no spike.

No significant change in the neutron yield is found when the low energy pre-ionization discharge is varied.

Effect of Impurities

Figure 6 shows a curve of the neutron yield against the percentage of nitrogen impurity. For comparison, the data of Anderson *et al.*⁹ are also shown. In view

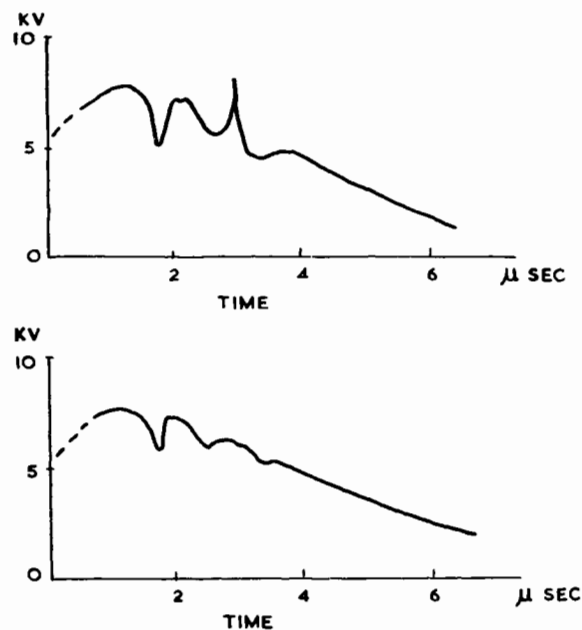


Figure 5. The upper waveform shows the sharp voltage spike observed at the third bounce in about 1% of shots. This spike is always accompanied by yields $\sim 10^6$. The lower shows the normal voltage waveform. Bank: 118 μ f, 9 kev. Initial tube pressure 48 microns Hg

of the fact reported by several workers that the neutron yield of, say, 10^7 – 10^8 was observed to be extremely sensitive to small amounts of impurity, it is of special interest to examine the neutron output at low values of the impurity. No tendency for any increase is observed from about 1% down to between 0.01 and 0.02%. Only a few measurements at very low impurity content have been made but there is slight evidence that the neutron yield is a little greater at 1–2% impurity than at any other value, but the data on this effect is not conclusive. Some correlation between the impurity content, the yield and the voltage waveform is observed. This is illustrated in Fig. 7. In general, at an impurity content that

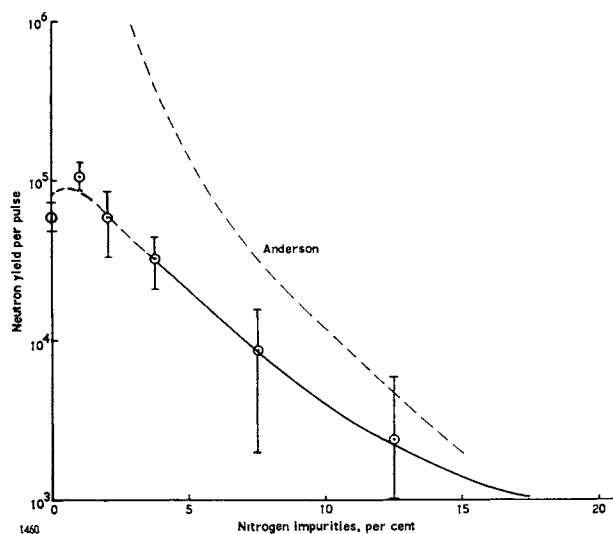


Figure 6. Effect of nitrogen impurity on the neutron yield. Bank: 118 μ f, 10 kv. Initial tube pressure 48 microns Hg. The results of Anderson *et al.*⁹ are shown for comparison

markedly affected the yield, the second and subsequent pinches are reduced in amplitude. At about 10% impurity, which causes a reduction in yield by ~ 100 , the first pinch is slightly affected and the later pinches are much reduced. At still higher impurity levels only one clear pinch is observed. Above 4% impurity the yield Y roughly follows an empirical law $Y = (1-7f)^2$ where f is the fraction by volume of impurity.

An attempt was made to assess the relative importance, in the discharge, of gas introduced into the tube and gas adsorbed on the walls and driven off in the early stages of the pinch. Only a few experiments have so far been performed but these have shown very clearly that gas on the walls plays an extremely important part in the discharge.

A few discharges were fired in hydrogen and helium, with a view to investigating possible X-ray emission, such as that reported by Lukyanov and Podgornyi.¹³ On some occasions small yields of X rays, equivalent to $\sim 10^3$ quanta of 100 kev energy, were observed. On other occasions, under the conditions which usually yielded $\sim 10^5$ neutrons in deuterium, no X radiation was observed although the minimum detectable was equivalent to approximately 100 quanta of energy 100 kev. Although only a few experiments on X-ray production have been carried out it has been established that high energy X-ray production is insignificant relative to the neutron production, in contrast with the work of Lukyanov and Podgornyi.

Effect of an Axial Magnetic Field

The effect of an axial magnetic field is of great importance from the point of view of understanding the processes which take place in the discharge, and may indeed help to elucidate the mechanism of production of the neutrons.

A plot (Fig. 8) of the neutron yield against the applied axial field shows a gradual decrease with increasing field, from about 10^5 to about 10^4 for a field of 10^3 gauss. The applied axial field was steady but not uniform—its value at the electrodes being about half that in the centre of the tube. The results of Anderson *et al.*⁹ are shown for comparison; a marked difference is apparent as they found that a field of 100 gauss reduces the yield from $\sim 10^7$ to less than 10^2 .

Another notable effect is that of the field on the dI/dt waveform as shown in Fig. 9. The first pinch is not greatly affected, but subsequent pinches are considerably damped and, generally, more pinches are observed.

Effects of Gas Pressure and Discharge Current

Figure 10 shows the variation of neutron yield with gas pressure. The yield generally fell below $\sim 10^3$ at a pressure of about 150 microns Hg. The maximum yield occurred towards the lowest pressures at which the tube would fire reliably (~ 25 microns Hg). The optimum pressures reported by other workers vary widely, and are probably dependent on the initial conditions of ionisation, and therefore depend on the applied voltage and the tube size and possibly wall material.

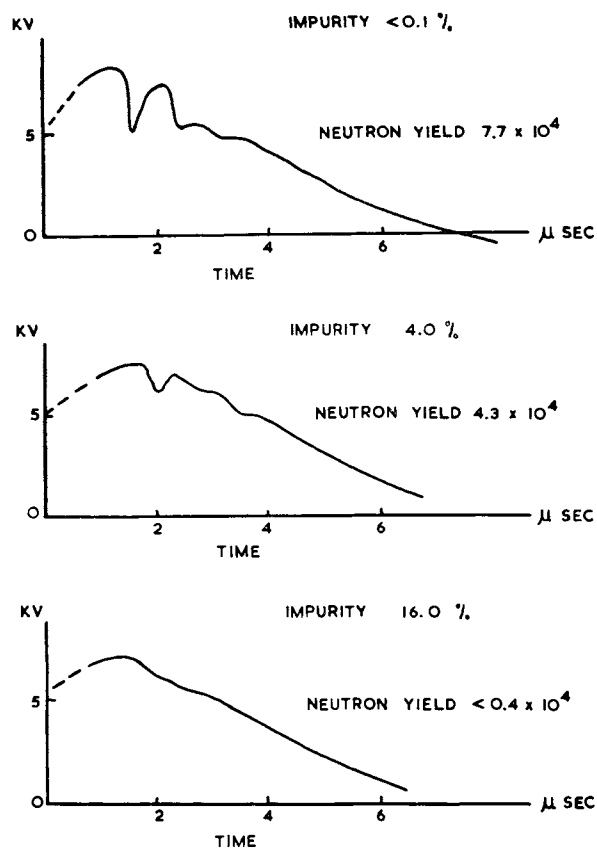


Figure 7. Correlation between the effect of impurities on the voltage waveform and the yield. Bank: 118 μ f, 10 kv. Initial tube pressure 48 microns Hg

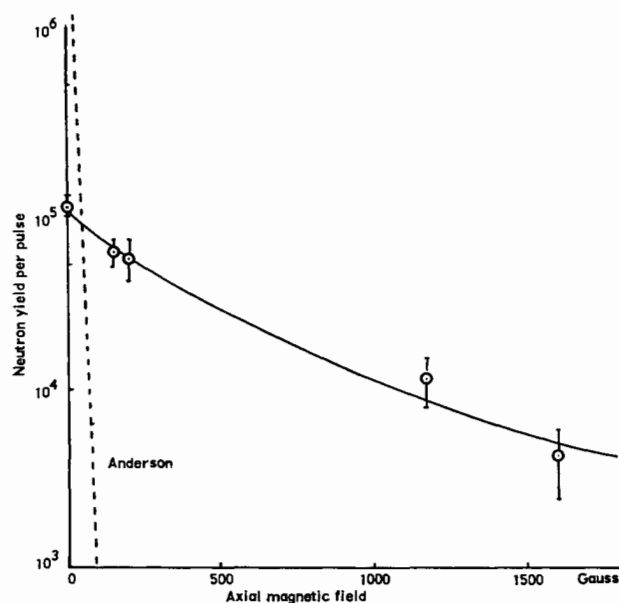


Figure 8. Effect of an axial magnetic field on the neutron yield. Bank: 118 μ f, 9 kv. Initial tube pressure 31 microns Hg. The results of Anderson *et al.*⁹ are shown for comparison

It is difficult to vary the maximum current without affecting other factors; e.g., varying the voltage also affects initial dI/dt and ionisation, while varying the capacity alters the inductance and the period. Some experiments were carried out varying the capacity, but the results were inconclusive. Figure 11 shows, for two particular pressures, the expected variation of neutron yield with current assuming a thermonuclear reaction and calculating the temperature from the pinch relation, $I = 2nk(T_e + T_i)$. It is further assumed that all the gas is collected up, that $T_e = T_i$ and that a compression of 100 is achieved. Any departures from the first two assumptions would tend to give a higher calculated temperature and a correspondingly greater neutron yield. The observed yields are superposed on the figure.

Effect of Varying Bank Inductance

The main difference between the AWRE equipment and that used elsewhere is in the low inductance of the bank. Experiments were carried out in which inductance was added to the bank to see if the results then obtained were in closer agreement with those reported by others.

The neutron yield is not significantly changed when the total circuit inductance external to the tube is increased from 8.5×10^{-9} h to 207×10^{-9} h; the initial tube inductance is 11×10^{-9} h. There is slight evidence for higher yields at the highest inductance but the wide variation yield from shot to shot makes this uncertain.

The effect of an axial magnetic field on the neutron yield, with and without the added inductance, is illustrated in Fig. 12. With high inductance added, the neutron yield is still $\sim 10^5$ per pulse, but now with small fields ~ 100 gauss applied the yield is reduced

by a factor of about 500. This marked reduction is very similar to the effect observed by others, including Anderson *et al.*,⁹ whose results are included in the figure.

The addition of inductance also affected the width of the neutron burst. The normal pulse length ~ 0.5 – 0.7μ sec was reduced to about 0.2μ sec when the total inductance was in the range 100 – 200×10^{-9} h. Here again the narrow pulse length is in agreement with the work of Anderson *et al.*⁹ and Burkhardt *et al.*,¹⁴ both of whom used higher inductance banks than that employed here.

The neutrons were emitted at much lower currents when extra inductance was added. In a typical run, the current at neutron time was 213, 138 and 107 ka, respectively, for a total bank inductance of 8.5, 97.5 and 207.5×10^{-9} h. At the low currents observed when extra inductance had been added, a thermonuclear origin for the neutrons would seem to be highly unlikely.

In general, with extra inductance added, the neutrons appear to have many of the properties of the instability neutrons reported by Anderson⁹ and others. The differences in behaviour of the basic equipment thus acquire much increased interest.

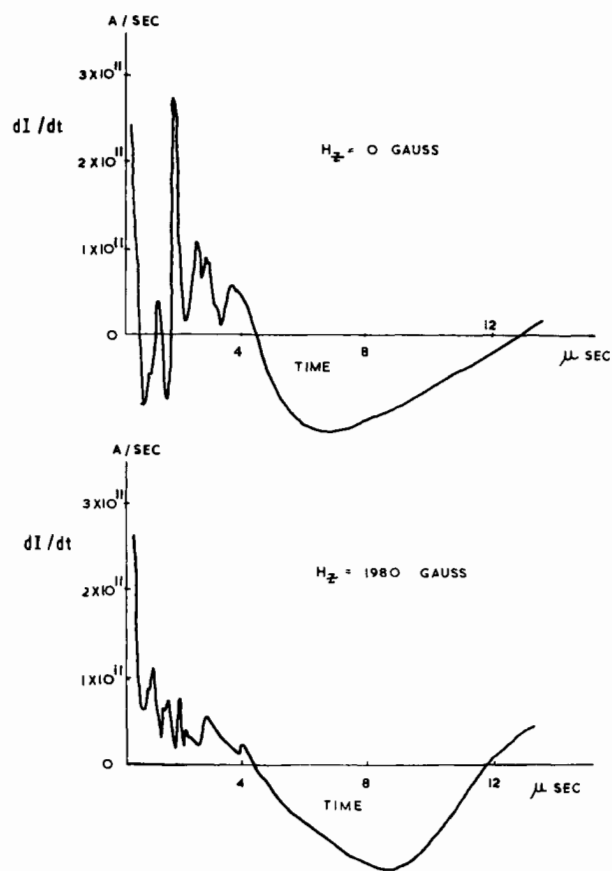


Figure 9. Effect of an axial magnetic field on the dI/dt waveform. Bank: 118 μ f, 9 kv. Initial deuterium pressure 31 microns Hg. The initial value of dI/dt , 4.5×10^{11} amp per second, is off the graph

PRELIMINARY RESULTS WITH THE 45 KILOJOULE BANK

No systematic measurements have yet been carried out with this machine but, rather recently, work has been done at 15 kv and between 20 and 25 kv. The initial rate of rise of current into the tube was just under 2×10^{12} amp per second, considerably greater than any previously published figure. A notable feature of the waveforms was the apparent absence of clear evidence of pinching. There was some indication of the first pinch, but practically none of subsequent bounces. In this respect the results were similar to those obtained with high power pinch apparatus at Los Alamos, reported by Hagerman and Mather.¹¹ The maximum current observed was ~ 540 ka.

In general, the neutron production seemed to be of the same type as that observed at 10 kv with the small bank, but the yield was greater by a factor of up to 100. The output rises steeply with bank voltage from $\sim 10^5$ at 10 kv up to a maximum $\sim 10^7$ at 25 kv. The duration of the neutron bursts is longer and probably averages between 1 and 2 μ sec. Yields $\sim 10^7$ have been obtained with 1% nitrogen impurity at 100 microns pressure. Again, the yield was observed to be relatively insensitive to an axial magnetic field, and a field of a few hundred gauss had practically no effect. About 1000 gauss produced rather less fractional reduction than with the small apparatus, and yields in excess of 10^6 have been recorded at fields of 2200 gauss.

The current at the start of the neutron burst was ~ 180 ka, roughly the same as in the small machine. As the neutron burst is longer and the current is rising much faster, compared to the 10 kv bank, the neutron emission usually extends up to currents of $\frac{1}{2}$ Ma.

Whereas with the small bank there were indications of difference between the AWRE results and those of Artsimovitch *et al.*,⁷ Anderson *et al.*,⁹ Burkhardt *et al.*,¹⁴ and others, the larger machine shows similar differences more markedly. Yields of $\sim 10^7$ neutrons with 1% impurity, and $\sim 10^6$ with 2000 gauss have not been reported elsewhere. These findings indicate that there may well be very significant differences in the mechanisms of neutron production and in the

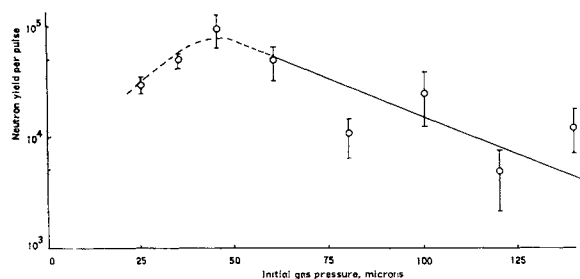


Figure 10. Effect of initial gas pressure on the neutron yield. The dashed part is intended to represent the possible trend at lower pressures. Bank: 120 μ f, 10 kv

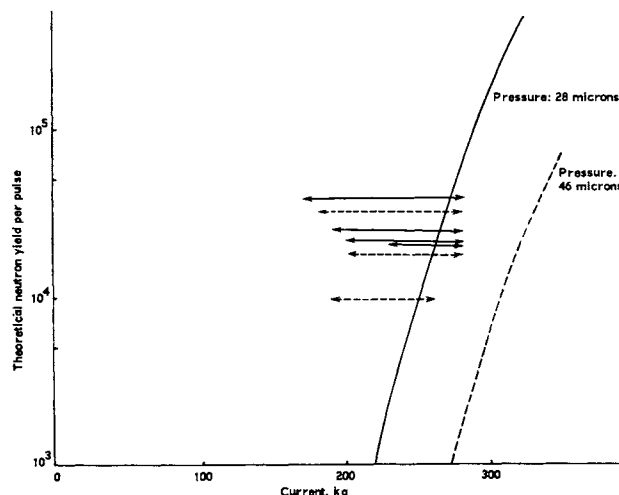


Figure 11. Comparison between observed yield and theoretical thermonuclear yield as functions of current. The curves represent theory for two pressures. The lines of constant yield show the range of current during the time neutrons are emitted, extending from the start to the finish of the neutron burst. Those lines are drawn at a yield corresponding to the total number of neutrons in the burst. The full and dotted lines are at initial gas pressures of 28 and 46 microns Hg respectively. Bank: 118 μ f, 9 kv

low inductance machines a quite different mechanism may be responsible.

DISCUSSION OF RESULTS OBTAINED WITH THE 6000 JOULE BANK

General

At this stage it is possible to consider the results only qualitatively rather than quantitatively. So little is known, at present, about the processes going on in the fast linear pinch that it is not profitable to speculate much on whether the neutrons may or may not be of thermonuclear origin.

As pointed out in a recent paper by Anderson *et al.*,¹⁵ neutrons which in many ways have the expected properties of a thermonuclear yield have been produced in linear tubes by several groups, and shown to be of non-thermonuclear origin. It is necessary to achieve a much more complete understanding of the basic phenomena in the discharge, in particular regarding the stability and energy distribution of the ions, before venturing an opinion on the origin of the neutrons discussed here.

In the discussion which follows, an attempt will be made to bring out some of the more interesting results which have been described, pointing out what features have no adequate explanation, and commenting on any aspects which appear different from the results obtained elsewhere.

Comparison with Theory of the Collapse Time

Rosenbluth¹⁶ has calculated the time taken to pinch for a plasma of infinite conductivity and a constant applied electric field E ; this implies a condenser bank of zero inductance, and the present case

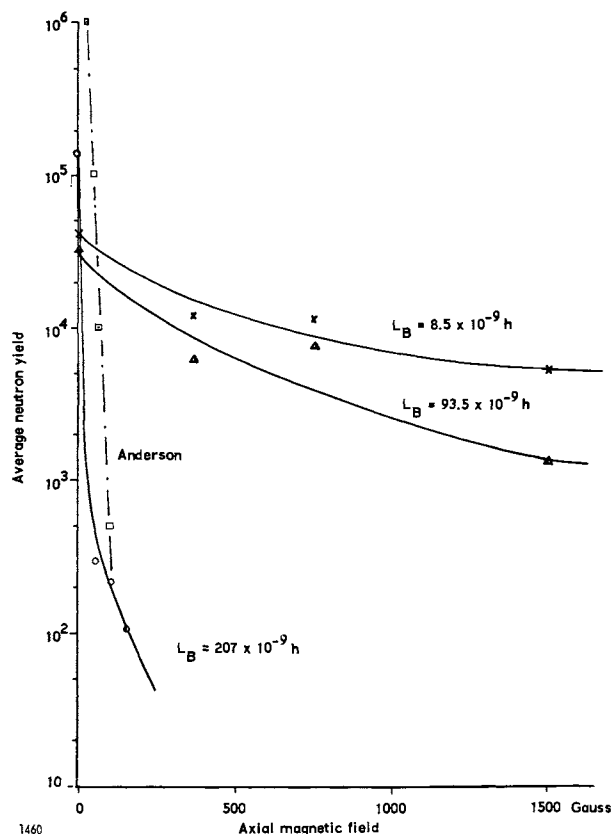


Figure 12. Variation of neutron yield with applied axial magnetic field, for different values of the total external inductance, L_B . Bank: 144 μ f, 9 kv. Initial tube pressure 25 microns Hg. The results of Anderson *et al.*⁹ obtained with a 12.5 μ f, 30 kv bank having an inductance $\sim 200 \times 10^{-9}$ h, are shown for comparison

is probably the best approximation to this in practice which has so far been achieved.

Rosenbluth found the time to the first pinch, t , given by

$$t = \tau R_0 (4\pi p / E^2)^{\frac{1}{2}}$$

where R_0 = tube radius (cm), p = initial gas density (gm/cm³), E = electric field (emu/cm) and τ = a function of the starting inductance of the tube.

He also showed that the results were not much different if one considered a shock wave model or a free particle model. The results shown in Fig. 3 agree with the theoretical dependence on density but the observed times are rather longer. Although this could possibly be due to wall hang-up, an adequate explanation has yet to be found.

General Properties of Neutron Production

One of the most striking observations was the absence, under all conditions which were investigated, of yields of 10^7 – 10^8 neutrons of non-thermonuclear origin, as found by Anderson *et al.*⁹ and Artsimovitch *et al.*⁷ and Burkhardt *et al.*¹⁴ Although the voltage used in the present experiments was generally rather lower, the small inductance of the bank enabled us to obtain maximum currents at least of the same order

as those reported by these workers. In fact, the current at neutron time was higher than that of some workers, e.g., Anderson.⁹ As the dynamic behaviour of the pinch depends basically on the magnetic field acting on it and this in turn depends on the current, high yields of instability neutrons might be expected to appear. It is, however, very likely that the initial conditions of ionisation and current sheath formation have an even greater influence on the whole progress of the pinch than has, till now, been anticipated. This could perhaps explain the differences because the low bank voltage and very high initial dI/dt essentially affect the early part of the discharge. Another possible explanation arises if there is very strong shock heating in the early stages of the collapse; this would lead to a large equilibrium pinch radius, which would delay the onset of $m = 0$ instabilities, and might even prevent the "necks" which Colgate (see Anderson *et al.*⁹) must assume to explain "accelerated neutrons" on his theory.

The length of the neutron burst is probably longer than that observed by Anderson *et al.*⁹ and Burkhardt *et al.*¹⁴, but of the same order as observed by Artsimovitch *et al.*⁷ This could be due to the fact that Anderson used smaller bore tubes. Another factor that is not adequately understood is why the neutron burst always stops before maximum current. On instability theory it could be argued that the pinch had broken up, but bounces have been observed, especially with an axial field, after the neutron burst. On a thermonuclear basis the yield should be greatest at maximum current, but cooling by the electrodes could quench the emission. The theoretical times for this are of the right order; Burkhardt and Lovberg¹² observed bursts of several microseconds duration, extending past the current peak, in a tube two feet in length.

Effect of Impurities

Except for the work of Anderson *et al.*⁹ there are no precise data on the effect of impurities. At impurity contents $\gtrsim 1\%$ the present results are similar to Anderson's, but no marked increase in yield at much lower values was observed. With the large bank, yields $\sim 10^7$ have been observed with an impurity content sufficient to reduce the yield in Anderson's experiments to $< 10^6$.

Effect of an Axial Magnetic Field

The effect of the field is very important because the strong quenching, by small magnetic fields, of the neutrons produced with linear pinch apparatus, observed for example by Anderson *et al.*,⁹ was one of the factors which were hard to reconcile with a thermonuclear origin (neglecting the unlikely occurrence of extremely high compressions $\sim 10^4$ – 10^6). Furthermore, Colgate's⁹ theory of the production of non-thermonuclear neutrons, from the $m = 0$ instability break-up of the pinch, should lead to very strong quenching by small fields. This arises because small fields will stabilise the $m = 0$ instability (as has been observed by Bezbatchenko *et al.*¹⁷ and others).

It follows, in fact, from the stabilising action of fields, that neutrons produced with an appreciable applied field have much less chance of arising from $m = 0$ instabilities, provided it can be shown that the field is trapped. If other evidence that the field is stabilising the discharge can be obtained, and this is perhaps suggested by the form of the dI/dt curve with a magnetic field (see Fig 9), then it is even less likely that the neutrons are produced from instabilities in the way Colgate has suggested.

Effect of Inductance of the Bank

This experiment was largely intended to show whether the differences observed between the present work and work done elsewhere were, in fact, due to the low inductance of the bank. It was also considered of interest to see if the addition of inductance caused the features of the results of Anderson *et al*⁹ to be reproduced.

The total yield of neutrons was not much altered by adding inductance up to 200×10^{-9} h, but there were differences in the behaviour of the yield in the two cases, most markedly in the effect of field, and also in the length of the neutron burst and the current at neutron time. With $\sim 200 \times 10^{-9}$ h, the phenomena observed were very similar to those reported by Anderson *et al*⁹. It will be interesting to investigate the neutron emission at low impurity levels and

also to search for X rays in hydrogen discharges, to see if there is any evidence for the effects reported by Lukyanov and Podgorny¹³.

It may appear strange that there is no very marked change in the yield over such a wide range of inductance, but, on the other hand, there is evidence for two neutron-producing mechanisms, one along the general lines of Colgate's theory, and one by some other process, not necessarily a thermonuclear one. It should be noted that the low-voltage bank was not expected to give a big improvement over other equipment in the shock heating phase. The addition of inductance to the larger machine may enable the two processes to be clearly separated.

CONCLUSION

Most of the results obtained with the small bank have been summarised, and several of the important questions yet to be answered have been indicated. It is hoped that the use of the 45 kJ machine will enable us to clarify our picture of the dynamic pinch as produced with a very low inductance bank, and so help to build up an overall picture of the fast pinch into which all the observations at present showing in several respects such wide divergencies, may be fitted. At the same time the neutron emission has now reached a level at which more definitive experiments bearing on their origin can be carried out.

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Development of a Powerful Discharge in Deuterium

By V. S. Komelkov, U. V. Skvortsov and S. S. Tserevitinov *

The investigations carried out at the Institute of Atomic Energy of the Academy of Sciences of the USSR resulted in the discovery of both neutron and hard X-ray radiation occurring under certain conditions in a powerful pulse discharge in deuterium.¹⁻³

In the present work, the investigations in this field were continued with a view to studying these processes at greater currents and higher rates of increase of the current, by minimizing the circuit inductance and the size of the discharge chambers. Studies were made of the current distribution in the chamber, neutron radiation, electrode-metal vapor movement, and the effect of pre-ionization on the initial stages of the process.

EXPERIMENTAL APPARATUS

The arrangement consisted of capacitors connected in parallel with a total capacity of 130 μ f. The total inductance of the circuit leads and the capacitors was about 15 cm. The spark gaps used for switching were of two types: air gaps, and gaps with a breakable hard dielectric. The construction of the latter switch is described elsewhere. Three types of chambers are used in the experiments:

The chambers of the first type had an inside diameter of 60.0 cm, a height of 17.5 cm, and porcelain walls 1.5 cm thick, with a number of holes for probes. It had two flat aluminum electrodes. In the lower electrode there was a cylindrical crater 4.0 cm deep, with a diameter equal to the inside diameter of the porcelain cylinder. This electrode was provided with a window 3.4 cm high, whose lower edge was at the level of the flat electrode surface. The total length of the window was 50.0 cm and it was divided into separate sections by four metal baffles.

The chambers of the second type had flat copper electrodes, were 4.7-cm high, and were of 18.5-cm inside diameter. The walls were 20.0-cm outer diameter glass cylinders, which were destroyed during each test.

The porcelain chamber of the third type had flat copper electrodes and the same diameter as the second

type, but its height was 25.0 cm. In optical measurements the porcelain cylinder was replaced by a glass one of the same diameter and height. The current and voltage were recorded by double-beam oscilloscopes with the aid of voltage dividers and Rogovsky belts. The accuracy of the measurements was $\pm 10\%$.

Measurements of the field and current distribution in the discharge were carried out by the magnetic probe method evolved by Andrianov, Bazilevskaya and Prokhorov.¹ The 3×6 mm measuring coil had 10 to 20 turns. One end of the coil was connected to a copper tube 4 mm in diameter and 40.0-cm long, serving both as a shield and a conductor. The second lead of the coil was inside the tube.

Both the tube and coil were placed in a capped porcelain tube to completely eliminate any contact with plasma. The connecting cable, amplifier and oscilloscope were insulated from ground. The coils were used to measure the magnetic field (accuracy $\pm 20\%$), and its time derivative, which determines the time at which the magnetic field appears at a fixed point in the discharge (accuracy $\pm 0.5 \mu$ sec). The plasma motion and the variations in the diameter of the pinch were registered by a high-speed camera (SFR-2M)[†] operated either as a streak camera or as a framing camera.

Kerr cells pulsed by a special circuit were also used. Explosive shutters⁴ were used to cut off the background light.

A 0.1 to 0.2-mm streak camera slit was placed at the center of the window (chamber 1) or of the translucent cylinder (chamber 2). To synchronize optical recordings with the current, a spark discharge with a duration of 0.2μ sec was recorded simultaneously by the streak camera and the oscilloscope.

The propagation of metal vapor was investigated as follows: Light from the discharge chamber window fell on the entrance slit of a monochromator (UM-2)[†] passed through the entrance lens, prism, collimator lens and outlet slit of the monochromator to the photocathode of a photomultiplier tube (FEU-39)[†] and through the oscilloscope amplifier (20 Mc band width). The vapor velocity measurements were

Original language: Russian.

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[†] Ed. note: Transliterated designation.

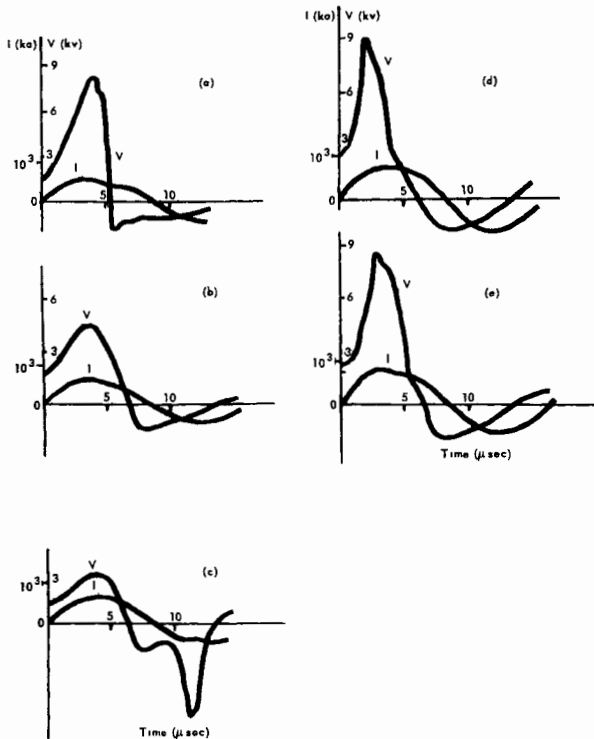


Figure 1. Current and voltage characteristics
(a) 20 kv, 0.05 mm Hg; (b) 20 kv, 0.1 mm Hg; (c) 20 kv, 0.5 mm Hg;
(d) 30 kv, 0.05 mm Hg; (e) 30 kv, 0.1 mm Hg

carried out with the λ 3961 Å aluminum line. The resolving power of the monochromator in this spectral region was 30 Å/mm.

Production of neutrons was observed with a stilbene scintillation counter using the FEU-19M[‡] photomultiplier. The discharge current was simultaneously recorded on the oscilloscope.

To protect the multiplier and the stilbene crystal from hard X-ray radiation, 20-mm thick lead screen was used. A B-type[‡] unit with a β -counter placed in a paraffin block was used as a control. The time variation of the diameter of the charge channel current, voltage across the electrodes, current distribution, neutron radiation and metal vapor velocity along the discharge channel were recorded in chamber 1.

The current, voltage across the electrodes, variations in the pinch diameter (by streak photographs and Kerr cell photographs) and neutron radiations were recorded in chamber 2.

The discharge in chamber 3 was investigated with a framing camera and with 12.0 and 16.0-cm diameter Rogovsky belts.

EXPERIMENTAL RESULTS

The experiments in chamber 1 were carried out at a working voltage of 20 and 30 kv at pressures between 0.05 and 0.5 mm Hg. The initial dI/dt and current

[‡] Ed. note: Transliterated designation.

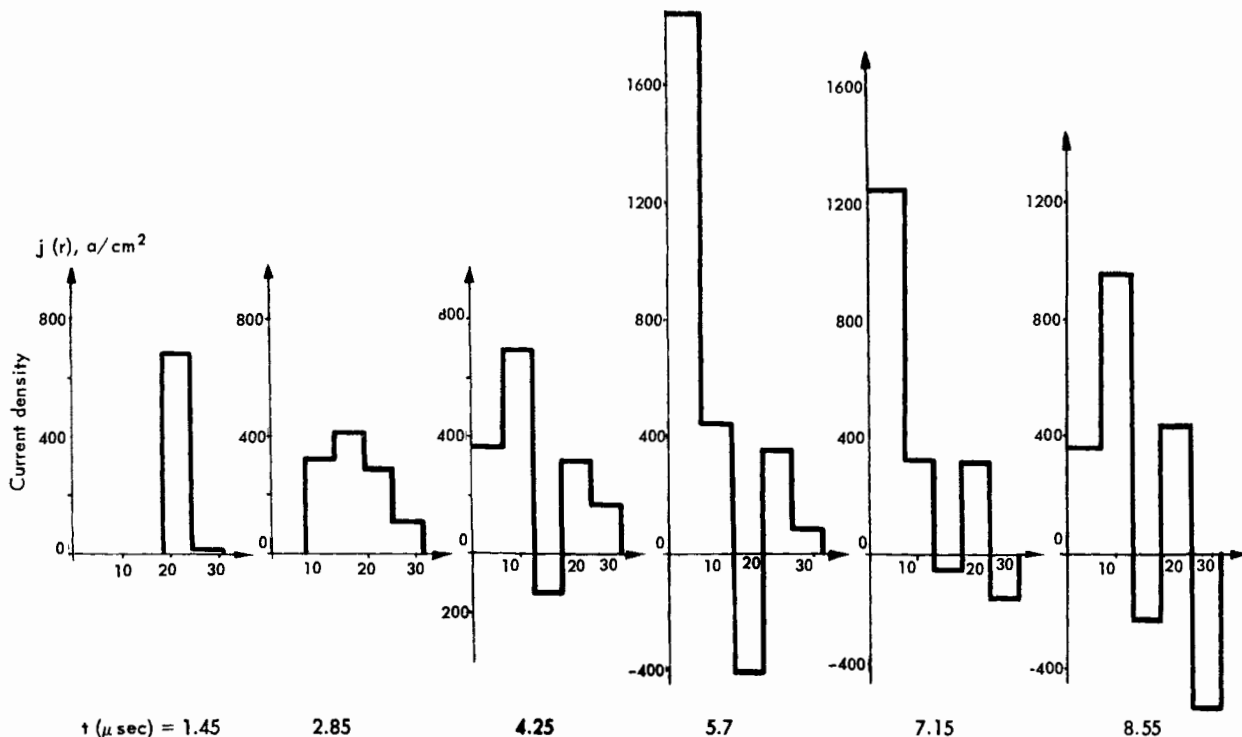
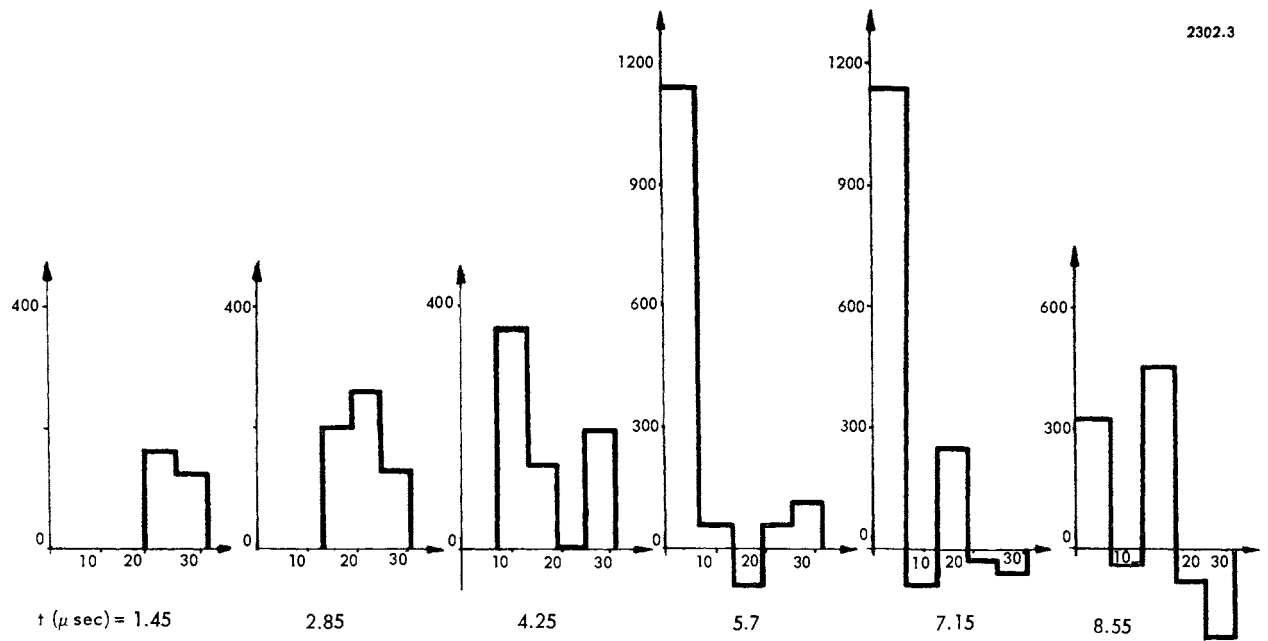


Figure 2. Radial distribution $j(r)$ of the current density; $V = 30$ kv, $P = 0.1$ mm Hg

Figure 3. $V = 20$ kV, $P = 0.05$ mm Hg

amplitude for these voltages varied between the following limits:

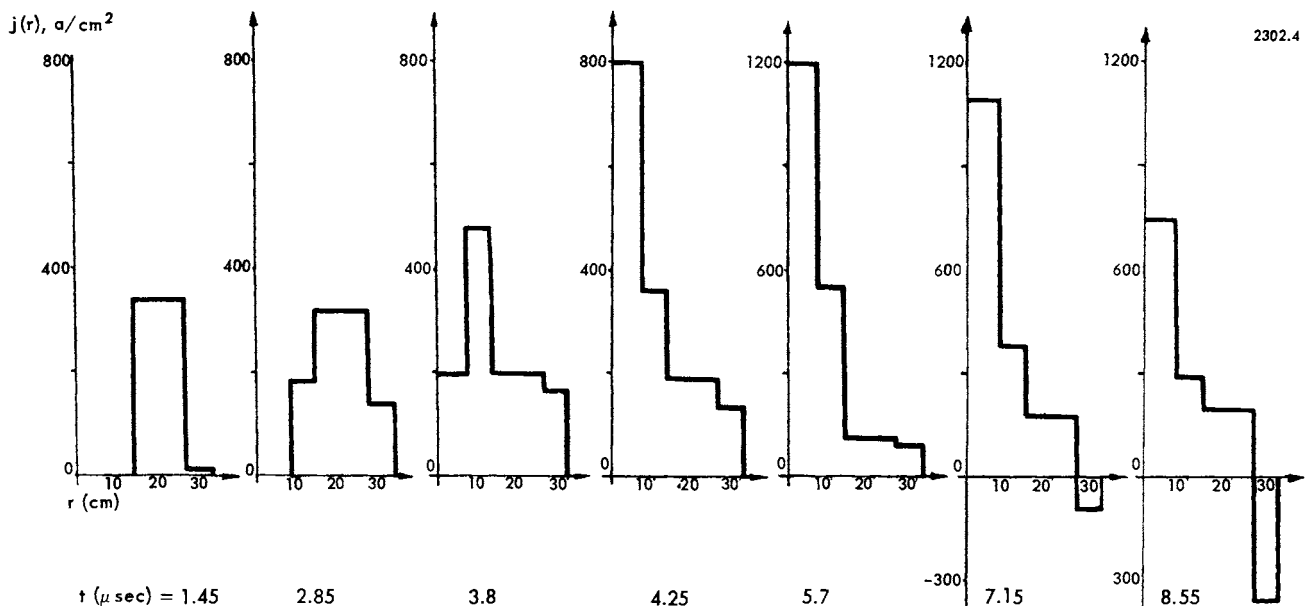
$$V = 20 \text{ kV}, (dI/dt)_{\max} = 3.5 - 4 \times 10^{11} \text{ amp/sec}, \\ I_{\max} = 540 - 600 \text{ kAmp}$$

$$V = 30 \text{ kV}, (dI/dt)_{\max} = 5.5 - 6 \times 10^{11} \text{ amp/sec}, \\ I_{\max} = 750 - 900 \text{ kAmp}.$$

The oscillograms of the current and voltage are given in Fig. 1 and show the typical characteristics for discharges which contract due to magnetic forces produced by the current distribution.

The results of measurements with the aid of integrating magnetic probes are shown in Figs. 2, 3, 4, 5

in the form of the radial distribution of the current density $j(r)$ in the chamber at different times. The oscillographic measurements were interpreted by assuming that cylindrical symmetry of the magnetic field and current are maintained during the discharge. About 50% of the discharge current remained in a layer 5-cm thick near the wall after the current started in the central region of the chamber. Typically, for pressures of 0.1 and 0.05 mm Hg, it is observed that negative currents are generated near the chamber walls. They appear before the main current reaches the center and before shock waves are reflected from the axis of the chamber.

Figure 4. $V = 30$ kV, $P = 0.05$ mm Hg

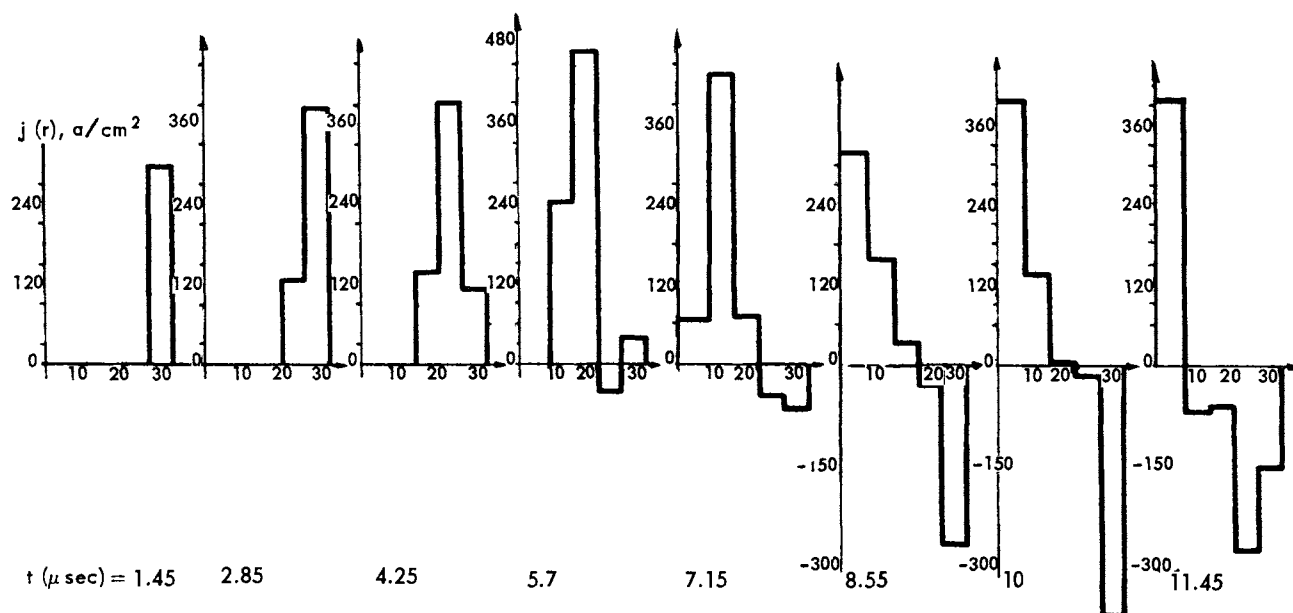
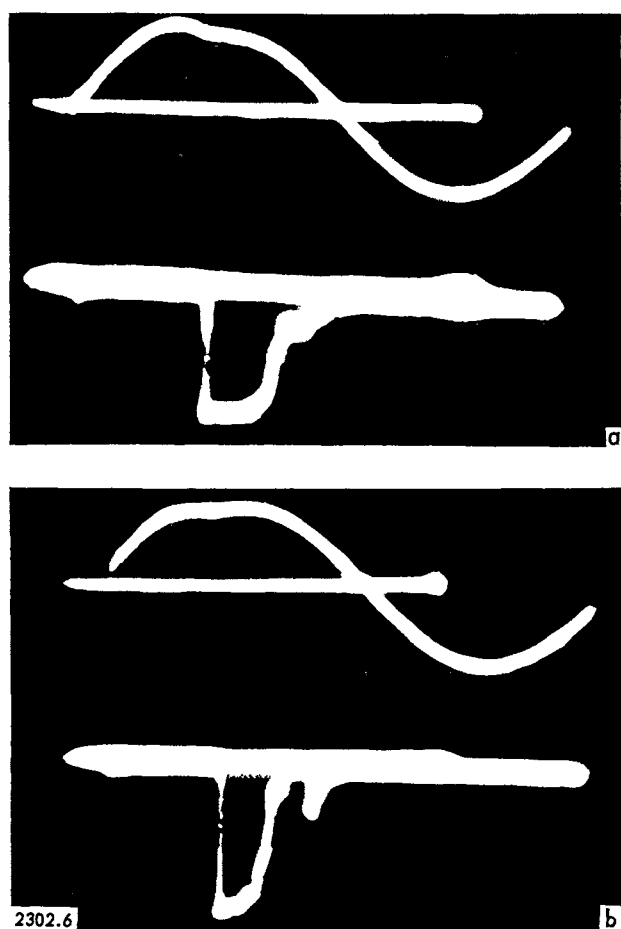
Figure 5. $V = 20$ kv, $P = 0.5$ mm Hg

Figure 6. Oscillograms of the discharge current and neutron pulse; (a) 30 kv, 0.1 mm Hg; (b) 30 kv, 0.05 mm Hg

The oscillograms of the discharge current and the neutron pulse are shown in Fig. 6. For $P = 0.5$ mm Hg and $V = 30$ kv, the current appears in the center of the chamber at $8 \mu\text{sec}$, at $9 \mu\text{sec}$ it reaches its maximum and remains at a high value until $12 \mu\text{sec}$. In the external circuit, the current changes its sign at $8 \mu\text{sec}$.

Streak camera photographs of the discharge show that the optical picture of the motion is in reasonable agreement with the magnetic probe data (Fig. 7). If one takes into consideration the fact that the magnetic probe records the time at which the internal boundary of the moving layer passes a fixed point and that the streak photograph shows its external boundary, the agreement between the two measurements is within the error of the probe measurements.

With $V = 30$ kv, the maximum rate of contraction is 10^7 cm/sec ($P = 0.05$ mm Hg) and 9×10^6 cm/sec ($P = 0.1$ mm Hg). In these experiments the intensity of neutron radiation was notably smaller than it had been when chambers 50 to 90 cm high were used in which the neutron yield sometimes reached 10^9 neutrons/pulse.¹ When the distance between the electrodes is reduced to 5.0 cm with $V = 40$ kv and $P = 0.1$ mm Hg, a small number of neutrons can still be observed within the sensitivity limit of the scintillation counter, which is 5×10^5 neutrons.

When the distance between electrodes is 17.4 cm, a neutron yield of 10^6 in one discharge was recorded with only 30 kv initial voltage, in the pressure range 0.05 to 0.4 mm Hg. The neutrons are produced at $3.8 \mu\text{sec}$ and $4.3 \mu\text{sec}$ after the current flow begins for $P = 0.05$ mm Hg and for $P = 0.1$ mm Hg, respectively. In all cases, the neutron radiation starts before the maximum of the current in the center of the

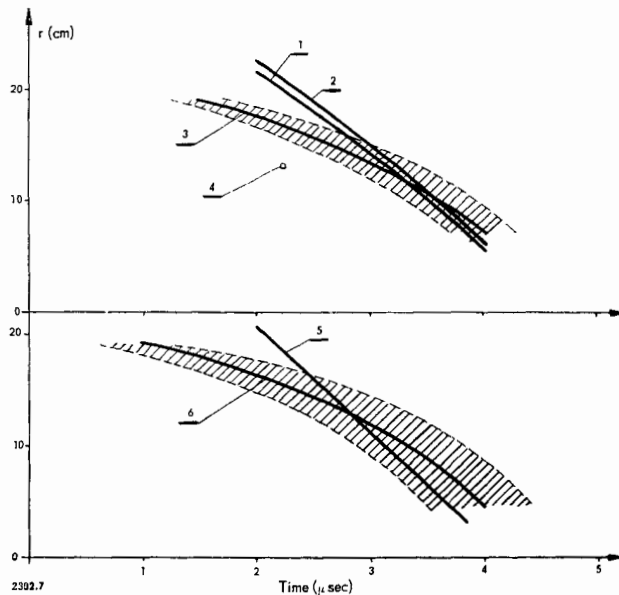


Figure 7. Variation of the pinch radius with time from optical and probe measurements

1. 20 kv, 0.1 mm Hg; optical. 2. 30 kv, 0.1 mm Hg; optical.
3. 20 kv, 0.1 mm Hg; probe. 4. 30 kv, 1.0 mm Hg; probe.
5. 20 kv, 0.05 mm Hg; optical. 6. 20 kv, 0.05 mm Hg; probe

chamber is reached. It can also be seen from the streak photographs (Fig. 8) that the neutrons are produced before the contracted pinch flies apart.

Spectroscopic measurements of the radiation from neutral aluminum atoms showed that the metal vapor in such discharges moves with velocities up to 10^6 cm/sec, and the velocity is independent of electrode polarity. The results of the measurements are given in Table 1, where T = time at which metal vapor appears at the electrodes, v_{av} = average velocity of metal vapor (0–1.5 cm from electrode).

It was noted that the aluminum vapor appeared at a time which corresponded to a definite current density in the discharge, 400 a/cm^2 . It was also found that the average velocities of the neutral aluminum atoms increased with increasing voltage. Typical oscillograms of the current and vapor luminosity in a discharge are shown in Fig. 9.

A discharge in small chambers 18.5 cm in diameter and 4.7 cm in height was investigated at a voltage of 40 kv, initial dI/dt of 1.4×10^{12} amp/sec and maximum currents of 1.4 – 1.8×10^6 amp. The initial pressure in the chamber varied between 0.1 and 10 mm Hg.

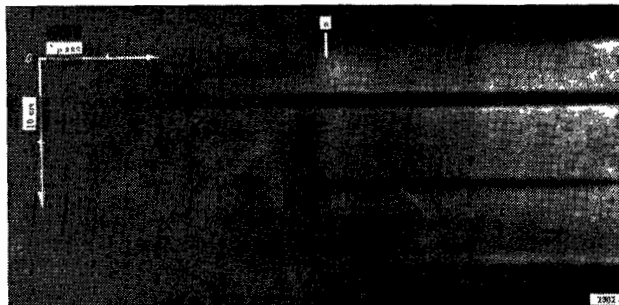


Figure 8. Streak camera photograph of a discharge with $V = 30 \text{ kv}$ and $P = 0.1 \text{ mm Hg}$

Table 1

P (mm Hg)	T , μsec	v_{av} cm/sec.	V , kev.	Polarity
0.03	3.0	6×10^5	30	—
0.1	3.3	7.5×10^5	30	—
0.6	3.0	5.5×10^5	30	—
0.03	3.8	1.2×10^6	30	+
0.1	4.1	8×10^5	30	+
0.6	3.3	6×10^5	30	+

For initial pressures of 10 and 1 mm Hg, many independent, relatively bright channels were observed in the constricting layer. Constriction of the gas occurs at an increasing speed with decreasing pressure. For a pressure of 10 mm Hg the maximum speed is 5.4×10^6 cm/sec. Full constriction is reached near the time of maximum current, 1.4×10^6 amp.

By the time constriction is complete, the gas velocity reaches 8×10^6 cm/sec; with a current of 1.6×10^6 amp, pinch diameter of 3.0 cm and initial pressure of 1 mm Hg.

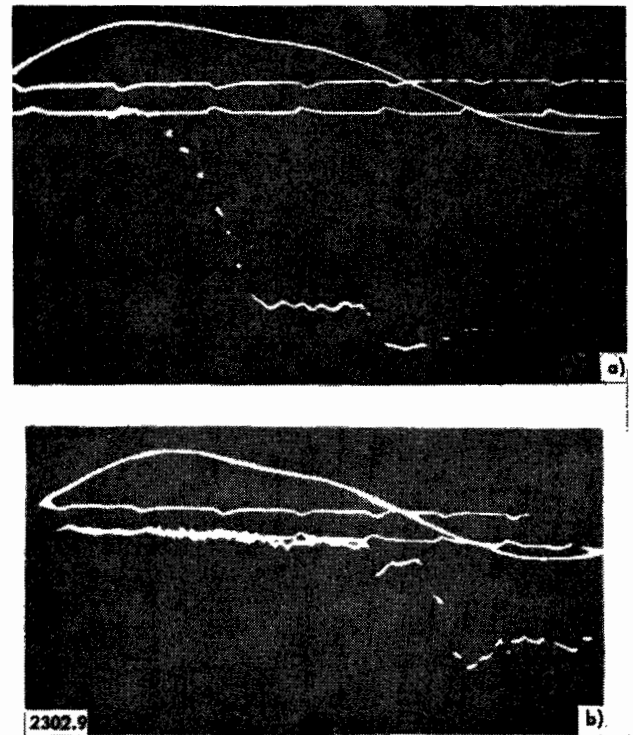


Figure 9. Oscillograms of the current and luminosity with $V = 20 \text{ kv}$, $P = 0.1 \text{ mm Hg}$
(a) near the electrode, (b) 1.5 cm from the electrode

Streak photographs, Kerr cell photographs, and oscillograms of the discharge current and voltages are shown in Fig. 10.

The maximum compression ratio of the plasma was estimated to be 140 from the square of the ratio of the initial to final optical pinch diameter. The expansion is accompanied by stratification of the gas, part of which moves back to the center of the chamber, where the luminous pinch and some current still remain. The stratification takes place three times with a time interval of 0.2 to 1.0 μsec . The Kerr cell

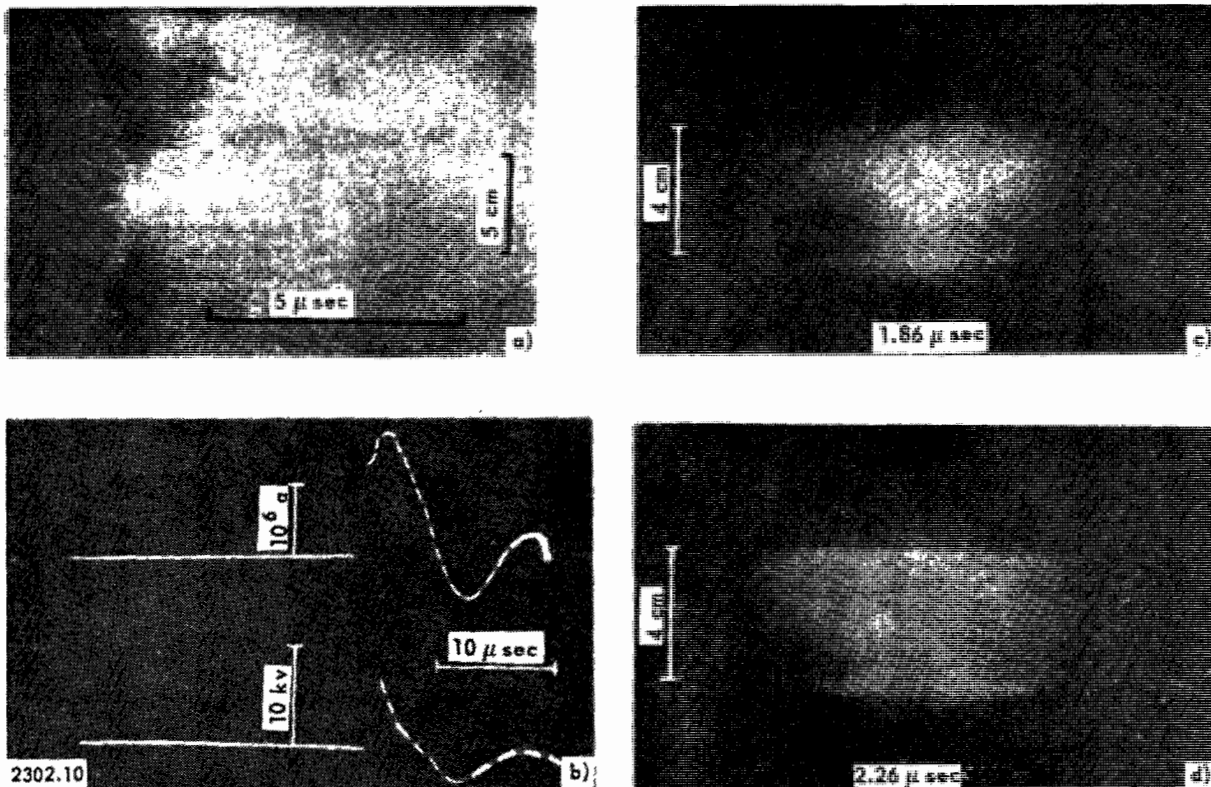


Figure 10. (a) Streak camera photographs, (b) current and voltage oscillograms, (c, d) Kerr cell pictures. $P = 1$ mm Hg

photographs show that the stratification is symmetrical. As the pressure fell to 0.1 mm Hg the rate of contraction rose to $1.1-1.3 \times 10^7$ cm. Three successive constrictions were observed, each originating from the wall. In the first constriction the minimum pinch diameter was 1.5 cm and the total current was 1.1×10^6 amp. In the second and third constrictions the currents were 1.2×10^6 and 1.3×10^6 amp, respectively. Neutrons were not observed with a lower limit of sensitivity of 10^6 neutrons/pulse.

Experiments with a discharge chamber 18.5 cm in diameter and 25.0 cm in height were aimed largely at finding the effect of pre-ionization on the subsequent development of the discharge. Pre-ionization was accomplished by means of a discharge from an auxiliary 22 μ f condenser bank rated at 20 kv.

After the pinch formed by the auxiliary current detached itself from the wall and traveled some distance towards the center, the main 20 kv bank was discharged. The maximum dI/dt and current amplitude for the first and second discharges were 1.4×10^{10} amp/sec, 5.5×10^4 amp, and 10^{11} amp/sec, 4.5×10^5 amp, respectively. The experiments were carried out at an initial pressure of 1 mm Hg. Typical oscillograms which show the total current and the current in the Rogovsky belt, with the main and auxiliary capacitor banks operating separately and jointly, are shown in Fig. 11. The experimental results are summarized in Table 2, where τ is the time lag between the auxiliary and main discharges, t_0 is the time at which current appears in the Rogovsky belt and I is the total discharge current at this time.

With pre-ionization, the period during which the conducting sheath remains near the wall is cut at least by half. The velocity of the current sheath, as obtained from oscillographic measurements of the time at which current appears in the belts and from optical measurements of the motion of the luminous layer, differs by not more than $\pm 15\%$.

DISCUSSION

In discharge chambers where the space between the electrodes is small, the influence of such processes as heat conduction from the constricted pinch to the electrodes and the occupation of the chamber by metal vapors should be taken into account. As has been demonstrated by S. I. Braginsky and V. D. Shafranov, heat conduction to electrodes during the current flow is insignificant for times

$$t < (3l/5)(e^2N/c^2kT)^{1/2},$$

where l is the chamber length, T the particle temperature, c the velocity of light, e the electronic charge,

Table 2

Belt diameter	Energy Source	$\tau(\mu\text{sec})$	$t_0(\mu\text{sec})$	$I(\text{amp})$
160	Auxiliary discharge		5.8 ± 0.4	42×10^3
	Main discharge		1.4 ± 0.1	190×10^3
	Joint	6.4 ± 0.1	0.7 ± 0.1	
120	Auxiliary discharge		10.0 ± 0.3	50×10^3
	Main discharge		2.7 ± 0.2	200×10^3
	Joint	6.9 ± 0.4	1.2 ± 0.3	

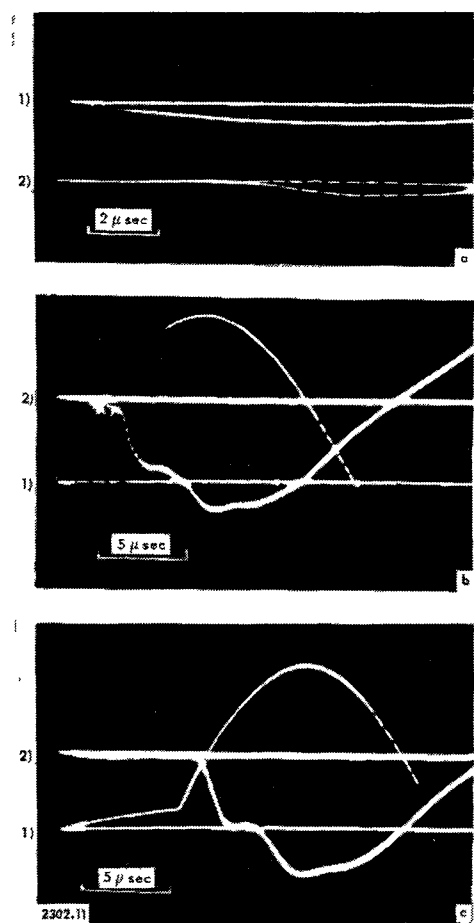


Figure 11. Oscillograms of (1) total discharge current and (2) current in a Rogovsky belt obtained with (a) auxiliary discharge and 16 cm belt, (b) main discharge and 12 cm belt and (c) joint discharge and 12 cm belt

k the Boltzmann constant, and N the number of particles in the discharge chamber. For chambers 4.7 cm in height, at a temperature of 300 eV and an initial pressure of 0.1 mm Hg, this time will be about 3 μ sec.

High-velocity metal vapors which move into the discharge are observed in regions where the current density is relatively high immediately before, or at the time of, maximum constriction. However, these velocities are such that in chambers of the first type, at a pressure of 0.03 mm Hg, the influence of the metal vapor is important 8 μ sec after the time of maximum constriction. In chambers of the second type at a pressure of 0.1 mm Hg, metal vapors appear 3 μ sec after the time of maximum constriction.

This process is roughly similar to that previously described.^{1,7} However, the experiments described here have provided the following additional information:

1. With the exception of the initial time when the discharge detaches itself from chamber walls, the luminosity and current boundaries coincide within the experimental accuracy. The initial discrepancy is explained by the fact that the differentiating magnetic probe responds to a build-up of current

inside the stationary conductor. Optical measurements with chambers of the second type show that the width of the luminous layer between the shock front and external boundary, for pressures of 10, 1.0, and 0.1 mm Hg are 2.7, 2.2, and 1.0 cm, respectively. Should the plasma conductivity be determined from the width of the luminous layer, which can be 2 to 3 times the skin-depth, it will be $0.5\text{--}1.4 \times 10^{14}$ cgse, which corresponds to an electron temperature of 3.5–6 eV.

The main discharge characteristics are in reasonable agreement with the hydrodynamic theory developed by Leontovitch and Osovetz⁵ and with calculations made by S. I. Braginsky, I. M. Gelfandt and R. P. Fedorenko.

2. The maximum compression ratio obtained from the optical measurements with chamber 2 was 140, which greatly exceeds the maximum compression ratio predicted by the hydrodynamic theory. The actual ratio of the initial and final (maximum) densities may be somewhat smaller because all the particles are not swept in by the contracting current sheath.

3. The current in chamber 1, with a pressure of 0.05 mm Hg and a voltage of 30 kV persists in the central zone (14 cm in diameter) for 4 μ sec which markedly exceeds the constriction time obtained from calculations. Optical measurements carried out with chamber 2 give similar results.

4. About 50% of the total current is concentrated in the center of the chamber. Characteristically this portion of the current grows after the maximum constriction is achieved. In measurements obtained with chamber 3, about 20% of the current remains near the wall in a layer 0.5-cm thick. This is probably due to the evaporation of materials with high atomic number from the chamber wall. Pre-ionization of the gas by a discharge shortens the time during which large currents flow near the wall and this reduces the energy losses to the wall.

The production of reverse currents at certain times will result in the formation of a closed circuit inside the chamber. This causes the current to grow in the center of the chamber at the end of the first and beginning of the second half cycles.

As follows from theory,⁵ the circuit parameters determine the rate of contraction and the gas temperature. In Fig. 12 the dependence of the maximum rate of contraction on voltage and the initial induct-

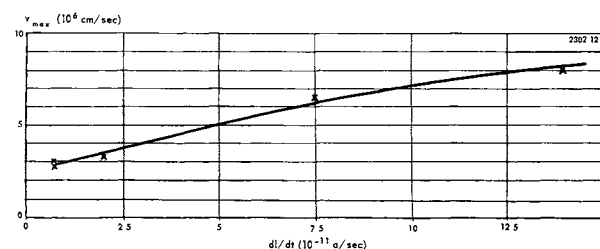


Figure 12. Maximum speed of contraction as a function of $(dl/dt)_{initial}$ for $P = 1$ mm Hg

ance is shown. This data was obtained with different set-ups in chambers with the same diameter at a pressure of 1 mm Hg.

According to existing theory, the thermal energy of the gas at the time of maximum constriction is approximately twice as great as the maximum kinetic energy during the constriction process. From these assumptions the maximum temperature in chamber 2 at pressures of 10, 1.0 and 0.1 mm Hg is estimated to be 40, 120 and 340 eV, respectively. In chambers of the first type, the gas temperature, estimated on the same basis, is 200 eV (at $P = 0.05$ mm Hg, $V = 30$ kV).

The neutron radiation from thermonuclear processes that would be expected at such temperatures could not be recorded by the available apparatus. Therefore, the radiation which was detected probably has the same origin as described earlier.^{1, 6} Its occurrence in pinches 5 cm long testifies to the fact that the accelerating processes which presumably take place occur in regions of this length or less. A decrease in the radiation intensity with decreasing pinch length probably testifies to the fact that the radiation sources are distributed along the entire pinch column, which is in accordance with data obtained previously.⁶

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Plasma Motion in Powerful Discharges

By V. S. Komelkov *

This work describes the results of research on plasma motion in powerful pulse discharges conducted with the aid of streak, Kerr cell, and framing camera photographs. Oscillograms were taken of the discharge current and the voltage between electrodes. A study was made of discharges in which plasma contracts along three axes and gas flows from the center of the chamber to the periphery. In some types of discharges there can be observed a relatively higher stability of the plasma than in the ordinary cylindrical constriction.

Up to now in the study of powerful discharges one has obtained and examined only discharges with a cylindrical skin-layer, which in its motion constricts gas to the cylinder axis.¹⁻³

In this work we relate the results of research on other kinds of motion, obtained mainly by optical methods. We assume, however, that conditions in our experiments are sufficiently close to conditions in the discharges with cylindrical symmetry, where a good correspondence is found between the motion of the current front and of the luminosity. This can be corroborated by the configuration of the plasma layers corresponding to the expected current distribution. The research was carried out in 1955-1957.

The work was accomplished through the efforts of D. C. Parfenov, junior scientist; B. P. Sournin, engineer; and N. F. Vladimirov, technician.

PROCEDURE

The discharge took place in a cylindrical glass chamber with an internal diameter of 190 mm, height of 250-150 mm.

The bank of capacitors (IM-50) † with a total capacity of 130 μ f was charged to the operating voltage of 20-40 kv. The maximum current in the discharge reached 9×10^5 amperes.

In a number of experiments the circuit inductance at the start of discharge did not exceed 55 cm. The discharge was photographed with the SFR-2M † high-speed camera, operating in the modes of framing camera and streak camera. In the former the

exposure of each frame was 0.5 microseconds. The central part of the chamber was photographed in the streak mode through a slit. The entrance slit of the camera was perpendicular to the axis of the chamber.

In some experiments one of the halves of the chamber was photographed by Kerr cells with a 0.2 μ sec exposure. Photographs were synchronized with the current oscillograms by means of a special light pulse of 0.2-0.5 μ sec duration.

The chamber was first evacuated to a pressure of 10^{-5} mm Hg and then filled with deuterium of 98.8% purity. With the aid of a Rogovsky belt and voltage divider, the discharge current and electrode voltage were registered by a double-beam oscilloscope, the OK-17M.†

RESULTS

In the first series of experiments a discharge with needle electrodes was investigated. To do this the metal flat flanges were covered with a rubber insulator with a hole in the center, where copper rods (needles) were clamped, projecting 1.5 cm above the dielectric surface. A series of experiments was carried out with flat electrodes for comparison.

In Figs. 1a, b, c Kerr cell and streak photographs of the discharge are illustrated, with current and voltage curves drawn on the latter. The discharge took place at an operating circuit voltage of 40 kv and an initial pressure of 10 mm Hg in chamber. The Kerr cells, which photographed half of the chamber, have registered skin-layer motion from one of the electrodes. The discharge is reshaped into a thin filament near the needle and begins to move along the chamber axis even before the current leaves the chamber side walls. The luminous current layer rapidly takes the shape of a cone with the apex directed into the chamber. Behind it a well-formed current cord is seen clearly. The two cones originating on different electrodes meet approximately in the middle of the chamber.

After their meeting the cord continues to be stable, while the cone continues to expand in the direction of the chamber walls. The average velocity of the cone apex in the period of 0.6-1.0 μ sec, as calculated from the Kerr cell photographs, is 5.3×10^6 cm/sec.

Original language: Russian.

* Academy of Science of the USSR.

† Transliterated designation.

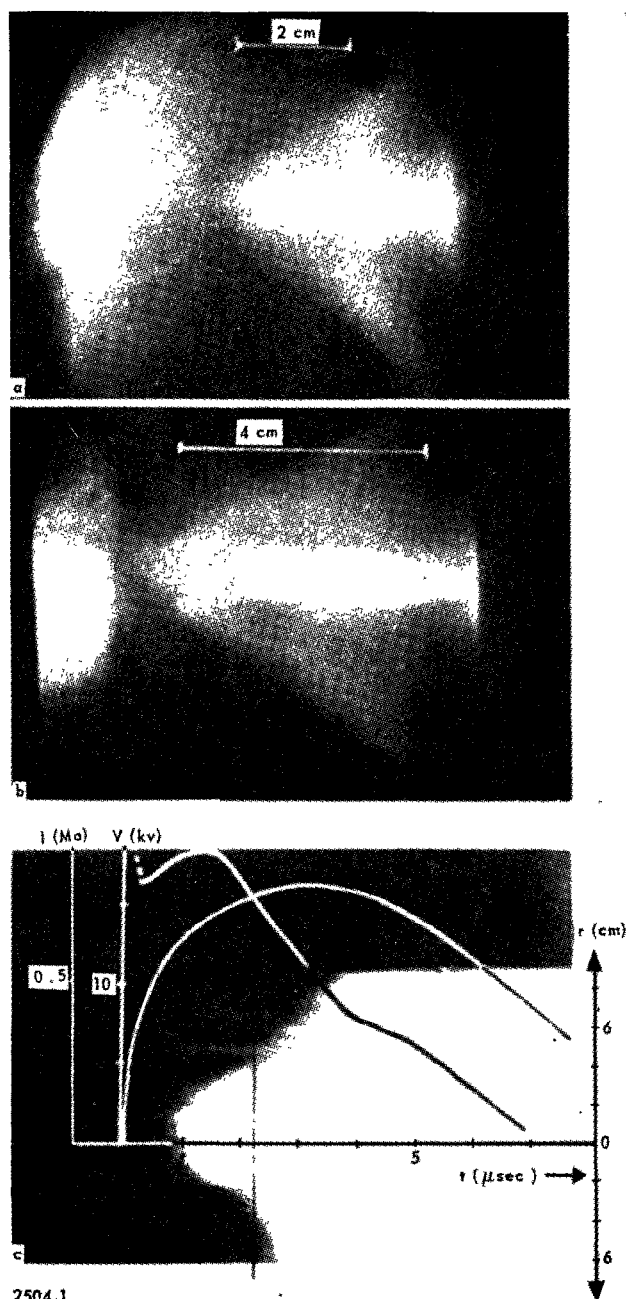


Figure 1. Discharge in the chamber with needle electrodes at a deuterium initial pressure of 10 mm Ng: (a) Form of the discharge near one of electrodes after 0.6 microseconds following the start of the discharge; (b) the same 1.0 microseconds after the start of discharge; (c) streak photography of the same discharge with current and voltage curves obtained from oscillograms plotted on it

The cone expansion rate (according to streak photography) is 2.7×10^6 cm/sec.

The discharge-column diameter alters slightly during the first two microseconds. Near the electrode it is 0.5 cm; at a distance of 3 cm it is 0.8 cm.

Despite the fact that the discharge current continues to grow, the column diameter decreases, becoming 1–2 mm smaller. From the study of current and voltage curves we learn that:

(1) at the initial moment the circuit inductance is low; at this time a weak luminosity can be observed in the entire chamber; (2) after $0.2 \mu\text{sec}$ the voltage increases and by $1.0\text{--}1.5 \mu\text{sec}$, when the cone surfaces meet, the discharge inductance reaches a maximum; (3) by the decline of current and voltage in the period of $1.5\text{--}7.0 \mu\text{sec}$, one can conclude that the circuit inductance remains high and consequently the greater part of the current flows in the centre of the chamber. In Fig. 2 is given a streak photograph of a discharge in the same chamber at an initial deuterium pressure of 1 mm Hg. Behind the luminous front a well-formed discharge cord is seen.

The expansion of waves from the centre to the chamber wall takes place at a rate of 7.7×10^6 cm/sec.

Data on the discharge regime in the case of flat and needle electrodes are given in Table 1.

Table 1

	Initial pressure 10 mm Hg	Initial pressure 1 mm Hg
Longitudinal (axial) velocity of plasma with needle electrodes, cm/sec.	5.2×10^6	7.6×10^6
Rate of plasma constriction with flat electrodes, cm/sec	3.0×10^6	4.8×10^6

In the first case the current-sheet velocity is twice as great. Similar tests were made with chambers 250 mm in height (of the same diameter) with porcelain flanges, in the centre of which copper needles were placed. The distance between the ends of needles was 240 mm. To save the chambers from destruction, the operating circuit voltage was lowered to 20 kv. The maximum current in the discharge did not exceed 3.1×10^5 amperes. The external circuit inductance was increased, and so the current in the discharge did not depart much from a sinusoidal wave-form.

As indicated by oscillograms, the inductance of the discharge in the chamber with needle electrodes in the first half-cycle, with pressures of 10.0, 1.0 and 0.1 mm Hg, is approximately 35% higher than in the chamber

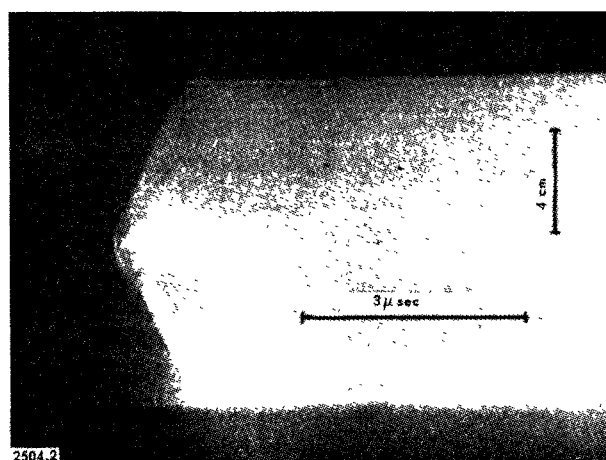


Figure 2. Streak photograph of the discharge in a chamber with needle electrodes, at an initial pressure of 1 mm Hg

with flat electrodes, in which a cylindrically symmetric constriction can be observed.

The difference in inductance shows that the mean effective diameter of the current cord in the first case is considerably smaller. In photographs obtained with the framing camera, it can be seen that at a pressure of 10 mm Hg the 2-cm diameter current filament is formed at needle electrodes and is moving inside the chamber at the first microsecond, while in the cylindrical constriction the diameter of the discharge channel exceeds 5 cm during the first 7 microseconds.

Beginning with the third and up to the twelfth microsecond the maximum diameter of the current filament in the case of needle electrodes is no more than 7 cm.

In a chamber with flat electrodes, a channel with such a diameter exists for about 2 microseconds.

At a pressure of 10 mm Hg the axial motion of the formed filament and the conical current surface is difficult to discern, while at a pressure of 1 mm Hg the process can be observed in detail. A filament and a conical surface, similar to those recorded by the Kerr cells, arise during the first 0.5 microseconds and are moving at a velocity of 3.4×10^6 cm/sec. The detachment of such a surface from the chamber walls occurs by the second microsecond.

The course of the process, as far as it can be depicted on the basis of optical observations, is shown in Fig. 3. The arrows indicate the probable direction of the current flow. The current surface, while constricting in the axial and transverse directions under the effect of magnetic forces, continually changes its form. The transverse constriction continues after the pinch motion has ceased in the third microsecond and axial conduction has taken place.

However, a full constriction, such as takes place in the case of cylindrical symmetry, would not occur; first because the axial current in the centre of the volume generates a magnetic field and secondly because the density of particles in the central region must be much higher than in the ordinary cylindrical constriction, since the plasma motion described is accompanied by a clustering of particles in a small volume in the centre of the chamber.

The equilibrium in the central part lasts more than two microseconds and is broken on the eighth microsecond, when the diameter begins gradually to increase. The length of current filaments joining the central region to the electrodes reaches 6.5–8 cm, and their stable existence lasts more than 12 microseconds.

In the chamber with flat electrodes the constriction begins by the fourth microsecond and lasts no more than 0.5 microsecond. The channel rapidly loses the cylindrical form in all its length, including the near electrode zone. With the cylindrical symmetry the average rate of constriction is 1.5×10^6 cm/sec, which is nearly three times as small as the rate of the current-filament advance in the axial direction in the chamber with needle electrodes at the same pressure.

At the pressure of 0.1 mm Hg (Fig. 4) the rate of axial motion (needle electrodes) exceeds 10^7 cm/sec, whereas in the cylindrical compression it is not more than 4.5×10^6 cm/sec. Lowering the pressure to 0.1 mm Hg barely changes the structure of the discharge. Current filaments forming at the electrodes are up to 10-cm long, their diameter not exceeding 1 cm. They advance into the interval at a rate of 5×10^6 cm/sec. After three microseconds from the beginning of the discharge the transverse constriction in the central part of the chamber is completed. The filament connecting two electrodes remains stable throughout its length for two microseconds, and then gradually begins to disintegrate in the centre, where major disturbances and a relatively high density of particles take place.

As we have said, a stable filament can be observed only in regions of the discharge where there is no sharp increase in particle density and no constriction waves.

A number of experiments were set up with chambers in which the region of disturbances was further reduced. To achieve this, one of the rod-shaped electrodes was replaced by a flat one. To this was fastened a cylinder 240-mm high and 160 mm in diameter, formed by sixteen 5-mm-diameter rods of stainless steel. The upper and lower ends of the rods were welded to a ring of the same diameter as the cylinder.

The discharge started between the rod-like electrode and cylinder. The central part of the luminous layer moved rapidly into the interior of the chamber, forming a conical current surface, but its periphery remained almost stationary. This being so, the transverse constriction ceased and the expulsion of gas to walls of the chamber followed.

When the initial pressure was 1 mm Hg the discharge behaved as follows: by the end of the first microsecond the initial flat surface of the luminous layer is transformed into a conical one. The top of the latter moves at a velocity of 4×10^6 cm/sec and on the sixth microsecond reaches the opposite flat electrode. The cone expands to the walls of the cylinder at a rate of 1.3×10^6 cm/sec. Inside the cone there is seen a current filament about 1 cm in diameter, which can be detected against the general background up to the tenth microsecond.

At a pressure of 0.1 mm Hg, the rate of advance of the cone into the interstice increases up to 8×10^6 cm/sec, and the rate of the expansion in the lateral direction up to 2.2×10^6 cm/sec, i.e., nearly twice as much compared with the preceding case.

When gas forced out of the central part impinges on the walls, the walls become luminous; it becomes difficult to discern the current filament against this background. The picture of the plasma motion can be clearly seen in the photograph (Fig. 5). The luminescence between the peak and the cylinder edge indicates that a portion of the current is short circuited by this shorter length, by-passing the current filament.

It is possible to produce a constriction which is principally in the longitudinal direction by means of insignificant changes in the chamber.

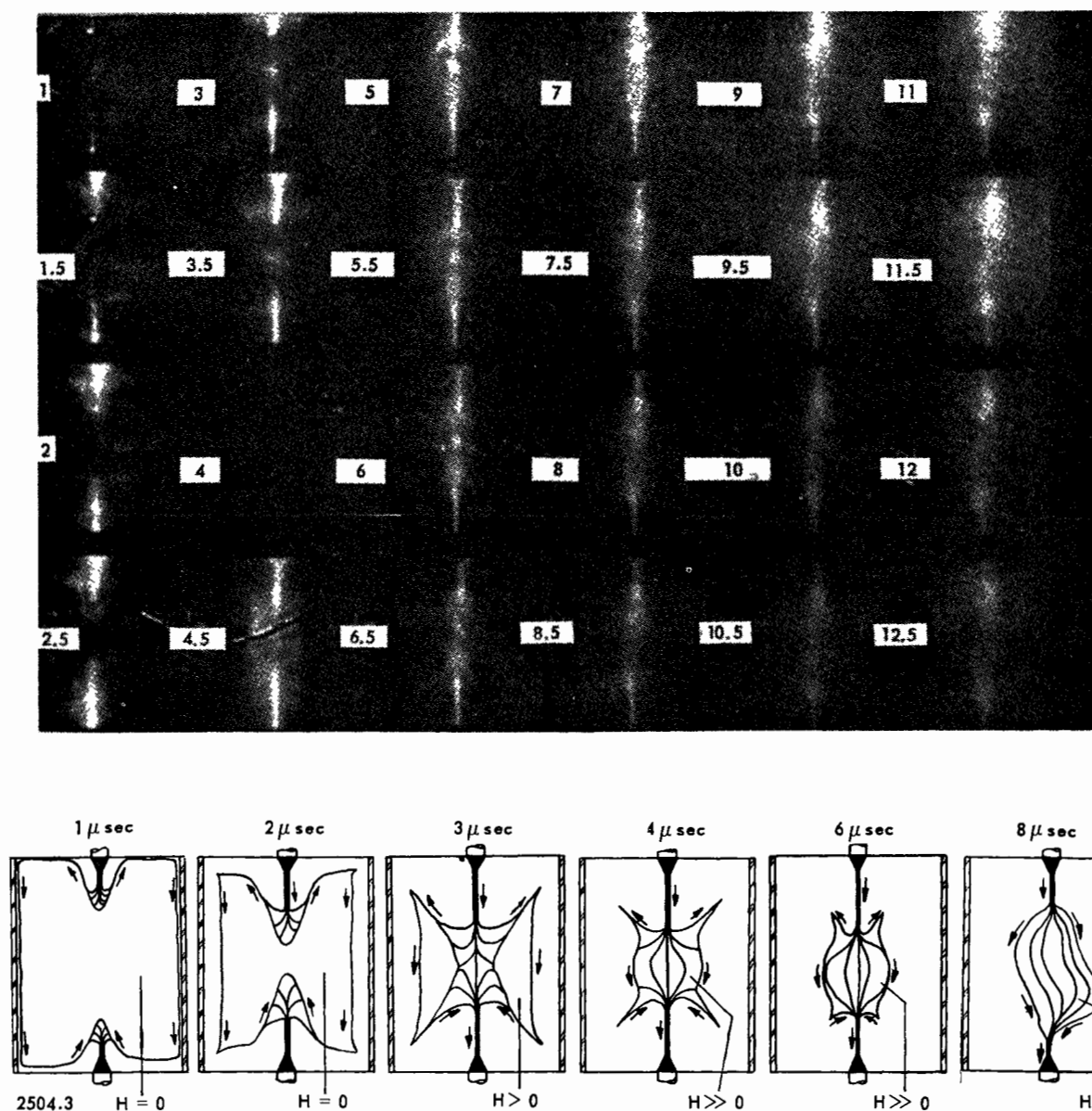


Figure 3. (a) photographs and (b) motion picture of the discharge plasma, in the chamber with needle electrodes, at an initial pressure of 1 mm Hg

For this, cylinders 80 mm in diameter and 85 mm in height, connected to one terminal of the capacitor bank, were placed on the insulating flanges of the chamber, instead of needle electrodes. The above-mentioned cylinder of rods was connected to the other terminal. The same result was obtained when the cylindrical electrodes were joined to different terminals of the bank, and the big cylinder was at a floating potential; the change in polarity of electrodes had a very small effect in both cases. In the whole range of pressures studied the initial current surface is formed and moves between the cylindrical electrode and the cylinder.

At a pressure of 10 mm the luminosity front is not homogeneous. The plasma consists of separate

independent channels. The average rate of motion is 1.8×10^6 cm/sec. The meeting of fronts takes place by the seventh microsecond. At the place of their contact there arises a bright luminous disc, 4-5-cm thick. The picture of the motion for a pressure of 1 mm Hg and the photograph of the same process are shown in Fig. 6.

The transverse constriction is observed only when the plasma reaches the edge of the cylindrical electrode and then it quickly comes to an end. By the time both fronts meet (by the fifth microsecond) their form becomes oval-shaped. Each front is advancing at a rate of 3×10^6 cm/sec, which is more than twice the rate of compression under conditions of cylindrical symmetry. The gas is contracted into

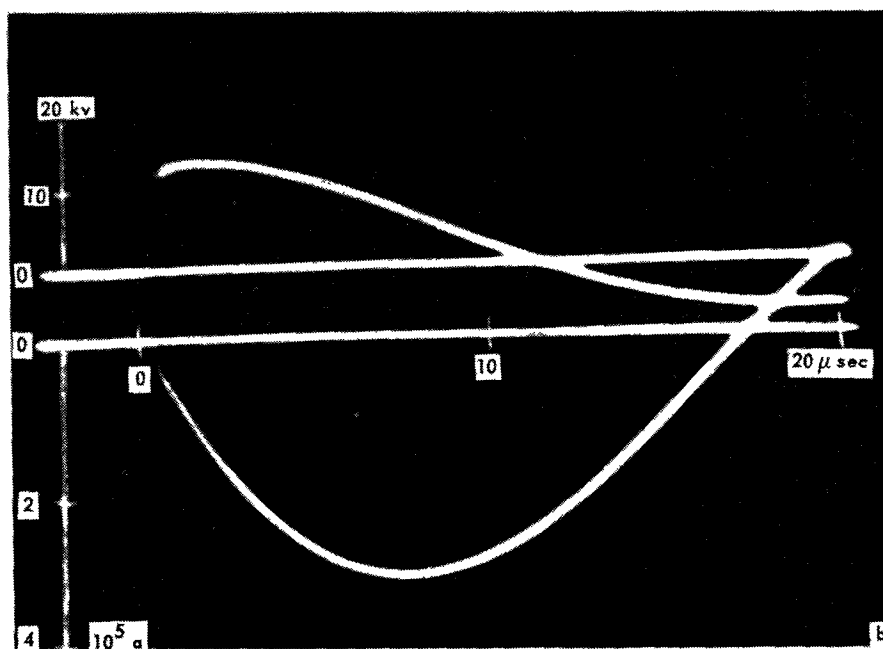
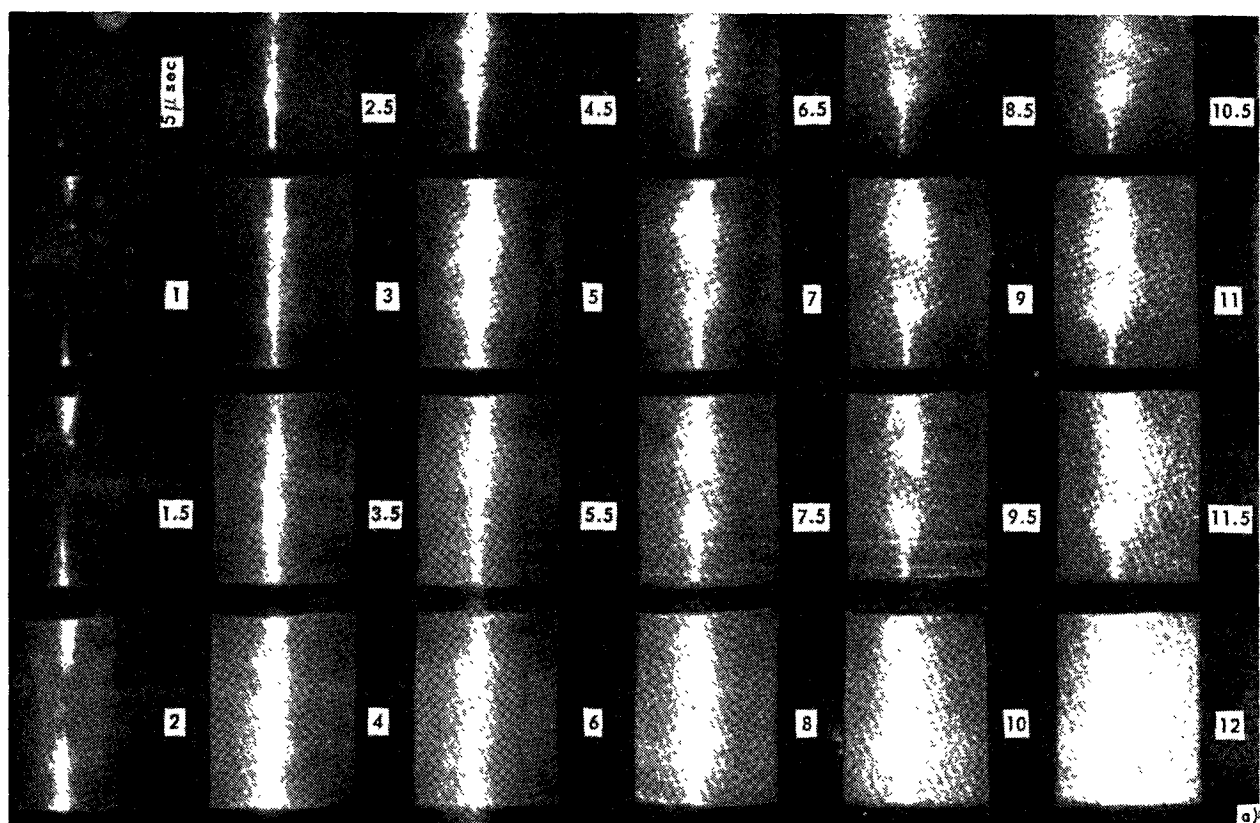


Figure 4. Discharge in the chamber with needle electrodes at an initial pressure of 0.1 mm Hg:
(a) Photograph obtained with a framing camera; (b) Oscillogram of voltage between the chamber electrodes (1) and discharge current (2)

a disc shape, first in the centre, and then in the more remote parts, a part of the neutral gas being forced to the chamber walls.

The disc is 2–2.5-cm thick, and persists for two microseconds. In cylindrical compression, the analogous period lasts not more than 0.5 microseconds.

The luminous disc, increasing its diameter, touches the chamber walls by the eighth microsecond. There then begins an expansion of the luminous layer at a rate of 1.5×10^6 cm/sec. It takes place primarily at the chamber walls where there are more neutral particles. At a pressure of 0.1 mm Hg the rate of

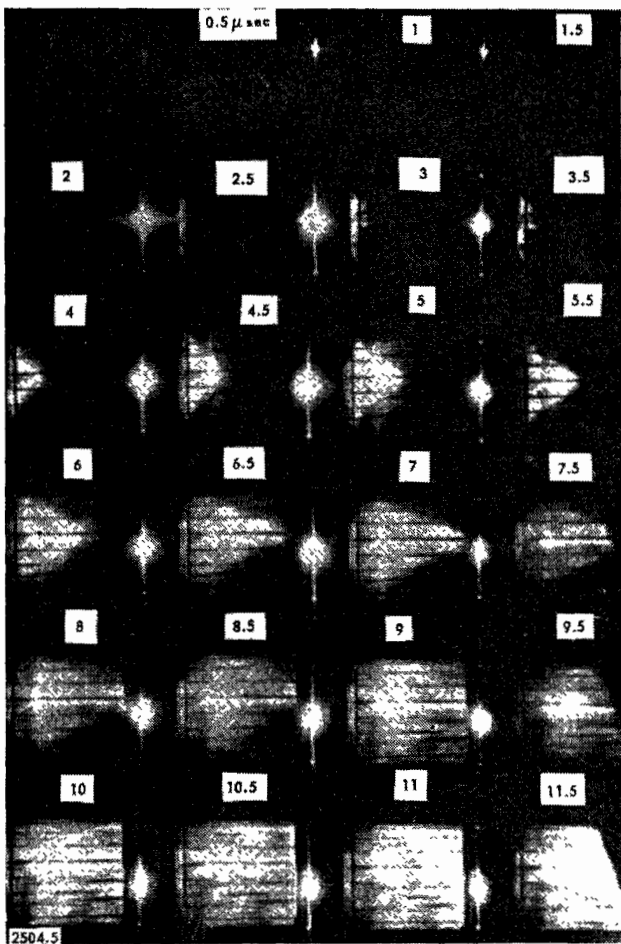


Figure 5. Photograph of a discharge in the chamber with needle and cylindrical electrodes at an initial pressure of 1 mm Hg

motion of the fronts has grown to 5×10^6 cm/sec. Their contact took place by the third microsecond; then the luminous disc was 1.5–2.5-cm thick and it existed in the compressed state for more than 1.5 microseconds. Just as in the preceding case the expansion of luminosity zone began at the wall of the chamber. In 4 microseconds luminosity filled up the whole volume between the ends of cylindrical electrodes. No success has been achieved in establishing the exact significance of the luminosity spreading rate, however it is known that it is more than 2×10^6 cm/sec.

In the following half-cycles the process is steadily repeated, but the plasma motion is slower because of the appearance of particles evaporated from electrodes and chamber walls.

Besides optical observations, in some experiments control measurements were made of neutron radiation in the discharge by means of a paraffin block and a beta counter with silver. In a number of cases a neutron radiation was discovered which was 3–4 times the background, that is $(4-6) \times 10^6$ neutrons per discharge.

DISCUSSION

In the above-described experiments the following peculiarities of the discharges of interest are:

1. Formation of the conical current surface, expanding and expelling the neutral gas from the centre to the chamber walls; a high rate of motion of this surface.

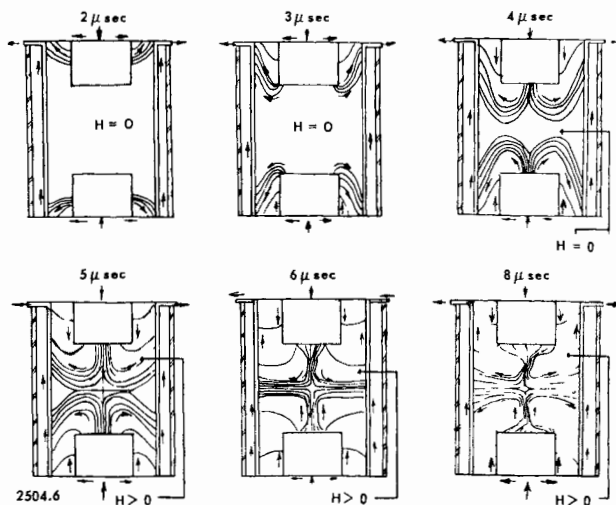
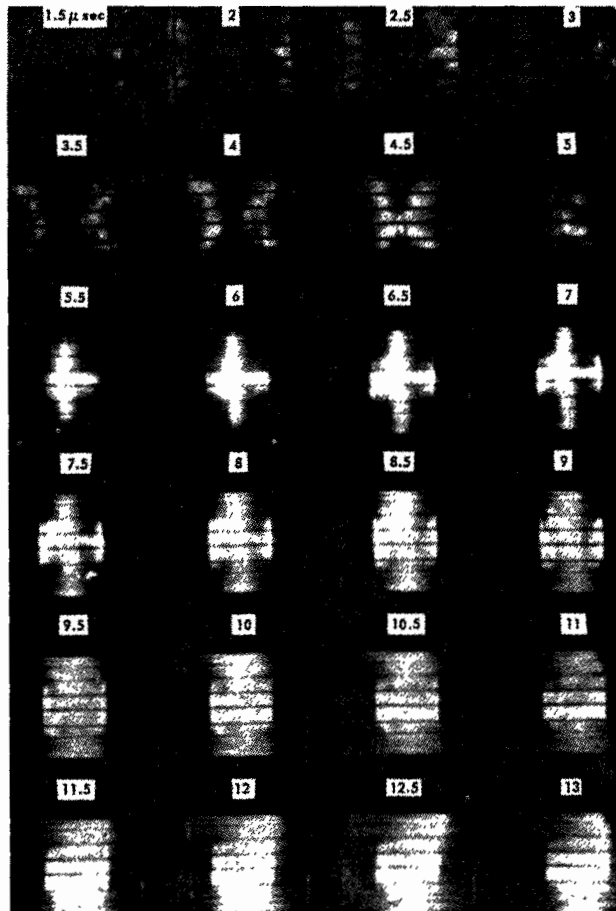


Figure 6. (a) photograph and (b) diagram of the discharge plasma pressure in the chamber with cylindrical electrodes, at an initial pressure of 1 mm Hg

2. A relatively high stability of the plasma filament.
3. Longitudinal constriction and the constriction time of the flat plasma layer.

The formation and the form of conic surface, as pointed out already, corresponds to the expected distribution of magnetic pressure on the conducting layer which is formed at the beginning of the discharge.

In the case of rod-shaped electrodes the magnetic field varies according to

$$H = 2I/cr$$

where r is the distance in cm from the electrode axis, and c is the velocity of light. Correspondingly the magnetic pressure P is

$$P = H^2/8\pi = I^2/2\pi^2 c^2 r^2$$

With these dimensions the pressure near the electrode is 90 times as great as at the chamber wall.

If the chamber diameter is increased still more, then the removed layers remain immobile and the discharge has a leader break-down character, though its nature is entirely different.

The relatively high rates of motion of the apex of the conical surface can be explained by the fact that apart from the high pressure (at $I = 300$ ka, $p = 10^9$ dyn/cm²) its motion takes place in a medium of constant particle density at the time when in the cylindrical symmetry the mass and density of gas in the constricted volume is continuously growing.

As we can see in the photographs, the current filament is formed some distance from the apex of the cone. A transitional zone exists between them, where the shock wave is slightly leading in its advance and where the current redistribution takes place. The plasma piston, which is the conical current surface, forces neutral gas to the chamber walls by means of the same charge-exchange mechanism which acts in gas constriction.

The further this surface is from the axis, the greater will be the mass of the gas and the smaller will be the specific pressure.

Acceleration or deceleration of this motion depends on the change of current in the discharge circuit. Between the plasma surface and the filament there is a residual neutral and partially ionized gas, through which a considerable portion of current can flow. The unusual brightness of the filament is to be explained by excitation of neutral particles advancing to its borders.

Instability of the current filament is observed in those regions where shock waves meet current surfaces and where there are high gradients of gas density. The plasma behaviour in such places is near to what we observe during the cylindrical constriction.

Where the perturbations are feeble or do not exist, the filament persists stably almost to the time the current passes through zero. The geometry of the constricting layer for the longitudinally moving plasma differs substantially from the one of cylindrical symmetry. The expansion of the disc begins in those places where magnetic pressure is relatively lower.

Experiments in longitudinal constriction are interesting because they demonstrate the possibility of focusing and producing plasma jets with the aid of guiding electrodes. Such a focusing was achieved in a number of tests. A change in the form of the vessels and the use of external magnetic fields will probably make it possible to strengthen this effect.

REFERENCES

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2. C. Iu. Lukianov and V. I. Sinitsin, *Atomnaya Energ.*, **1**, 88 (1956).
3. H. A. Borsunov, D. V. Orlinsky and S. M. Osovets, *Atomnaya Energ.*, **4**, 149 (1958).

Record of Proceedings of Session A-6

Experimental Aspects of Plasma Physics

WEDNESDAY AFTERNOON, 3 SEPTEMBER 1958

Chairman: Mr. H. D. Smyth (USA)

Vice-Chairman: Mr. U. Facchini (Italy)

Scientific Secretaries: Messrs. T. Coor and C. Sánchez del Río

PROGRAMME

P/360 Ion cyclotron resonance T. H. Stix and R. W. Palladino
(Presented by Mr. T. H. Stix, who also included material from paper
P/361 in this volume.)

P/2211 Ion cyclotron oscillations and new injection methods L. V. Dubovoi *et al.*
(Presented by Mr. K. D. Sinelnikov.)

DISCUSSION

P/345 Magnetic compression of shock preheated deuterium A. C. Kolb

P/1790 Intense gaseous discharges J. S. Luce

P/344 The Oak Ridge thermonuclear experiment C. F. Barnett *et al.*
(Presented by Mr. P. R. Bell.)

P/2301 High-current pulse discharges A. M. Andrianov *et al.*
(Presented by Mr. A. M. Andrianov.)

P/2302 Development of a powerful discharge in deuterium V. S. Komelkov *et al.*
(Presented by Mr. E. I. Dobrokhotoov.)

P/1460 Studies of high-current gas discharges at high rates of build up S. C. Curran *et al.*
(Presented by Mr. S. C. Curran.)

P/1708 Liquid conductor model of instabilities in a pinched discharge A. Dattner *et al.*
(Presented by Mr. A. Dattner.)

DISCUSSION

DISCUSSION OF P/360 AND P/2211

The CHAIRMAN (USA): This is a beautiful example of similar work being done by good and thoughtful people in different countries. It is a very happy circumstance that they are both able to report at the same meeting and that they will be able to discuss these things.

DISCUSSION OF P/345, P/1790, P/344, P/2301, P/2302, P/1460 AND P/1708

Mr. S. COLGATE (USA): In connection with Mr. Curran's paper, I should like to suggest that the series inductance enhances the $m = 0$ instability by faci-

litating sharp pinch boundary formation. Low inductance gives a broad pinch boundary with resulting $m = 1$ instability. With the H_z field quenching of the latter instability, that is to say of the $m = 1$ kink instability, is the neutron yield expected to be small as observed in all stabilized pinch work?

Mr. S. C. CURRAN (UK): Yes.

Mr. M. HIGATSBERGER (Austria): I should like to address a question to Mr. Andrianov relating to paper P/2301: You have measured the velocity or energy of the deuterons with a Thompson spectrogram. With this same photographic plate, I imagine, you must have gained some additional information on the

ceramic or metallic impurities from the ion sources or on the ratio of single to double charge deuterons.

Mr. A. M. ANDRIANOV (USSR): Yes, we have observed impurity parabolas in exposures of a very large number of discharges at low pressures. However, for the regime I specified, at 40 kv and 200 ka, the impurity parabolas are almost indiscernible in a tenfold exposure.

Mr. KNECHTLI (USA): Dr. Luce, how large is the energy lost per centimeter by a D^+ ion of 100 to 300 kev when it passes through your carbon arc in vacuum,

the electron density in the arc being of the order of $10^{14}/\text{cm}^3$?

The CHAIRMAN: As Mr. Luce is not here at the moment perhaps Dr. Bell will answer.

Mr. P. R. BELL (USA): I cannot remember just what this was, but during 100,000 rotations of our ions through the arc, they would not lose enough energy actually to get outside the arc column. As I recall, something like 10 kev total energy was lost out of 300 kev for 100,000 rotations through the arc, which is about 1.5 cm in thickness.

PEACEFUL USES OF ATOMIC ENERGY

Proceedings of the Second United Nations International Conference in Geneva, September 1958

The following is a listing of the thirty-three volumes in the English-language edition of the Proceedings. The titles of the sessions included in each volume are given to show the main subjects dealt with therein.

VOLUME 1

Progress in Atomic Energy

Programme of the Conference

1. Opening of the Conference

2. The Future of Nuclear Power

23a. The Supply and Training of Technical Personnel

23b. Recent Developments in International Collaboration

23c. Closing of the Conference

Evening Lectures

Exhibit Catalogue

VOLUME 2

Survey of Raw Material Resources

E-5. Raw Material Supplies

E-7b. Geochemistry

E-9. Mineralogy, Geology and Prospecting

VOLUME 3

Processing of Raw Materials

E-10. Mining Aspects and Instrumentation

E-6 & E-7a. Ore Treatment

VOLUME 4

Production of Nuclear Materials and Isotopes

E-11 & E-12. Production of Nuclear Materials

C-14 & C-15. Methods of Separating Isotopes

VOLUME 5

Properties of Reactor Materials

E-14 & E-15. Properties of Reactor Materials

VOLUME 6

Basic Metallurgy and Fabrication of Fuels

E-13^f Basic Studies in Metallurgy and Ceramics

E-17 & E-18. Fabrication of Fuels

VOLUME 7

Reactor Technology

E-19, E-21 & E-22. Reactor Technology

VOLUME 8

Nuclear Power Plants, Part 1

3. Experience with Nuclear Power Plants

6. The Use of Nuclear Energy for Purposes other than Generating Electricity

7. Plans for the Construction of Nuclear Power Plants
Reactor Catalogue

VOLUME 9

Nuclear Power Plants, Part 2

B-9. Power Reactors

B-10. Power Reactor Experiments

B-11. Experience with Reactors

VOLUME 10

Research Reactors

B-5. Research and Engineering Test Reactors

B-12. Experience with Reactors

VOLUME 11

Reactor Safety and Control

B-13 & B-14a. Safety, Location and Containment

A-14. Reactor Kinetics and Control

VOLUME 12

Reactor Physics

B-17, B-18 & B-21. Reactor Physics

VOLUME 13

Reactor Physics and Economics

B-19. Reactor Physics

B-15. Fuel Cycles

B-14b. Economics of Nuclear Power

VOLUME 14

Nuclear Physics and Instrumentation

- A-18. Nuclear Instrumentation and Techniques
- A-19. Basic Physics in Nuclear Energy

VOLUME 15

Physics in Nuclear Energy

- A-21 & A-22. Basic Physics in Nuclear Energy

VOLUME 16

Nuclear Data and Reactor Theory

- A-11. Nuclear Data
- A-12. Nuclear Data and Reactor Theory
- A-13. Reactor Theory and Computing Methods

VOLUME 17

Processing Irradiated Fuels and Radioactive Materials

- C-17 & C-18. Processing Irradiated Fuels
- C-19. Handling Radioactive Materials

VOLUME 18

Waste Treatment and Environmental Aspects of Atomic Energy

- C-21 & C-22. The Treatment of Radioactive Wastes
- D-19. Environmental Aspects of the Large-Scale Use of Atomic Energy

VOLUME 19

The Use of Isotopes: Industrial Use

- 5b. Progress in the Use of Isotopes
- D-7. The Use of Isotopes in Industry

VOLUME 20

Isotopes in Research

- D-6. Isotope Production and Use in Research

VOLUME 21

Health and Safety : Dosimetry and Standards

- 5a. Operational Experience in Health and Safety
- D-15. Dosimetry and Standards

VOLUME 22

Biological Effects of Radiation

- D-9 & D-10. Biological Effects of Radiation

VOLUME 23

Experience in Radiological Protection

- D-11 & D-12. Experience in Radiological Protection

VOLUME 24

Isotopes in Biochemistry and Physiology, Part 1

- D-13. Use of Isotopes in Biochemistry and Physiology

VOLUME 25

Isotopes in Biochemistry and Physiology, Part 2

- D-14. Use of Isotopes in Biochemistry and Physiology

VOLUME 26

Isotopes in Medicine

- D-17 & D-18. Use of Isotopes in Medicine

VOLUME 27

Isotopes in Agriculture

- D-21 & D-22. Use of Isotopes in Agriculture

VOLUME 28

Basic Chemistry in Nuclear Energy

- C-9, C-10 & C-11. Basic Chemistry in Nuclear Energy

VOLUME 29

Chemical Effects of Radiation

- C-12 & C-13. Chemical Effects of Radiation

VOLUME 30

Fundamental Physics

- 15. Recent Developments in Fundamental Physics
- A-17. Physics of Elementary Particles

VOLUME 31

Theoretical and Experimental Aspects of Controlled Nuclear Fusion

- 4. Possibility of Controlled Fusion
- A-5. Theoretical Aspects of Plasma Physics
- A-6. Experimental Aspects of Plasma Physics

VOLUME 32

Controlled Fusion Devices

- A-7 & A-9. Controlled Fusion Devices
- A-10. Special Topics and Instrumentation in Fusion

VOLUME 33

Index of the Proceedings

- Subject Index
- Numerical Index
- Author Index